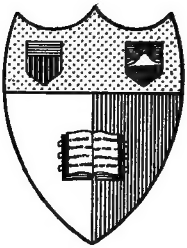

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A TEXT-BOOK

OF

ELECTRICAL ENGINEERING

A TEXT-BOOK
OF
ELECTRICAL ENGINEERING

TRANSLATED FROM THE GERMAN OF

DR ADOLF THOMÄLEN

BY

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FIFTH EDITION

LONDON
EDWARD ARNOLD
1920

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PREFACE TO THE FIRST EDITION

GREAT difficulty has always been experienced in recommending a text-book to the second and third year students of electrical engineering at the Central Technical College. There appeared to be nothing to bridge the gap between the elementary text-books and the specialised works on various branches of electrical engineering. I have no doubt that the same difficulty has been experienced by all who have lectured to the more advanced students in our Universities and Technical Colleges.

It was while contemplating the preparation of a suitable text-book that my attention was drawn to Dr Thomälen's work, written with the same object and covering much the same ground as I had contemplated. The present work is a translation of the second edition of this "Kurze Lehrbuch der Elektrotechnik," but includes additional matter which it is intended to introduce into the third German edition now in preparation. Generally speaking, it is a close translation of the original, but I have not hesitated to use other methods where they appeared preferable. This applies especially to Section 101, which differs entirely from the corresponding Section in the German edition. A Section has been added to Chapter XX, dealing with the Cascade Converter.

The book is concerned almost exclusively with principles and does not enter into details of the practical construction of apparatus and machines. It is not intended to take the place of the standard works on the design of dynamo machinery, both direct and alternating, but rather to lay a thorough foundation which shall make the study of such works more profitable.

In the preface to the first German edition, Dr Thomälen expresses his desire to lead the student to enter mentally into the various phenomena and to give him a physical conception of the underlying principles. This "*elektrotechnisches Denken*" will enable the student to predetermine with confidence, either graphically or mathematically, the behaviour of various electrical machines and apparatus.

The higher mathematics employed never goes beyond the simplest elements.

G. W. O. HOWE.

CENTRAL TECHNICAL COLLEGE,
SOUTH KENSINGTON.
June, 1907.

PREFACE TO THE FIFTH EDITION

In preparing this new edition a number of minor alterations have been made and several subjects introduced which were not dealt with in the earlier editions. Among these may be mentioned the differential booster and the automatic voltage regulation of alternators. The sections dealing with single-phase commutator motors have been extended and rewritten. Descriptions of machines of obsolete types have been omitted.

The symbols adopted throughout the book have been modified so as to bring them into agreement with the recommendations of the International Electrotechnical Commission. For the same reason the alternate current vector diagrams have been redrawn so that the direction of rotation is anti-clockwise. Capital letters are used to signify continuous currents and voltages and also the root-mean-square values of alternating currents and voltages. Small letters signify instantaneous values except in the case of magnetic flux and its density, the root-mean-square values of which are rarely used, and for the instantaneous values of which capital letters can therefore be employed. Maximum or crest values of alternating quantities are represented either by a circumflex being placed over the symbol or by the latter being printed in heavy (Clarendon) type.

G. W. O. HOWE.

South Kensington.
June, 1920.

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A TEXT-BOOK OF ELECTRICAL ENGINEERING

CHAPTER I

1. The electric current.—2. Electromotive force.—3. Ohm's law.—4. Resistance, dependent on material, cross-section, length, and temperature.—5. Kirchhoff's rules.—6. Resistances in parallel.—7. Series and parallel arrangement of cells.—8. Wheatstone's bridge.—9. Potentiometer method of measuring electromotive force.—10. Joule's law, electrical energy and electrical power.—11. Potential difference.—12. Energy lost in heating conductor.

1. The Electric Current.

THE heating of a glow lamp filament, the power exerted by an electric motor, the magnetisation of the iron in an electromagnet, the decomposition of liquids, and many other phenomena, are said to be due to the effect of the electric current. The question arises: Why do we ascribe these phenomena to a current, i.e. a continuous movement of electricity?

The electricity with which we are here concerned is essentially the same as static or frictional electricity. If a glass rod is rubbed with silk it becomes electrified and attracts small light bodies. In the same way a rod of resin, when rubbed with a woollen cloth, becomes electrified, but in the opposite sense to the glass rod. This difference of sense or sign is based on the fact that the electrical charges of the glass and resin mutually neutralise each other. Because of this, the electricity of the glass is called positive, and that of the resin negative. The equalising exchanges between positive and negative electricity follow in exactly the same way as those between heat and cold, high pressure and low pressure. As water flows from the higher to the lower level, or heat from the hot to the cold body, so electricity is said to flow from the higher positive to the lower negative level. It is now known that what really occurs is a movement of negative electricity in the opposite direction. This equalising flow of electricity is what we call the electric current. There is no essential difference between the flow of electricity between the terminals of a cell, or of a dynamo, and that between two metal balls charged with electricity of opposite sign.

Although the same in principle, there is a great difference as regards quantity between the frictional electricity with which we are familiar and the current of electricity from a cell or dynamo. The quantity of electricity stored in a Leyden jar, which flows through the spark on discharge, is very small compared with the quantity flowing in a short time through a Daniell cell, and quite negligible compared with the quantity flowing in a day through the supply mains of a town. The pressure of the frictional electricity is, on the other hand, very much higher than that of the ordinary dynamo. This can be seen from the ease with which frictional electricity sparks across large air-gaps. Moreover, in the case of frictional electricity, the equalising flow is, as a rule, a sudden momentary rush known as the discharge, and can hardly be called a current.

The strength of the current is defined as the quantity of electricity that flows through the cross-section of the conductor in one second. It is immaterial whether it is a movement of positive electricity in one direction or of negative in the other, or a combination of both. When one speaks of the direction of the current it is always assumed to be a movement of positive electricity. To measure the current we make use of its electrolytic or of its magnetic effect. We say that one current is twice as strong as another when it deposits twice as much silver from a silver solution in the same time, or when it exerts double the force on a magnetic needle under exactly the same conditions.

Now, the absolute determination of a current from its magnetic effect is a very laborious procedure, whereas the electrolytic method is simple, involving merely the accurate weighing of a silver deposit. For this reason the legal definition of the unit of current is based on its electrolytic effect*. According to this definition, **that current has unit strength which deposits 1·118 milligrammes of silver per second from a silver solution. This current is called an Ampere.**

For practical purposes the strength of a current is more conveniently measured by means of its magnetic effect, a current-carrying coil acting on a magnet or piece of iron with resulting attraction or change of direction. Such instruments are called Ammeters. If intended for the measurement of very small currents they are called Galvanometers.

The quantity of electricity that passes through the cross-section of a conductor per second, when the current strength is one ampere, is called a Coulomb. If we assume, for example, that the current passing through a glow lamp is 0·5 ampere, then 0·5 coulomb of electricity passes per second through any cross-section of the conductor. The quantity of electricity passing in an hour is 0·5.3600 or 1800 coulombs. In general, if

Q = the quantity of electricity in coulombs,
I = the current in amperes,
and t = the time in seconds,
then

$$Q = I \,.\, t \text{ coulombs} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We see therefore that an ampere-hour is equal to 3600 coulombs.

2. Electromotive Force.

We have already mentioned the idea of electricity flowing from a higher to a lower level, the word "level" being used in a special sense. This difference of electrical level can be produced by machines in which windings are continually passed before the poles of magnets, or by galvanic cells consisting of two chemically different plates in a liquid. The Bichromate cell, for example, consists of zinc and carbon plates dipping into a mixture of chromic acid (CrO_3) and sulphuric acid (H_2SO_4). If the cell be examined with the aid of an electrometer, the upper end of the carbon is found to be positively charged, while the upper end of the zinc is charged negatively. Hence, a difference of electrical level exists between the upper ends of the carbon and zinc or between the terminals of the cell, and if these terminals be joined by means of a metallic or liquid conductor, electricity will flow from the higher to the lower level, that is, an electric current will flow through the conductor. Positive electricity will flow from the carbon *C* through the external conductor to the zinc *Zn* (Fig. 1).

---

* This is not strictly true for England, as the ampere is legally defined as the current which gives a certain reading on an ampere balance at the Board of Trade. This balance was, however, standardised by deposition of silver and not from its dimensions. It has recently been removed to the National Physical Laboratory.

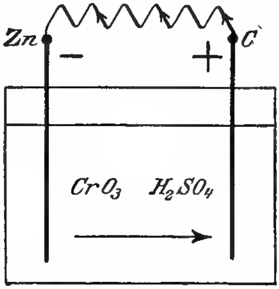

Fig. 1

It is found, however, that a steady flow of electricity continues through the external conductor so long as it connects the terminals. Hence, we must assume that the electricity, which flows externally from the carbon to the zinc, passes back in the cell from the zinc to the carbon, being lifted, as it were, from the lower to the higher level. In a similar manner the water which flows from the mountains to the sea is raised again to the higher level by evaporation due to the heat of the sun. We see, therefore, that there is something in the cell which causes a movement of electricity and is able to maintain a steady flow or current round the circuit, through the internal resistance of the cell itself, and through the resistance of the external conductor. **The cause of this movement of electricity is called the electromotive force of the cell.** It is analogous to a pump which raises the water from a lower to a higher level. The electromotive force of the cell is equal to the difference of level or pressure measured between its terminals when on open circuit, i.e. with the external circuit broken.

The question arises: How can we determine the magnitude of this difference of pressure, or compare it with any other pressure? We could use a gold-leaf electroscope, and determine the pressure from the deflection of the gold leaves. As a rule, however, we determine the pressure from the strength of the current it produces. If, for example, two vessels communicate with each other by means of a pipe having a known constant resistance, the quantity of water flowing through it per second is proportional to the difference of pressure or of level in the two vessels. In the same way we connect one and the same conductor, viz. the coils of a galvanometer, successively between the terminals of the various cells which are to be compared, or between the two points between which the pressure difference is to be measured. The strength of the current in the galvanometer is then a measure of the magnitude of the electrical pressure difference, i.e. of the electromotive force of the cell.

In this way we find that the electromotive force of a given type of cell is an essentially constant magnitude, quite independent of the size of the cell. In this respect a cell resembles a high-level reservoir. The difference of level and thereby also the pressure is fixed, once for all, by the difference in altitude between the reservoir and the point at which the water power is used; it is quite immaterial whether the reservoir is large or small, so long as it is kept full. The current or flow of water is entirely in the hands of the consumers, depending only on the number of cocks opened. Hence, we see that it is very misleading to speak about the current strength of a cell; it suggests the idea of a certain fixed current in the cell, which must always flow when the cell is in use. This idea is entirely wrong, for the cell has only one essentially characteristic property and that is its electromotive force, which is present whether the cell is in use or standing idle. The strength of the current is, however, entirely in our hands, depending, as it does, on the convenience of the path open to it, that is, on the magnitude of the external resistance connected between its terminals.

We are now at liberty to choose any pressure as the unit with which to compare all others. **The unit used in practice is roughly half as large as the electromotive force of the above-mentioned Bichromate cell, and is called a Volt.**

The following table gives particulars of the cells commonly employed:

Type of Cell	Composition of Cell	Electromotive force in volts
Bichromate	Zinc and carbon in sulphuric and chromic acid	2
Daniell	Zinc in dilute sulphuric acid, copper in solution of copper sulphate	1·07
Bunsen	Zinc in dilute sulphuric acid, carbon in strong nitric acid	1·8 to 1·9
Leclanché	Zinc and carbon in solution of sal-ammoniac, the carbon being surrounded with manganese dioxide	1·4
Clark Standard	Zinc in zinc sulphate solution, mercury in solution of mercurous sulphate	1·433 (15° C.)
Weston Standard	Cadmium amalgam in solution of cadmium sulphate, mercury in mercurous sulphate	1·019 (15° C.)

The zinc, or the cadmium, which is chemically very similar to zinc, forms the negative pole in all these cells, and the current flows from the positive pole to the zinc through the external circuit.

3. Ohm's Law.

We have already seen that if the same external conductor be connected successively across cells having different electromotive forces, the strength of the current, in each case, is proportional to the electromotive force. We go now a step further and connect the terminals of the same cell successively by means of wires of different materials, lengths, and cross-sections. We observe that the current differs in each case, and conclude therefore that the various wires offer different resistances to the current. If the current is small, we conclude that the wire has a large resistance, whereas, if the current is large, the resistance of the wire must be small. Hence, the strength of the current is inversely proportional to the resistance of the path. Similarly, the quantity of water, forced per second through a narrow pipe by means of a given pressure, is smaller, the longer the pipe, the greater the friction against the walls of the pipe, and the smaller the bore. In other words, the quantity of water per second is inversely proportional to the total resistance of the pipe. We come thus to the conclusion that the **current is proportional to the electromotive force and inversely proportional to the electrical resistance.** Hence, if

E = the electromotive force in volts,
I = the current in amperes,
and R = the resistance of the whole circuit,

then
$$I = \frac{E}{R} \quad \text{..............................(2).}$$

This fundamental law of electrical technology is known as Ohm's law. At first sight, it might appear necessary to introduce a coefficient after the sign of equality; this, however, is unnecessary, or rather it becomes unity if the resistance be expressed in terms of a suitable unit. Inversely, the unit of electrical resistance is determined by Ohm's law as expressed in (2), and cannot now be arbitrarily chosen. If we put $I = 1$ and $E = 1$, then from (2) we have $R = 1$. Hence, **the unit of resistance is that resistance through which a pressure of 1 volt produces a current of 1 ampere. This unit of resistance is called the Ohm.** It is found experimentally that a column of mercury 106·3 cm long and 1 sq. mm in cross-section has a resistance of 1 ohm.

To take an example, let the electromotive force of a dynamo be 115 volts, its resistance 0·05 ohm, and the external resistance 1·1 ohms; then

$$E = 115 \text{ volts},$$
$$R = 0{\cdot}05 + 1{\cdot}1 = 1{\cdot}15 \text{ ohms},$$
$$I = \frac{E}{R} = \frac{115}{1{\cdot}15} = 100 \text{ amperes.}$$

The legal definition of the electrical units follows, however, a different procedure from that which we have followed. We have defined the units of pressure and current, and have derived the unit of resistance from them. Legally, however, the ampere is defined as that current which deposits 1·118 mg of silver per second, and the ohm as the resistance at 0° C. of a column of mercury having a length of 106·3 cm and a mass of 14·4521 grammes. From these two legally defined units, it follows that **a volt is that pressure which produces a current of 1 ampere in a resistance of 1 ohm***

Equation (2) can be written thus,

$$R = \frac{E}{I} \text{..................................(3).}$$

---

* Considerable confusion and difference of opinion exists at the present time on the subject of legal definition of electrical quantities. It is now being considered by an international committee.

This equation expresses mathematically the fact which we have already mentioned, viz. that if, in an experiment, we find the current to be small, in spite of a large electromotive force, we can conclude that the resistance is large. Hence, we see that resistance can only be accurately defined as the ratio of electromotive force to strength of current. One must be very careful, however, not to regard resistance as a back pressure. The back pressure is only obtained when we multiply the resistance by the current flowing through it. Thus, equation (2) can be written thus,

$$E = I.R \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The left-hand side of this equation is the pressure or electromotive force of the source of supply, while the right-hand side is the pressure necessary to send the current I through the resistance R. They are both equal to one another.

Ohm's law, however, is not only applicable to the circuit as a whole, but to each individual part of the circuit. Suppose, for example, that the current I flows successively through the resistances R_1, R_2 and R_3 (Fig. 2). If the differences of potential between the terminals of the resistances R_1, R_2 and R_3 be denoted by V_1, V_2 and V_3 respectively, it is found by experiment that

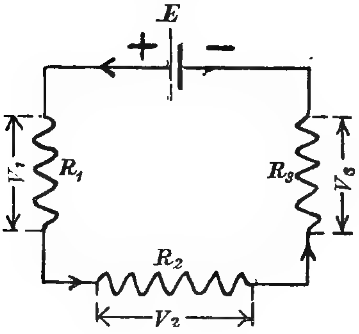

Fig. 2

$$V_1 = I.R_1, \qquad R_1 = \frac{V_1}{I},$$

$$V_2 = I.R_2, \qquad R_2 = \frac{V_2}{I},$$

$$V_3 = I.R_3, \qquad R_3 = \frac{V_3}{I}.$$

We see, therefore, that although we may make the current as large as we like, by using, say, a greater number of cells, the ratio of the terminal pressure V_1 to the current I has always a constant value. This is on the assumption that neither dimensions, nature nor temperature of the material undergoes any change. This constant ratio we call the resistance R_1. It is really nothing more than a coefficient by which the proportionality of the drop of pressure in a conductor to the current can be expressed as in the equation $V_1 = I.R_1$.

If, for example, a resistance of 1 ohm is placed in circuit with two arc lamps connected in series, and the current is 15 amperes, then the drop of pressure in the resistance is $1.15 = 15$ volts. If the supply pressure is 110 volts, there remains only $110 - 15 = 95$ volts for the two lamps.

In general, if V denote the potential difference between the terminals of a resistance R, then

$$V = I.R \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

This equation enables us to understand the difference between ammeters or current strength meters and voltmeters or potential difference meters. Generally speaking, both classes of instruments depend on the magnetic effect of the current. They differ, however, in details of construction and in the method of connection. Ammeters are put in series with the main current and have as a rule few turns of thick wire. The heat developed, and the pressure drop in the instrument, are consequently very small, even with heavy currents. If, on the other hand, an instrument is to be used as a voltmeter, it is wound with many turns of fine wire and connected, in series, if necessary, with a high constant resistance, across the points between which the potential difference is to be measured, e.g. the dynamo terminals in Fig. 3. The voltmeter is thus connected in parallel with the resistance across which the terminal pressure is being measured.

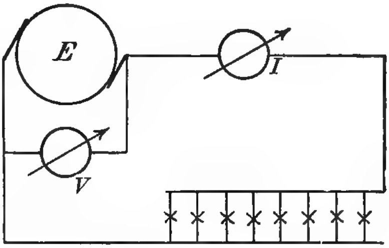

Fig. 3

The current that flows through the voltmeter is a loss to the supply. The large resistance of the instrument is for the object of keeping this current as small as possible. This is also necessary in order that the supply pressure and current may not be affected by the connection of the voltmeter. To obtain a powerful magnetic effect it is necessary to employ a large number of turns in its coils, and we are compelled, from considerations of economy and space, to use fine wire. The instrument acts really as an ammeter in that its deflection is caused by the current flowing through it. If, however, this current be multiplied by the resistance of the voltmeter, we obtain the potential difference between its terminals, and the scale can therefore be marked in volts. That ammeters and voltmeters are the same in principle can best be seen from the fact that there are many instruments which can be used for both purposes. If, for example, the sensibility of an ammeter be such that a deflection of one degree is produced by a thousandth of an ampere, and external resistance be added bringing the total resistance up to 1000 ohms, then a deflection of one degree will correspond to a potential difference of $\frac{1}{1000} \cdot 1000 = 1$ volt.

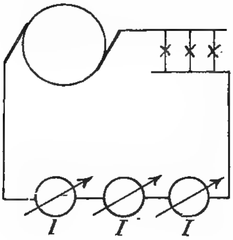

Fig. 4

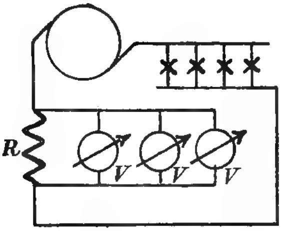

Fig. 5

From the above it is evident that, to compare two or more ammeters, we must connect them in series (Fig. 4). If, on the other hand, we wish to compare several voltmeters, we must connect them in parallel, each with its terminals connected to the ends of the resistance R, the pressure across which is to be measured. Such an arrangement is shown in Fig. 5, in which the lamps enable us to regulate the pressure across the resistance R by dropping a larger or smaller fraction of the terminal pressure of the dynamo.

Above all, when comparing two voltmeters, one must be careful not to connect them in series as, owing to their unequal resistances, the total pressure is distributed unequally between them.

Having thus explained the principles underlying the measurement of current and pressure, mention must be made of their combined use to determine resistance. This indirect method of resistance measurement by observing the pressure and current is specially suitable for very small or very large resistances. If, for example, a current of 10 amperes be passed through the armature of a stationary electric motor and the potential difference between the brushes be measured and found to be 2 volts, then the armature resistance R_a is given by the equation

$$R_a = \frac{V}{I} = \frac{2}{10} = 0{\cdot}2 \text{ ohm.}$$

The advantage of this method is that the resistance is measured exclusive of the resistance of any connecting leads. We have assumed that the resistance to be measured does not include an electromotive force; if it does, the calculation of the resistance is not so simple.

4. The Relations between the Resistance of a Conductor and its Material, Cross-section, Length, and Temperature.

The importance of Ohm's law lies largely in the fact that the resistance of a conductor, that is, the constant ratio of the potential difference between its ends to the current flowing through it, stands in a simple relation to its cross-section and its length; it depends largely on the material and to a far lesser degree on the temperature. Thus, if

l = length of conductor in metres,
A = cross-section in sq. mm,
ρ = a coefficient depending on the material,

it is found experimentally that

$$R = \rho \frac{l}{A} \quad \text{.................................(6).}$$

Hence, the resistance of a wire is proportional to its length and inversely proportional to its cross-section. The coefficient ρ is very different for different materials. Its meaning can be seen by putting $l = 1$ and $A = 1$ in equation (6). R is then equal to ρ, so that **the coefficient ρ is the resistance of a wire of the given material 1 metre long and 1 sq. mm cross-section. This is called the specific resistance of the material**[*]. To determine its value for a given wire, the values of l, A and R are found from experiment and ρ calculated from them. In this way we find the following values:

---

[*] It is customary in England to define the specific resistance as the resistance of a wire 1 cm long and 1 sq. cm cross-section. In this case the values of ρ given in the table must be multiplied by 10^{-4}.

Copper at 15° C. ... ... $\rho = 0\cdot017$,
Mercury ... ... $\rho = 0\cdot94 = \frac{1}{1\cdot063}$,
German Silver ... ... $\rho = 0\cdot2 - 0\cdot4$,
Carbon ... ... $\rho = 100 - 1000$,
Sulphuric Acid (25—30 %) ... $\rho = 14{,}000$.

Resistances for the purpose of weakening the current or for causing a drop in the pressure are made, therefore, of German silver, whereas the windings of machines and the leads are made of the best conducting material, viz. copper, to avoid all unnecessary pressure losses. Even then the pressure losses are not inconsiderable. Consider, for example, the supply of 60 amperes to a point 20 metres distant, through wires with a cross-section of 50 sq. mm. The total length of wire, go and return, is

$$l = 2.20 = 40 \text{ metres},$$

and we have
$$R = \rho\frac{l}{A} = \frac{0\cdot017 \,.\, 40}{50} = 0\cdot0136 \text{ ohm}.$$

The pressure drop V in the leads is then found as follows:

$$V = I.R = 60.0\cdot0136 = 0\cdot8 \text{ volt}.$$

The specific resistance is, however, not constant, but increases in all metals with increasing temperature, the increase of resistance being directly proportional to the rise of temperature. **The temperature coefficient of a material is defined as the increase in resistance for a temperature rise of 1 degree Centigrade, the increase being expressed in terms of the resistance at 0° C.** Thus, when we say that experiment gives a mean temperature coefficient for copper of 0·00427, we mean that for every degree rise of temperature the resistance increases by 0·00427, that is 0·427 per cent. of its value at 0° C. If, then,

R_1 be the resistance at the initial temperature T_1,
R_2 the resistance at the final temperature T_2,
R_0 the resistance at 0° C.,
and α the temperature coefficient,

it follows that
$$R_1 = R_0\,(1 + \alpha . T_1),$$
$$R_2 = R_0\,(1 + \alpha . T_2),$$
and
$$R_2 - R_1 = R_0\,(T_2 - T_1)\,\alpha;$$
therefore
$$\frac{R_2 - R_1}{R_1} = \frac{\alpha\,(T_2 - T_1)}{1 + \alpha T_1},$$
and
$$T_2 - T_1 = \frac{R_2 - R_1}{R_1}\left(\frac{1}{\alpha} + T_1\right) = \frac{R_2 - R_1}{R_1}(234\cdot5 + T_1)$$

for copper. In practice, when considering the heating of dynamos and motors, T_1 is usually about 15° C. This leads to the approximate equation

$$\frac{R_2 - R_1}{R_1} . 100 = 0\cdot4\,(T_2 - T_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

which corresponds to an increase of 0·4 per cent. in the resistance for each degree Centigrade. This is strictly accurate when $T_1 = 15\cdot5$° C.

Under working conditions the temperature of electrical machines rises about 50° C., which corresponds to an increase in resistance of $0{\cdot}4 . 50 = 20$ per cent. The specific resistance of warm copper is, for this reason, usually taken as 0·02 instead of 0·017.

With the help of the temperature coefficient 0·004, we are able to determine the temperature rise of parts of a machine which are quite inaccessible to a thermometer. Suppose, for example, that the resistance of the field winding at 15° C. is 50 ohms, that the field current after running for several hours is 2 amperes, and that the pressure across the terminals of the field winding is 114 volts. The resistance of the field winding under working conditions is therefore $\frac{114}{2} = 57$ ohms, and we have

$$R_1 = 50,\ R_2 = 57,\ T_1 = 15, \text{ and } \alpha = 0{\cdot}004.$$

From equation (7) we have

$$\frac{57 - 50}{50} . 100 = 0{\cdot}4 (T_2 - 15),$$

whence

$$T_2 = 50^\circ \text{ C.}$$

The temperature rise is therefore 35° C.

As the temperature coefficient of all single metals, with the exception of mercury, is relatively large, they cannot be used for standard resistances. For such purposes we use an alloy, such as German silver, an alloy of copper, nickel and zinc, which, in addition to the advantage of a high specific resistance, possesses a very small temperature coefficient, viz. 0·0002 to 0·0004. The resistance of manganin, an alloy of copper (84 per cent.), manganese (12 per cent.), and nickel (4 per cent.), is practically independent of the temperature.

In contrast to the metals, the resistance of carbon and of liquid conductors decreases with rising temperature, that is, their temperature coefficient is negative. The temperature changes to which a glow-lamp filament is subjected are so great that its resistance must necessarily be determined while burning, i.e. by the ammeter and voltmeter method.

Calculations are sometimes simplified by the use of the *conductance*, that is, the reciprocal of the resistance of a conductor. The specific conductance or the conductivity of copper, for example, is $\frac{1}{0{\cdot}017} = 59$ or roughly 60. It was formerly the custom to define the conductivity of liquids in another way, viz. by comparison with mercury; modern electro-chemical works, however, follow the above method and use the reciprocal of the ohm. The reciprocal of the ohm is sometimes called a "mho."

5. Kirchhoff's Rules.

(*a*) KIRCHHOFF'S FIRST RULE.

The sum of the currents flowing to any point is equal to the sum of the currents flowing away from that point. Calling the currents flowing to the point positive and those flowing away negative, we have for any point in a wire or network,

$$\Sigma I = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

This rule is of fundamental importance. The idea of many beginners that the electricity is gradually used up as it flows through the circuit from the positive to the negative terminal is entirely wrong. The whole quantity of electricity that leaves the positive terminal flows through the circuit to the negative terminal and passes through the battery or dynamo to the positive terminal whence it started. Hence, there is no continual generation of electricity but simply a setting in motion around the whole circuit of a quantity of electricity, the rate of flow, that is the current, being the same at every point in the circuit. Similarly, water supplied from a reservoir to a house is not used up in the sense of ceasing to exist; the whole quantity flows from the higher level to the lower level, to be raised again, if by no other means, by evaporation, to the higher level. When we speak in practice of a loss of current, we simply mean that a part of the current, by taking a bypath, has prevented its useful employment, and represents therefore a monetary loss. It is, therefore, not the current, or a quantity of electricity, which is used up, but pressure.

By means of Kirchhoff's first rule, we are enabled, from a knowledge of two or more currents at a point in a network, to calculate the other current. For example, let the terminal pressure V of a shunt dynamo be 220 volts, the current I in the external circuit (Fig. 6) 100 amperes, and the resistance R_m of the shunt field winding 50 ohms. We wish to find the armature current I_a. By Ohm's law,

$$I_m = \frac{V}{R_m} = \frac{220}{50} = 4{\cdot}4 \text{ amperes.}$$

By Kirchhoff's first rule,

$$I_a = I + I_m = 100 + 4{\cdot}4 = 104{\cdot}4 \text{ amperes.}$$

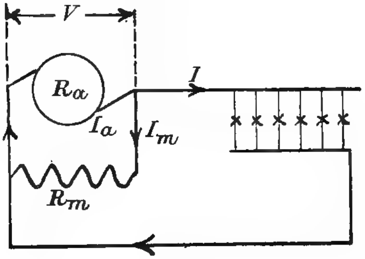

Fig. 6

(*b*) KIRCHHOFF'S SECOND RULE.

In every closed circuit, the sum of the products of current and resistance taken round the whole circuit is equal to the sum of the electromotive forces in the same circuit. Hence, for any closed circuit

$$\Sigma (IR) = \Sigma E \quad\text{.............................(9).}$$

In applying this rule to any circuit, or closed mesh forming part of a network, we can carry out the summation in either direction, all currents or electromotive forces in the opposite direction being given the negative sign. If the direction of any current or electromotive force is unknown it must be assumed. If the calculation then gives a negative result for this quantity it means that its direction is opposite to that which had been temporarily assumed.

As an example, suppose a dynamo giving an E.M.F. (electromotive force) of E = 116 volts to be connected up in opposition to a battery of 50 accumulators, each having an E.M.F. of 2 volts, in order to charge it (Fig. 7). Let the dynamo resistance R_a be 0·1 ohm, the battery resistance R_b 0·18 ohm, and the resistance of the leads R_l 0·12 ohm. We are required to find the current I and the terminal pressure across both dynamo and battery.

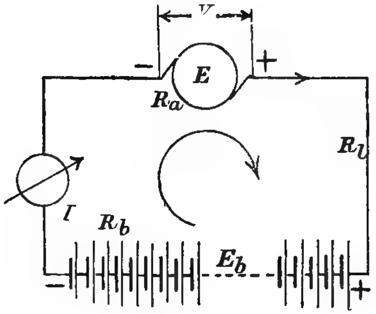

Fig. 7

The back E.M.F. of the battery

$$E_b = 50 . 2 = 100 \text{ volts.}$$

If now we go round the circuit in a clockwise direction, as indicated by the arrow, and carry out the summations of equation (9), we get

$$IR_a + IR_b + IR_l = E - E_b,$$

or $$I = \frac{E - E_b}{R_a + R_b + R_l}$$

$$= \frac{116 - 100}{0{\cdot}1 + 0{\cdot}18 + 0{\cdot}12} = 40 \text{ amperes.}$$

The various ohmic pressure losses, or drops, are then as follows:

$$\begin{aligned} I . R_a &= 40 . 0{\cdot}1 = 4 \text{ volts,} \\ I . R_b &= 40 . 0{\cdot}18 = 7{\cdot}2 \text{ ,,} \\ I . R_l &= 40 . 0{\cdot}12 = 4{\cdot}8 \text{ ,,} \\ \text{adding the back E.M.F.} &= 100 \text{ ,,} \\ \hline \text{Total} &= 116 \text{ volts.} \end{aligned}$$

We see that the E.M.F. of 116 volts is exactly used up in overcoming the back E.M.F. of the battery and in supplying the pressure losses, or drops, in the various resistances.

In finding the terminal pressure of the dynamo, which is also the pressure across the external circuit, we have to remember that a part of the electromotive force of the dynamo is used up in the dynamo itself in driving the current through its internal resistance. The greater part, however, remains available for the external circuit. Similarly, in a water supply the full pressure corresponding to the difference of level is only obtained when all cocks are closed. Directly water is drawn, a pressure loss occurs due to friction between the water and the inner surface of the pipes, and the pressure available for the consumer drops below that corresponding to the difference of level. To find the terminal pressure V we must therefore subtract the internal drop from the electromotive force, thus,

$$V = E - IR_a = 116 - 4 = 112 \text{ volts.}$$

For the battery, which is being charged, the relations are, however, quite different. The terminal pressure V_b across the battery has a double duty, viz. to overcome the back E.M.F. E_b and to drive the current through the ohmic resistance of the battery. Hence

$$V_b = E_b + IR_b = 100 + 7{\cdot}2 = 107{\cdot}2 \text{ volts.}$$

6. Resistances in Parallel.

It is self-evident that the resistance of several conductors connected in series is equal to the sum of their individual resistances. If, on the other hand, two resistances R_1 and R_2 are connected in parallel, as shown in Fig. 8, the combined resistance is evidently less than that of either, for the current has a more convenient path than would be offered by either resistance alone. The conductance of the loop is, as can be shown experimentally, equal to the sum of the conductances of the parallel branches, that is,

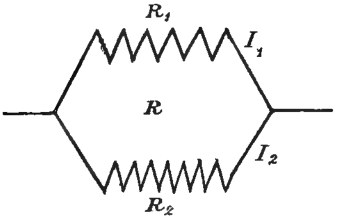

Fig. 8

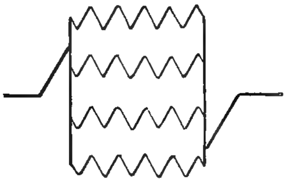

Fig. 9

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

From this we have
$$R = \frac{R_1 R_2}{R_1 + R_2} \quad \text{..............................} (10).$$

In making very low resistances, for example, it is almost impossible to cut the wire to the exact length required. It is therefore cut so as to have a resistance R_1, slightly higher than that required, which is then brought down to the correct value by connecting another resistance R_2 in parallel with it. Suppose that R_1 has been made 0·102 ohm. The question is: What resistance must be put in parallel with R_1 so that the total resistance R may be exactly 0·1 ohm? From equation (10) it follows that

$$0{\cdot}1 = \frac{0{\cdot}102 \, . \, R_2}{0{\cdot}102 + R_2},$$
$$\therefore R_2 = 5{\cdot}1 \text{ ohms.}$$

Suppose now that an error of 2 per cent. is made in the adjustment of this 5·1 ohms, so that R_2 is really only 5 ohms. The total resistance will then be

$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{0{\cdot}102 \, . \, 5}{0{\cdot}102 + 5} = 0{\cdot}09996 \text{ ohm},$$

an error of only 0·04 per cent.

If more than two resistances be connected in parallel, we find in the same way that

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots \quad \text{....................} (11).$$

If the individual resistances are equal, the calculation is much simpler. Suppose, for example, that the armature of a 4-pole machine is parallel wound, so that there are four equal parallel paths through it (Fig. 9). If the total length of wire on the armature is 200 metres and its cross-section 10 sq. mm, the resistance of each branch, taking the specific resistance of warm copper as 0·02, is equal to

$$\rho \cdot \frac{l}{A} = 0{\cdot}02\,\frac{200/4}{10}\text{ ohm.}$$

The combined resistance of the four equal parallel branches is only a quarter of this, so that

$$R_a = \frac{0{\cdot}02 \, . \, 200}{4^2 \, . \, 10} = 0{\cdot}025\text{ ohm.}$$

To find the ratio of the currents in the two branches in Fig. 8 we observe that both resistances R_1 and R_2 have the same terminal pressure V. If I_1 be the current in the resistance R_1, and I_2 that in the resistance R_2, then we have

$$V = I_1R_1 = I_2R_2,$$

wherefore

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} \quad \text{..................................(12).}$$

Hence, when a current divides between two or more parallel paths, the currents in the various branches are inversely proportional to their resistances. It is on this principle that large currents are measured by means of sensitive galvanometers carrying only very small currents. The galvanometer is put in parallel with a known low resistance, and is said to be shunted. The main current I will divide, a small portion I_g flowing through the galvanometer of resistance R_g, while the remainder I_s flows through the shunt R_s. We have then (Fig. 10),

$$\frac{I_g}{I_s} = \frac{R_s}{R_g},$$

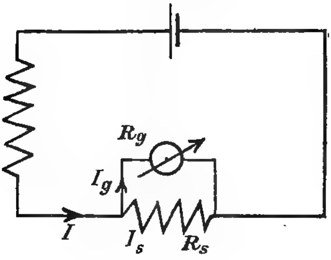

Fig. 10

or

$$\frac{I_g}{I_g + I_s} = \frac{R_s}{R_s + R_g},$$

but $I_g + I_s = I$, the main current; therefore

$$I = I_g\,\frac{R_s + R_g}{R_s} \quad \text{............................(13).}$$

To simplify calculation the resistance of the shunt is usually made equal to $\frac{1}{9}$, $\frac{1}{99}$ or $\frac{1}{999}$ of the resistance of the galvanometer. If, for example, the galvanometer has a resistance of 100 ohms, and is shunted with $\frac{100}{999}$ ohm, we have

$$I = I_g\,\frac{\frac{100}{999} + 100}{\frac{100}{999}} = 1000\,I_g.$$

Hence the main current is 1000 times the galvanometer current.

7. Series and Parallel Arrangement of Cells.

When cells are connected in series, the positive terminal of one cell is connected to the negative of the next, and so on (Fig. 11). The same current flows through all the cells one after another, and all the electromotive forces act in one direction. Hence, not only must the electromotive forces of all the cells be added together, but also their internal resistances. The series arrangement is used, therefore, where a large current is required in spite of a high external resistance. The increased battery resistance due to the series arrangement is then negligible, whereas the great electromotive force thus obtained is important. To connect cells in parallel, all their positive terminals are joined together, on the one hand, and all their negatives on the other. By this means the battery resistance is made very small, but the pressure across the external circuit is only that due to a single cell. We can no more add parallel connected electromotive forces than parallel connected hydraulic pressures. If, for example, two pipes come down from two equally high reservoirs, the pressure obtained is no greater than if a single reservoir were used, if the resistance of the pipes be neglected. Cells are therefore connected in parallel to obtain a large current in an external circuit of low resistance, that is, when a large current can be obtained without a high pressure. The parallel arrangement has here the advantage of keeping the battery resistance, which is now considerable compared with the low external resistance, as small as possible, thus enabling a larger current to be obtained.

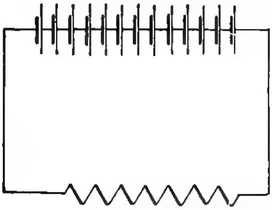

Fig. 11

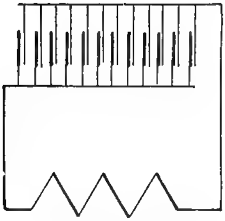

Fig. 12

A combination of these two methods leads to the mixed arrangement shown in Fig. 13, in which several cells are connected in series, and then a number of such rows connected in parallel. The question arises: How should a given number of cells be arranged so as to give the maximum current? Let

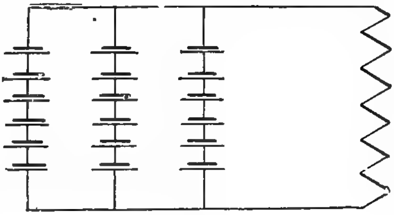

Fig. 13

R be the external resistance,
R_b the battery resistance,
R_i the internal resistance of one cell,
x the number of cells in series,
E the E.M.F. of each cell,
z the total number of cells.

The number of parallel paths is $\frac{z}{x}$, and the resistance of each branch $R_i x$. The battery resistance is therefore

$$R_b = \frac{R_i x}{z/x} = \frac{R_i x^2}{z}.$$

Since the E.M.F. of the battery is equal to that of a single branch, that is $E.x$, we have, by Ohm's law, for the current I through the whole battery,

$$I = \frac{E.x}{R + \frac{R_i x^2}{z}} = \frac{E}{\frac{R}{x} + \frac{R_i x}{z}}.$$

For a maximum value of this expression the denominator must be a minimum. The denominator is a function of the variable x, which fact can be expressed thus:

$$f(x) = \frac{R}{x} + \frac{R_i x}{z}.$$

Differentiating with regard to x, we have

$$f'(x) = -Rx^{-2} + \frac{R_i}{z}.$$

The function $f(x)$ will have a minimum value when $f'(x) = 0$, that is when

$$R = \frac{R_i x^2}{z}.$$

The left-hand side of this equation represents the external resistance, while the right-hand side is equal to the internal resistance of the whole battery. Hence, to obtain the maximum current through a given external resistance with a given number of cells, these should be so arranged that the internal battery resistance is, if possible, equal to the external resistance*.

8. Wheatstone's Bridge.

The measurement of resistance with the Wheatstone bridge is an important and instructive application of Kirchhoff's rules.

The cell E (Fig. 14), whose E.M.F. need neither be known nor constant, is connected across the ends of a bare straight wire AB. Under the wire is a scale by means of which the exact position of the sliding contact C can be read. Connected in parallel with this wire is a branch consisting of the unknown resistance x, the value of which is to be determined, and a known resistance R. The resistance of the connecting wires is assumed to be negligible. A galvanometer is connected between the sliding contact C and the connecting point between x and R. The sliding contact is moved until no current flows through the galvanometer. The current I_1 then flows uniformly through the resistances a and b, and the current I_2 through the resistances x and R. We now apply Kirchhoff's

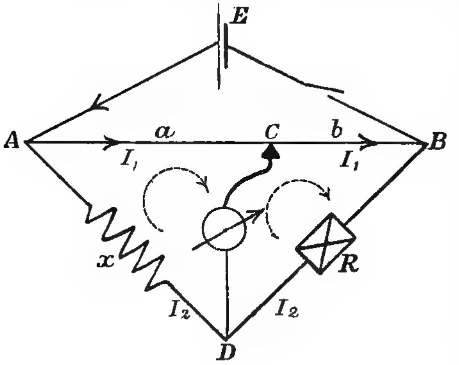

Fig. 14

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* For further information on this subject, see *The Grouping of Electric Cells*, by W. F. Dunton. Spon. 1906.

second rule to the left-hand lower mesh, going round it in a clockwise direction as indicated by the dotted arrow. This circuit contains no E.M.F. and the current in the galvanometer is zero; therefore

$$I_1a + 0 - I_2x = 0.$$

Similarly in the right-hand mesh

$$I_1b - I_2R + 0 = 0.$$

Hence it follows that

$$\frac{a}{b} = \frac{x}{R} \quad \text{..............................} (14).$$

The ratio $a : b$ in this equation refers to the resistances of the two portions of the slide wire; these are, however, proportional to their lengths, so that the ratio $a : b$ can be read off directly on the scale.

It is instructive to deduce this result, without using Kirchhoff's rules, from the analogy of a water main which divides into two parallel vertical mains. If a horizontal pipe joins the two mains at any level, no water will flow through this connecting pipe, because there is no difference of level or pressure between its ends. This is exactly analogous to the Wheatstone bridge. When no current flows through the galvanometer there is no difference of pressure between the points C and D, and the pressure drop from A to C is equal to that from A to D. Writing this in the form of an equation, we have

$$I_1 . a = I_2 . x,$$

and

$$I_1 . b = I_2 . R.$$

Equation (14) follows directly from this.

This form of Wheatstone's bridge is known as the slide-wire bridge or metre-wire bridge, the wire AB generally being a metre long.

The lengths of wire a and b can be replaced by accurately known resistances, and it is then more convenient to obtain a balance by altering the value of R. The well-known Post Office box is of this type.

The method outlined above is suitable for resistances of medium size.

In the case of very small resistances, the connecting wires would introduce considerable error. To avoid this, the cell could be connected directly to the ends of the resistances x and R, while a and b could be represented by accurate resistances of such values that the end connections would be quite negligible. In the ordinary form of bridge, however, it is impossible to eliminate the effect of the connection between x and R. If, in Fig. 14, the galvanometer be connected to the end of x, R will be increased by the resistance of the connection between x and R, whereas, if the galvanometer be connected to R, x will appear too large.

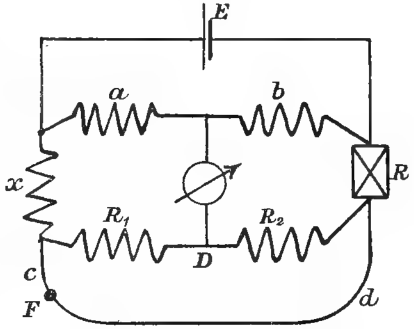

Fig. 15

In *Thomson's* (*Kelvin's*) *double bridge* this difficulty is surmounted by connecting the galvanometer through resistances R_1 and R_2 to both x and R (Fig. 15). The former direct connection between x and R, shown at the bottom of Fig. 15, still remains. We see that there are now two leads connected to each end of both x and R. The resistances R_1 and R_2 are so chosen that their ratio $R_1 : R_2$ is equal to the ratio $a : b$. If, for example, $a : b = 1 : 10$ then $R_1 : R_2$ must be made 1 : 10. The point D will then have the same potential as the point F which divides the direct connection between x and R in the ratio $c : d = 1 : 10$. We can therefore imagine the galvanometer directly connected to F instead of to D, and when, by altering the value of R, the current in the galvanometer has been reduced to zero, we have

$$\frac{a}{b} = \frac{x + c}{R + d}.$$

In our case, where $a : b = 1 : 10$,

$$\frac{x + c}{R + d} = \frac{1}{10},$$

and as $c : d$ is also in the ratio 1 : 10, it follows that $x : R = 1 : 10$. In this way the resistance of the connecting leads is eliminated and it is therefore possible to measure very low resistances, such as the armature resistance of a large dynamo, with great accuracy*.

9. Measurement of Electromotive Force by the Potentiometer Method.

A cell of constant but unknown E.M.F. is connected across the ends of a calibrated wire AB (Fig. 16). The P.D. (potential difference) between A and B must be greater than the E.M.F. x which has to be measured, and also greater than the E.M.F. E_0 of the standard cell. A Bunsen cell, for example, could be used for E, and a Daniell cell for E_0. A galvanometer, the standard cell and a high resistance are connected in series between the end A of the calibrated wire, and the sliding contact, the cells E and E_0 having similar poles connected to A. The sliding contact is moved along the wire until a point C is found where no current flows through the galvanometer, the final adjustment being made with the high resistance short-circuited. The cell to be tested is now put in the place of the standard cell so that the electromotive forces E and x oppose each other as before. The sliding contact is again adjusted until no current flows through the galvanometer, its new position being C'.

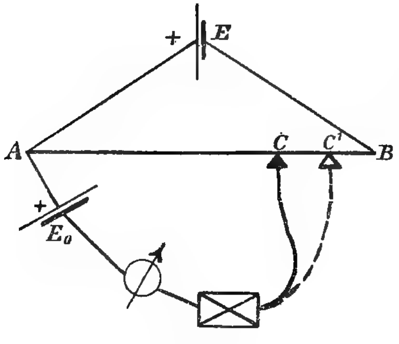

Fig. 16

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* For further information on the subject of measurement of resistance, see Price's *Measurement of Resistance*, Kempe's *Handbook of Electrical Testing*, or Chapter xxvi. in Watson's *Practical Physics*.

In order to find the relation between the electromotive forces E_0 and x from the measured lengths AC and AC', we observe that the terminal pressure of the Bunsen cell E is used up along the whole length of the wire AB. Across half the wire, for example, there is half the pressure, for, with the same current, pressure differences are proportional to resistances. The ratio of the P.D. between A and C to the P.D. between A and C' will therefore be equal to the ratio of the lengths of wire, that is $AC : AC'$. We have seen, however, that the P.D. between A and C exactly balances the E.M.F. E_0, while the P.D. between A and C' exactly balances the E.M.F. x. If now we replace potential differences by corresponding lengths of wire, we have

$$\frac{AC}{AC'} = \frac{E_0}{x} \quad \text{..............................(15).}$$

This method, which brings home very clearly to the beginner the fall of potential along a resistance, is one of the most convenient methods of calibrating instruments, and it may be said that almost all instruments are now calibrated in this way. The calibrated slide wire is replaced, either entirely or for the most part, by accurate resistance coils, an accumulator takes the place of E, and a Weston or Clark cell is used as the standard.

In a similar way it can be shown that the pressure y across the points A and B is such that

$$\frac{AB}{AC} = \frac{y}{E_0}.$$

It must be carefully noticed, however, that y is not the E.M.F. of the upper cell but only the P.D. between its terminals. The cell E differs from the cells x and E_0 in that it is carrying a current, and its terminal pressure is therefore less than its E.M.F. by an amount equal to its internal pressure drop.

10. Joule's Law, Electrical Energy and Electrical Power.

A conductor through which an electric current is flowing becomes heated. The amount of heat so developed was measured by the English physicist Joule, who determined the relation between the heat and the current, pressure and time. The quantity of heat taken as the unit in Electrical Engineering is the gramme-calorie, which is the amount of heat required to raise the temperature of 1 gramme of water from 0° to 1° C., or, what is practically the same thing, the heat required to raise 1 gramme of water through 1° C. If Q_h represents the quantity of heat in gramme-calories, V the pressure across the terminals of the conductor in volts, I the current in amperes, and t the time in seconds, then it is found experimentally that

$$Q_h = 0{\cdot}24 . V . I . t \quad \text{......................(16).}$$

The experiment can easily be repeated by immersing a platinum spiral in a measured quantity of water, the spiral being soldered to two thick copper leading-in wires of negligible resistance (Fig. 17). The inner glass vessel, which contains the water, rests on cork knife-edges and is separated by an air space from the external vessel to avoid as far as possible any loss of heat. A current is passed through the spiral and readings are taken of the pressure across the terminals, the current and the resulting temperature rise $T_2 - T_1$ in a certain time. The amount of heat is determined by multiplying the weight of water w in grammes by the temperature rise $T_2 - T_1$. It is found that this quantity of heat is proportional to the product $V.I.t$.

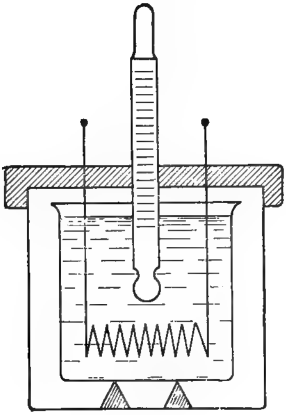


Fig. 17

$$Q_h = w\,(T_2 - T_1) = k\,.V.I.t.$$

With a little care the value 0·24 will be obtained for k. It is a good plan to start the experiment with the water a few degrees below the temperature of the room and to continue it until the water is an equal number of degrees above the room temperature. The amount of heat radiated out in the second half of the experiment will be equal to the heat received from the surroundings during the first half, thus correcting any radiation error. It must be remembered moreover that the value of w is made up of the actual weight of the water and the water equivalent of the inner glass vessel. The latter is found by multiplying the weight of the empty glass vessel in grammes by the specific heat of glass, i.e. 0·19.

Joule's experiment is as fundamentally important in Electrical Engineering as his determination of the mechanical equivalent of heat, according to which a kilogramme-calorie is equivalent to 427 metre-kilogrammes or 1 British Thermal Unit (Fahrenheit) to 778 foot-lbs. According to Joule's law the product $V.I.t$ is proportional to a quantity of heat. Now, as heat is nothing more than a special form of energy, the product $V.I.t$ must represent an amount of energy, and it is therefore a measure of the electrical energy or work.

The unit of electrical energy is the work done in one second when a current of 1 ampere flows under a pressure of 1 volt. This amount of energy is called a joule or a watt-second. The electrical work W is given therefore by the equation

$$W = V.I.t \text{ joules} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

We have now to find the relation between the electrical work in joules and the mechanical work in metre-kilogrammes or in foot-pounds. Putting V, I and t to equal 1 in equations (16) and (17) we find that the electrical energy is one joule while the heat Q_h is 0·24. Hence 1 joule is equivalent to 0·24 gramme-calorie. According to the mechanical equivalent of heat a gramme-calorie is equivalent to 0·427 metre-kg and therefore

$$1 \text{ joule} = 0{\cdot}24 \text{ gr-cal} = 0{\cdot}24\,.\,0{\cdot}427 \text{ met-kg} = 0{\cdot}102 \text{ met-kg},$$

or

$$1 \text{ met-kg} = 9{\cdot}81 \text{ joules} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Similarly, we could show that

$$1 \text{ foot-pound} = 1{\cdot}356 \text{ joules}.$$

It is no mere coincidence that we get the figure 9·81, the acceleration due to gravity; it follows necessarily from the units employed.

Let us consider, as an example, a dynamo giving a current of 50 amperes at a pressure of 220 volts for 10 hours, i.e. for 36,000 seconds; the electrical work done or energy given out is

$$W = 220.50.36{,}000 = 396.10^6 \text{ joules,}$$

which corresponds to $40{\cdot}5.10^6$ metre-kg or 292.10^6 foot-lb. Again, if a Daniell cell gives a current of 0·55 ampere for 1 hour at a terminal pressure of 1 volt, the energy given out is

$$W = 1.0{\cdot}55.3600 = 1980 \text{ joules.}$$

This is the same amount of work as is required to raise $\frac{1980}{9{\cdot}81} = 200$ kg one metre.

From the unit of work we can obtain the unit of electrical power, i.e. the rate at which work is done. If the product $V.I.t$ represents the electrical work, the product $V.I$ must be the electrical power. Hence, **the unit of electrical power is given by one ampere under a pressure of 1 volt. This unit of electrical power is called a watt.** If the electrical power in watts be represented by P, we have

$$P = V.I \text{ watts} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

From equation (18) we obtain the relation between the watt and a metre-kg per second. One metre-kg per second is equal to 9·81 joules per second, that is, to 9·81 watts. Hence

$$1 \text{ H.P.} = 75 \text{ met-kg per second} = 75.9{\cdot}81 \text{ watts} = 736 \text{ watts.}$$

This is the continental or metric horse-power, which is about 1·4 per cent. smaller than the British horse-power.

$$1 \text{ British H.P.} = 550 \text{ ft-lb per second} = 550.1{\cdot}356 = 746 \text{ watts.}$$

The power taken by a 10 H.P. motor, having an overall efficiency of 0·85, will thus be

$$P = \frac{10 . 736}{0{\cdot}85} = 8700 \text{ watts.}$$

At a pressure of 220 volts, the current taken will be

$$I = \frac{8700}{220} \approx 40 \text{ amperes.}$$

The above units of energy and power are often, in practice, found inconveniently small. Larger units are therefore used, as follows:

1 hectowatt = 100 watts (rarely used).
1 kilowatt = 1000 watts.
1 watt-hour = 3600 watt-seconds or joules.
1 kilowatt-hour = $3{\cdot}6.10^6$,, ,,

The kilowatt-hour is also known as a Board of Trade Unit.

11. Potential Difference.

In the preceding section we based our consideration of electrical energy on the experimental evidence of Joule, as it appeared advisable to do this before giving a theoretical proof. We have repeatedly made use of the analogy between the electric current and a current of water. The analogy is also very close with regard to electrical work. The work done by a waterfall in a given time is obtained in metre-kilogrammes by multiplying the weight of water which has fallen in the given time by the height of the fall in metres. We proceed in a similar manner to calculate the electrical work by forming the product $V.I.t$. For V is the pressure or difference of level, while $I.t$ is the quantity of electricity which has passed during the time t from the higher to the lower level or potential. From this we obtain a more accurate definition of that which we have, up to the present, represented by V and designated difference of pressure or difference of level. Thus, if in the equation

$$W = V.I.t$$

we make the product $I.t = 1$, that is, we make the quantity one coulomb, we have $W = V$. In other words, **the difference of pressure V in volts is the work, measured in joules, which is done when one coulomb flows from the higher to the lower electrical level.** While this coulomb of electricity was at the higher level it possessed potential energy or capacity for doing work, just as a raised weight possesses potential energy, which is greater the greater the difference of level. We speak therefore of an electrical **difference of potential,** and define it as **the work done when a unit of positive electricity flows from the higher to the lower electrical level.**

Inversely we could define potential difference as the work in joules which we must do to raise one coulomb of electricity from the lower to the higher potential. Referring to Section 2, we see that this raising of the electricity from the lower to the higher electrical level or potential is the special function of the electromotive force. Electromotive force and potential difference or pressure are thus, in a sense, identical and are measured in the same units.

The above definition of potential difference is identical with the well-known definition met with in frictional or static electricity. Fig 18 represents a positively charged insulated metal sphere. Imagine a small freely moving body with unit charge of positive electricity to be placed on the surface of the sphere. Since like charges repel, this small body will be repelled to an infinite distance by the large sphere. By this means work will be done equal to the summation over the whole distance of the product of force and distance, or this amount of kinetic energy will be imparted to the body. Then the potential at the surface of the sphere is the work done by the electric forces in repelling the unit of positive electricity to an infinite distance.

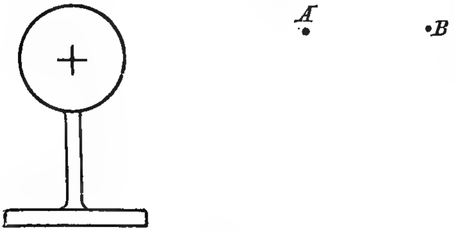


Fig. 18

When the small body is at A, it possesses with respect to the point B a capacity for doing work, i.e. it possesses potential energy analogous to that of a raised weight. The potential at the point A is, therefore, higher than that at B, and there is a difference of potential, or of electrical level, between the two points. This difference of potential is equal to the work done on a unit positive charge in repelling it from A to B.

In principle it is immaterial whether the unit of electricity which is driven from A to B is a charge on a small body, moving with this body, or a quantity of electricity moving along a stationary conductor as in an electric current. One point to which attention must be specially drawn is this, that if, in the case represented in Fig. 18, the coulomb be taken as the unit charge on the small body and the work be measured in joules, then the potential difference will be obtained directly in volts.

12. Energy lost in heating conductor.

Substituting for V its value $I.R$, the formula for the electrical power can be written as follows:

$$P = V.I = I^2.R \quad \text{..........................(20)}.$$

The power expended in the useful resistance, such as lamps, for example, is proportional to the square of the current and to the resistance of that portion of the circuit, while the same holds true for the leads. This fact has had considerable influence on the development of electrical engineering. To make this clear let us consider the case in which 10,000 H.P. has to be transmitted over a distance of 30 kilometres (about 19 miles). We can transmit this power by means of a large current at a low pressure or by means of a small current at a high pressure. In either case, to transmit 10,000 H.P. or 7,460,000 watts, we have

$$I = \frac{P}{V} = \frac{7{,}460{,}000}{V}.$$

If, now, we try successively pressures of 100, 1000 and 10,000 volts, we obtain as the corresponding currents 74,600, 7460 and 746 amperes.

Allowing 10 per cent., i.e. 746,000 watts for the loss in the leads or cables, the resistance of which is R_l, we have

$$I^2.R_l = 746{,}000,$$

or

$$R_l = \frac{746{,}000}{I^2}.$$

The total length of cable is 2.30 km = 60,000 metres, and using equation (6) on page 8, the cross-section can be found as follows:

$$A = \frac{0{\cdot}017 \,.\, 60{,}000}{\dfrac{746{,}000}{I^2}} = 13{\cdot}7 \,.\, 10^{-4} \,.\, I^2.$$

Hence, the cross-section of the copper is directly proportional to the square of the current, or inversely proportional to the square of the pressure employed. The above equations give the following values:

V	I	$A = 13{\cdot}7 . 10^{-4} . I^2$
100	74,600	$760 . 10^4$ sq. mm.
1,000	7,460	$760 . 10^2$,,
10,000	746	760 ,,

It is evident that the last case is the only one of the three which is at all practicable.

We shall now endeavour to make the effect of a higher pressure on the loss in the mains, or on the cross-section of the conductor, clearer by a consideration of the 3-wire system. Imagine a glow lamp to be constructed so as to burn normally with a current of 0·5 ampere and to have a resistance, when so burning, of 220 ohms. To maintain a current of 0·5 ampere the pressure across the lamp terminals must be $0{\cdot}5 \,.\, 220 = 110$ volts. We shall assume that 200 such lamps have to be supplied at a distance of 2 kilometres, with an allowable loss in the mains of 10 per cent. of the power transmitted, and we shall calculate the necessary cross-section of cable, using pressures of 110 and 220 volts respectively.

1st case. Pressure 110 volts. All lamps in parallel (Fig. 19).

The total current taken by 200 lamps at 0·5 ampere each is

$$I = 0{\cdot}5 \,.\, 200 = 100 \text{ amperes.}$$

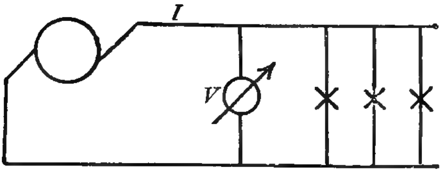


Fig. 19

The total power transmitted is

$$P = V \,.\, I = 110 \,.\, 100$$
$$= 11{,}000 \text{ watts.}$$

With 10 per cent. loss 1100 watts will be dissipated as heat in the mains. If R_l be the resistance of the mains, then

$$I^2 \,.\, R_l = 100^2 \,.\, R_l = 1100$$

and

$$R_l = \frac{1100}{100^2} = 0{\cdot}11 \text{ ohm.}$$

The total length l is $2 \,.\, 2000 = 4000$ metres. From the equation $R = \rho \,.\, \frac{l}{A}$ we have

$$A = \frac{\rho \,.\, l}{R} = \frac{0{\cdot}017 \,.\, 4000}{0{\cdot}11} = 620 \text{ sq. mm.}$$

2nd case. Pressure 220 volts.

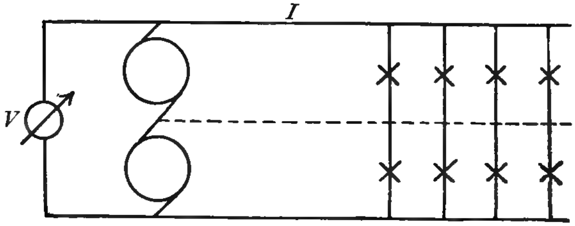


Fig. 20

To obtain this pressure we connect two 110 volt dynamos in series (Fig. 20). If, now, we connect the lamps 2 in series, the pressure across each lamp is 110 volts. We have, then, for 200 lamps 100 parallel paths each taking 0·5 ampere, the total current being $I = 100 . 0{\cdot}5 = 50$ amperes. The power transmitted is therefore

$$P = V . I = 220 . 50 = 11{,}000 \text{ watts.}$$

This is the same power as in the first case. The loss in the mains shall, of course, remain the same as before, viz. 1100 watts. We have therefore

$$I^2 . R_l = 50^2 . R_l = 1100$$

and
$$R_l = \frac{1100}{50^2} = 0{\cdot}44,$$

wherefore
$$A = \frac{\rho . l}{R_l} = \frac{0{\cdot}017 . 4000}{0{\cdot}44} = 155 \text{ sq. mm.}$$

Thus by doubling the pressure on the mains, the cross-section of the conductor has been reduced to a quarter of its previous value. To enable each lamp in the second case to be switched on and off independently of the others, a balancing wire or neutral wire connects the middle point of the dynamos to the middle point of every two lamps. The conductor is usually made of half the cross-section of each of the outer conductors. There is a great saving in copper in spite of the three wires. It is evident, moreover, that, with the same weight of copper in the mains, doubling the working pressure enables nearly four times the power to be transmitted with the same percentage loss in the mains.

The above example has been worked out in detail because of the difficulty experienced by the average beginner in forming clear and tangible ideas on this question of power transmission at high or low pressures. It is evident that in both the above cases the same number of lamps were supplied, each lamp having the same current and burning, therefore, with the same brilliancy. The result obtained is therefore exactly the same in both cases.

CHAPTER II

13. Electrolysis.—14. Quantitative laws of electrolysis.—15. Polarisation.—16. Accumulators.—17. Primary cells.—18. The Voltameter.

13. Electrolysis.

Conductors of the first class are those through which the electric current passes without producing any chemical change. The metals and carbon belong to this class. Second class conductors, on the other hand, undergo chemical changes or decomposition when a current passes through them. This decomposition is called *electrolysis* and such conductors are known as *electrolytes*. To this class belong the bases, acids and salts, either in solution or in the molten state.

The vessel or apparatus, in which electrolysis is carried on, is called an *electrolytic cell*, and the immersed conductors, by means of which the current is led in and out of the liquid, are called the *electrodes*. The positive electrode, by which the current enters the liquid, is called the *anode*, while the negative electrode, by which the current leaves the cell, is called the *cathode*.

The constituents into which the liquid is split up appear at the electrodes, one constituent wandering with the current to the cathode, while the other makes its way against the current to the anode. These constituents are therefore called *ions*, i.e. wanderers. According to modern views the molecules of an electrolyte are already split up into ions before the application of any P.D. It has been observed that the pressure with which water forces its way through a porous wall into a solution on the other side, is proportional to the number of molecules in solution. Very weak solutions, containing equal numbers of molecules dissolved in equal volumes, have the same "osmotic" pressure. Their boiling points are equally raised and their freezing points equally lowered. When we come to examine electrolytes, however, we find striking exceptions. Hydrochloric acid has an osmotic pressure twice, and sulphuric acid three times, as great as what one would expect. Arrhenius explains this by the assumption that the molecules are subdivided into ions which act as molecules in producing osmotic pressure and influencing the boiling and freezing points. The molecule of HCl will be split up into the ions H and Cl, while a molecule of H_2SO_4 gives three ions, H, H, and SO_4.

The conductivity of the electrolyte is due to the electric charges carried by the ions. Those charged positively are called *cations*, and are, of course, attracted towards the negative electrode, to which they give their charges; their motion is in the direction of the current through the liquid. The negatively charged ions are called *anions*, and move towards the positive electrode, to which they give their negative charges. It is perhaps difficult for the beginner to reconcile these views with the general idea that an electric current consists only in a movement of positive electricity. It is probable, however, that the latter is correct when applied to first class conductors, except that it is the negative electricity which moves and not the positive.

To obtain a better understanding of electrolysis we will now turn our attention to the difference between metals and non-metals. Metals, such as potassium, magnesium, iron and gold, have a characteristic lustre; they are good conductors of heat and electricity, and their compounds with hydrogen and oxygen are bases, e.g. caustic soda ($NaOH$), caustic potash (KOH), slaked lime ($Ca(OH)_2$). The basic character of the metallic oxides is shown by the facts that they turn red litmus blue, act as caustics, have an alkaline taste and neutralise acids. The basic nature of many metallic oxides is not very pronounced, being only shown by their ability to neutralise acids.

In addition to the above characteristics, all metals show a remarkable peculiarity when present in a solution through which an electric current is flowing, in that their ions always move with the current, towards the cathode. The metals are therefore electro-positive, that is, their ions are the carriers or transporters of positive electricity (*cations*). Since hydrogen also moves with the current towards the cathode, it is also to be classed as a metal, more especially as it can be chemically replaced in its compounds by metals.

To the non-metals belong chlorine, bromine, iodine, nitrogen, oxygen, sulphur, etc., which are without metallic lustre, and which, in so far as they occur in the solid state, are bad conductors of heat and electricity. Finally, they combine with hydrogen and oxygen to form acids, e.g. hydrochloric acid (HCl), sulphuric acid (H_2SO_4), nitric acid (HNO_3) and phosphoric acid (H_3PO_4). Acids are characterised by a sour taste, their ability to turn blue litmus red, to dissolve metals and to neutralise bases with the formation of salts.

Of the non-metals, chlorine, bromine, iodine and fluorine move against the electric current and belong, therefore, obviously, to the *anions*; moreover, the hydroxyl group (OH) among the bases, as well as the acid radicals SO_4, PO_4, NO_3, etc., are anions.

We will now endeavour to make the processes of electrolysis clearer by means of a few characteristic examples.

1. ELECTROLYSIS OF BASES.

When a solution of caustic potash (KOH) is electrolysed, the metal potassium moves with the current and causes hydrogen to be liberated at the cathode, in accordance with the equation

$$2K + 2H_2O = 2KOH + H_2.$$

The hydroxyl group OH, on the other hand, moves against the current and causes oxygen to be liberated at the anode in accordance with the equation

$$4OH = 2H_2O + O_2.$$

From the result it would appear that water only was being decomposed. The decomposition of the water, however, is purely chemical and takes no part in the passage of the current, which is due entirely to the decomposition of the caustic potash. Pure water is a perfect insulator.

2. ELECTROLYSIS OF ACIDS.

The simplest case is that of hydrochloric acid (HCl). Hydrogen is liberated at the cathode and chlorine at the anode. We have in this case a simple decomposition without secondary reactions. In the electrolysis of sulphuric acid (H_2SO_4) hydrogen is likewise liberated at the cathode. The acid radical SO_4, on the other hand, is not liberated at the anode, but if the anode consists of carbon or platinum, oxygen is liberated, the chemical reaction being as follows:

$$2SO_4 + 2H_2O = 2H_2SO_4 + O_2.$$

The result is therefore the same as if water alone were decomposed.

If, however, the anode consists of copper, the radical SO_4 attacks it, forming copper sulphate in accordance with the equation

$$SO_4 + Cu = CuSO_4.$$

3. ELECTROLYSIS OF SALTS.

When a solution of potassium chloride (KCl) is electrolysed, the potassium ions wander to the cathode, where caustic potash is formed and hydrogen liberated, as above; the chlorine goes to the anode. If the two electrodes are separated by a porous partition, e.g. an unglazed earthenware pot, the final products of electrolysis will be caustic potash and chlorine. In the absence of such a partition, a tertiary reaction ensues between the chlorine and caustic potash with the formation of potassium chlorite ($KClO$) in accordance with the equation

$$2KOH + 2Cl = KClO + KCl + H_2O.$$

If, however, the solution be hot, potassium chlorate is formed, thus:

$$6KOH + 6Cl = KClO_3 + 5KCl + 3H_2O.$$

As a further example of the decomposition of a salt solution, we may take the electrolysis of copper sulphate ($CuSO_4$). The copper wanders with the current and electroplates the cathode. The group SO_4 moves, as before, to the anode and, if the latter consists of carbon or platinum, causes a liberation of oxygen. If the anode is of copper it will be eaten away, with the formation of copper sulphate, thus:

$$Cu + SO_4 = CuSO_4.$$

14. Quantitative Laws of Electrolysis.

The amount of an element or of a chemical compound deposited on an electrode was found by Faraday to be proportional to the current and to the time, or, in other words, proportional to the quantity of electricity which has passed. Thus, if

w be the weight of deposit in milligrammes,
I the current strength in amperes,
t the time in seconds,
and c a coefficient,

then it is found experimentally that

$$w = c.I.t.$$

The coefficient c is different for various ions. Faraday found that it was proportional to the atomic weight and inversely proportional to the valency. Thus, if

a be the atomic weight,
and k the valency,

then Faraday found experimentally that

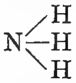
$$w = 0{\cdot}010386\,\frac{a}{k}\,.\,I\,.\,t\ \text{mg}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

To understand this equation it is necessary to define the terms "atomic weight" and "valency." The *atomic weight* of an element is the weight of its atom relatively to the weight of the hydrogen atom. An atom is defined as the smallest particle of matter which can take part in chemical action, or enter into the chemical structure of a compound. The atomic weight of chlorine, for example, is 35·4, since hydrochloric acid (HCl) contains 35·4 parts of chlorine to one part of hydrogen. Water, on the other hand, contains 8 parts of oxygen to one part of hydrogen. At first sight it would appear, therefore, that the atomic weight of oxygen was 8, but on examining the relative volumes in which hydrogen and oxygen combine to form water, we find that its formula must be written H_2O. One atom of oxygen is therefore combined with two atoms of hydrogen, and the atomic weight of the former must be 16, in accordance with the ratio $1 : 8 = 2 : 16$. From the atomic weights of the elements and the formula for the compound, the percentage of any element in a compound can be calculated. It is evident that the atomic weights are of prime importance in calculating the amounts of various elements deposited by electrolysis.

Another term used in equation (21) is *valency*. By this is understood the number of atoms of hydrogen which can combine with, or be replaced by, one atom of the element. Chlorine, for example, is univalent, because it combines with a single atom of hydrogen to form hydrochloric acid (HCl). Similarly, potassium is univalent, since it combines with a single atom of the univalent element chlorine to form potassium chloride (KCl), or, in other words, it replaces the hydrogen atom of hydrochloric acid. Oxygen, on the other hand, is divalent, since one of its atoms combines with two hydrogen atoms, forming water (H_2O). It is best to look upon the valency as the number of arms or bonds by which the atom is linked with the arms of other atoms. Copper, for example, is divalent in most of its compounds, and links its two arms with the two arms of the divalent oxygen atom, forming copper oxide, as shown by the equation

$$\mathrm{Cu + O = CuO.}$$

Nitrogen, on the other hand, is trivalent and links its three arms with the single arms of three hydrogen atoms, forming ammonia (NH_3), a molecule of which is therefore built up in the following manner:



We are now in a position to illustrate Faraday's law by means of a simple experiment. By means of platinum electrodes the same current is passed through a row of electrolytic cells containing, respectively, sulphuric acid (H_2SO_4), hydrochloric acid (HCl), copper sulphate ($CuSO_4$), cupric chloride ($CuCl_2$), and cuprous chloride (CuCl) (Fig. 21). We will assume that the experiment is carried on until 2 mg of hydrogen have been liberated in the first cell. The amounts liberated or deposited in the same time in the various cells will be found to be as follows, the atomic weights being given in brackets:

In the 1st cell (H_2SO_4):
2 mg hydrogen (H = 1), 16 mg oxygen (O = 16).
In the 2nd cell (HCl):
2 mg hydrogen (H = 1), 70·8 mg chlorine (Cl = 35·4).
In the 3rd cell ($CuSO_4$):
63·2 mg copper (Cu = 63·2), 16 mg oxygen (O = 16).
In the 4th cell ($CuCl_2$):
63·2 mg copper (Cu = 63·2), 70·8 mg chlorine (Cl = 35·4).

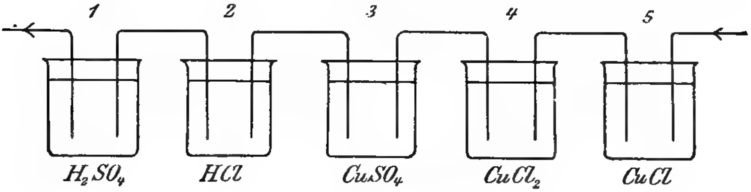


Fig. 21

So far, everything is quite regular, and it appears that a given current flowing for a given time liberates the same weight of a given element in whatever combination it may exist in the electrolyte. The actual weight of any element liberated is seen to depend primarily on its atomic weight. We obtain in each cell either atomic weights or multiples of them. From this fact we could have calculated from the weight of hydrogen liberated in the first cell, the results obtained in the next three cells. We meet a difficulty, however, when we come to the fifth cell. We cannot say beforehand if we shall obtain the same weight of copper as in the third and fourth cells, viz. 63·2 mg. If so, the electrolyte being CuCl, the amount of chlorine liberated would only be 35·4 mg. If, however, the chlorine liberated is the same as in the second and fourth cells, viz. 70·8 mg, the amount of copper deposited must be 126·4 mg. Experiment proves this latter to be the case.

For the same current flowing for the same time, twice as much copper is deposited from a cuprous chloride solution as from a cupric chloride solution.

In cuprous chloride one atom of copper is bonded with one atom of chlorine; the copper is therefore univalent. Assume for a moment that an atom of hydrogen weighs 1 mg, then in the first cell we have separated 2 atoms of hydrogen from the group SO_4, that is, we have broken two bonds or valencies. Similarly, we find that two bonds have been broken in each cell, leading us to the simplest expression of Faraday's law: **The same current flowing for the same time breaks up always the same number of bonds or valencies.**

We can, however, express this law in another form. The number obtained by dividing the atomic weight by the valency is called the chemical equivalent of the element. The weights of various elements liberated by a given quantity of electricity are therefore proportional to the chemical equivalents, i.e. the quantities are chemically equivalent. Now, according to modern views of electrolysis, the electric pressure applied to the cell causes those electrically charged ions which exist in a free state in the solution to wander in or against the direction of the applied pressure. These ions give up their charges on arrival at the electrodes, and act therefore as the carriers of the electricity through the liquid. We see, then, that equivalent weights of various elements carry or transport equal quantities of electricity, or, in other words, **equivalent weights of different ions carry equal electric charges.**

The weight in milligrammes of an element which is deposited in one second by a current of one ampere is called its electro-chemical equivalent. This can be calculated from equation (21) if we know the atomic weight and valency of the element. For example, silver is univalent and has an atomic weight of 107·6, whence its electro-chemical equivalent is $\frac{0{\cdot}010386\,.\,107{\cdot}6}{1} = 1{\cdot}118$; for copper (cupric) we get $\frac{0{\cdot}010386\,.\,63{\cdot}2}{2} = 0{\cdot}328$. In the same way the weight of hydrogen and oxygen liberated by an ampere in one second can be calculated. From a knowledge of its specific gravity or density the volume of the gas liberated can also be found. In this way we find that an ampere liberates in one second 0·174 cubic centimetre of the explosive mixture of hydrogen and oxygen, measured in a dry state at 0° C. and at a pressure of 760 mm of mercury.

The relative volumes of hydrogen and oxygen evolved by the electrolysis are proportional, according to Avogadro's law, to the number of molecules of each gas. Since the molecules of both gases contain two atoms, the volumes are proportional to the number of atoms, that is, from the formula H_2O, in the ratio of 2 to 1.

15. Polarisation.

If dilute sulphuric acid be electrolysed in a cell with platinum electrodes, there is found to be a potential difference of two or three volts between the terminals of the cell. If a smaller pressure than this be applied the electrolytic action ceases, while the application of a larger pressure of, say, 10 or

12 volts to the cell and an external resistance R in series simply causes such a current to flow that the drop in the external resistance leaves a pressure of 2 or 3 volts across the cell. The current is then given by the equation

$$I = \frac{E - V}{R},$$

where E is the E.M.F. of the generator in Fig. 22, V the P.D. between the terminals of the cell and R the external resistance, including that of the generator in the figure. It is remarkable that, although the P.D. V is applied externally to the cell and it is this P.D. which drives the current through the cell, yet the magnitude of the P.D. is determined entirely by the cell. It appears to be a function of the cell which we cannot arbitrarily alter, which remains practically constant, for example, when we so alter E and R that the current varies over a wide range. It even remains very nearly constant when the resistance of the liquid is considerably changed by varying the distance between the electrodes. We see, then, that the P.D. across an electrolytic cell cannot be found, as in the case of a metal resistance, by multiplying the internal resistance R_i by the current I. The P.D. V is very much larger than this product.

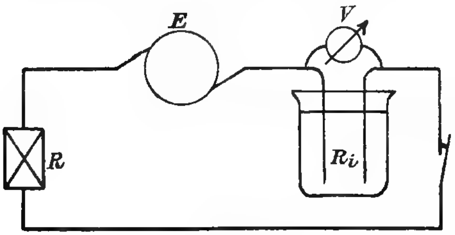


Fig. 22

The reason for this is obvious when we notice that, on breaking the circuit, the electrolytic cell sends a current through the voltmeter. It acts now as a generator or source of electrical energy, the current coming out from that terminal of the cell at which it previously entered. This current is called the polarisation current and the electromotive force producing it is the E.M.F. of polarisation. Its effects are only noticeable so long as the electrodes are covered with bubbles of the liberated gases. We have thus in the electrolytic cell a galvanic cell made up of hydrogen, sulphuric acid, and oxygen, the electromotive force of which is opposed to the original current.

This galvanic cell was, however, active during the passage of the original current. The terminal pressure V had therefore a double function to perform, viz. to overcome the electromotive force of polarisation E_1, and to supply the pressure drop due to the internal ohmic resistance of the cell. We have then for the terminal pressure of the cell the equation

$$V = E_1 + I.R_i.$$

Hence, we define the E.M.F. of polarisation, or the back E.M.F., as the electromotive force of the new element or galvanic cell produced by the chemical changes of the electrodes. If, as is usual, the polarisation is very large compared with the ohmic resistance, it follows that the terminal pressure will depend almost entirely on the value of E_1, and very little on the current or internal resistance.

The phenomenon of polarisation will be made clearer by considering it in the light of the principle of the conservation of energy. By multiplication with I the above equation becomes

$$V.I = E_1.I + I^2.R_i.$$

The product $V.I$ is the total power supplied to the cell, while $I^2.R_i$ is the power dissipated as heat in the liquid. The power $E_1.I$ must therefore be used in decomposing the water. This is obviously so, for the mixture of gases produced possesses a store of potential energy. We can explode the mixture at any moment and obtain in the form of heat or of mechanical work the energy used in its formation.

Now, the energy stored in the gas is proportional to the quantity of gas, which, as we have seen, is proportional to the quantity of electricity which liberated it. The energy stored in the gas is therefore

$$W = k.I.t,$$

where k is a constant for the gas concerned, viz. water-gas. On the other hand, the energy stored must be

$$W = E_1.I.t.$$

From this it follows that $E_1 = k$, that is, that the E.M.F. of polarisation for a given electrolytic process has a constant value k. We assume, of course, that the liquid and electrodes are similar in each case, so that the final products are of exactly the same chemical composition. For other types of cells the E.M.F. of polarisation is different, depending, as it does, on the energy of chemical composition, or calorific value, as it were, of the electrolytic products. When copper sulphate is decomposed with platinum electrodes the back E.M.F. will be quite different to that found for sulphuric acid. In the first case, the products of electrolysis are copper and oxygen. When copper and oxygen combine to form copper oxide the heat evolved is quite different to the heat produced by the explosion of equivalent amounts of hydrogen and oxygen.

It is very interesting to calculate the back E.M.F. from a knowledge of the chemical data. We know, for example, that when 18 grammes of water-gas are exploded to form 18 grammes of water, 68,000 calories of heat are evolved. The electrical energy required for the decomposition of 18 grammes of water must be equal to this; it is given by the equation

$$68{,}000 = 0{\cdot}24.E_1.I.t.$$

From equation (21) on page 29, 1 ampere flowing for 1 second liberates

$$\frac{0{\cdot}010386\,.\,1}{1}\text{ mg hydrogen and }\frac{0{\cdot}010386\,.\,16}{2}\text{ mg oxygen,}$$

or, together,

$$\frac{0{\cdot}010386\,.\,18}{2}\text{ mg} = 9{\cdot}35\,.\,10^{-5}\text{ grammes of water-gas.}$$

The quantity of electricity necessary for the liberation of 18 gr of water-gas is therefore

$$I\,.\,t = \frac{18}{9{\cdot}35\,.\,10^{-5}} = 1{\cdot}92\,.\,10^5\text{ coulombs.}$$

Putting this value of $I.t$ in the above equation we get

$$E_1 = \frac{68{,}000}{0{\cdot}24 \,.\, 1{\cdot}92 \,.\, 10^5} = 1{\cdot}47 \text{ volts.}$$

We see, then, that it would be quite impossible to electrolyse acidulated water by means of a Daniell cell. It is important to remember, however, that the actual back E.M.F. is much higher than the calculated theoretical value. The foregoing considerations were merely for the purpose of illustrating the main principles underlying the phenomenon of polarisation.

We will now consider a case in which there is practically no polarisation, because the nature of the electrodes is not changed by the electrolysis and no energy is stored in the final products. If copper sulphate solution be decomposed with a plate of pure copper as anode, chemically pure copper will be deposited on the cathode, while an equal weight of copper is dissolved from the anode. Both electrodes are therefore always of the same chemical nature and cannot form a polarising element or cell with the liquid. The back E.M.F. is therefore zero and V is equal to $I.R_i$. We come to the same conclusion when we consider that the energy used in depositing copper on the cathode is equal to the energy liberated at the anode where copper is dissolved, i.e. oxidised. The decomposition requires therefore no supply of energy, except to overcome the internal ohmic resistance.

16. Accumulators.

The first accumulators or secondary cells were made by Planté, who electrolysed dilute sulphuric acid between electrodes of lead, the surfaces of which became thereby chemically changed, or formed, as the change is technically termed. To increase the capacity Faure made the plates with deep grooves or holes into which pastes of various oxides of lead, or finely divided metallic lead, were introduced. The positive plates are now largely made of deeply grooved lead plates having an enormous surface (Fig. 23). These plates are "formed" in the works by electrolysis of sulphuric acid, their surface being thus converted into lead peroxide. The negative plates (Fig. 24) consist of lead grids into the meshes of which lead oxide paste is forced. In this unfinished condition they are delivered and, after fitting up the battery, it is necessary to charge it continuously for about 40 hours, thus converting the paste in the negative plates into metallic lead. A cell consists of a number of positive and negative plates placed alternately, and

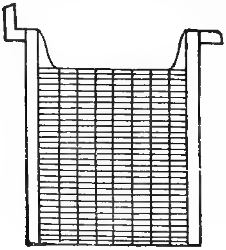


Fig. 23

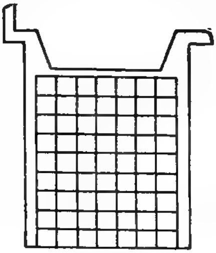


Fig. 24

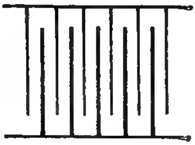


Fig. 25

supported on the sides of the glass vessel by means of the lugs seen in Figs. 23 and 24. Small portable cells are somewhat differently constructed, and the plates are completely enclosed in a vessel of ebonite or celluloid. The number of plates in a cell is usually odd, there being a negative plate at each end. All the positive plates are connected together by a lead bar running along the top, and similarly the negative plates by another bar, as shown in Fig. 25, which represents the arrangement of plates as seen from above.

To understand the working of the cell we may assume that we have two pure lead plates and a vessel of sulphuric acid from which we can construct an accumulator by the original method devised by Planté. On passing a current through our electrolytic cell, hydrogen will be liberated at the cathode, which will therefore be unchanged. At the positive plate or anode, however, oxygen will be evolved which will attack the lead and produce a coating of the brown peroxide of lead (PbO_2). On stopping the current we have now a cell consisting of lead, sulphuric acid and lead peroxide, and having an electromotive force of 2 volts. This E.M.F. was also acting during the passage of the current, that is when charging, its direction being opposed to the current. If now the terminals of the cell be joined through a resistance, the cell discharges and a current flows through the resistance, leaving the cell by the terminal at which the charging current entered. The terminal which was positive while charging is also positive when discharging, for in an electrolytic cell that terminal is called positive at which the current enters, while in any source of electrical energy the positive terminal is that by which the current leaves.

When discharging the changes in the cell are as follows:

	+		−
Before discharging:	PbO_2	$2H_2SO_4$	Pb
Direction of current in cell:	$\longleftarrow$		
Movement of the ions:	$H_2 \longleftarrow$		$\longrightarrow SO_4$
Reactions at the electrodes:	$PbO_2 + H_2 + H_2SO_4$ $= PbSO_4 + 2H_2O$		$Pb + SO_4$ $= PbSO_4$
Fully discharged:	$PbSO_4$	$2H_2O$	$PbSO_4$

Both plates thus become converted into lead sulphate; the positive through reduction of the peroxide by means of the liberated hydrogen, the negative through oxidation of the metallic lead by the oxygen, the lead sulphate being considered as a compound of lead oxide (PbO) and SO_3, obtained by displacing water from sulphuric acid. The result is that the excess of oxygen in the peroxide plate is transferred to the lead plate, the oxidation of which is the source of the electrical energy, just as the oxidation of coal in a furnace is the source of heat energy. When, finally, the positive plate has given up all its excess oxygen, and the negative plate has been thereby coated with sulphate, the store of potential energy is used up and the cell is discharged. This is evident from the fact that the plates are now similar and can therefore no longer make up a voltaic cell, one essential of which is the presence of two chemically different electrodes. We must not forget to mention that the concentration of the acid gradually decreases as the cell becomes discharged, its specific gravity indicating to a certain degree the extent to which the cell is discharged. This change of concentration must be taken into account, in addition to the reduction and oxidation, when considering the energy supplied by the cell.

The discharged accumulator is treated again as an electrolytic cell, that is, it is charged. The following changes then take place:

	+		−
Before charging:	$PbSO_4$	H_2SO_4	$PbSO_4$
Direction of current in cell:	$\longrightarrow$		
Movement of the ions:	$SO_4 \longleftarrow$		$\longrightarrow H_2$
Chemical reactions:	$PbSO_4 + SO_4 + 2H_2O$ $= PbO_2 + 2H_2SO_4$		$PbSO_4 + H_2$ $= H_2SO_4 + Pb$
Fully charged:	PbO_2	$3H_2SO_4$	Pb

The positive plate is again converted into peroxide and the negative plate to metallic lead, and the electrolytic cell has become once more capable of being used as a source of electrical energy.

It is evident that the process of charging an accumulator is quite different from the charging of a condenser, for in the latter case the energy is stored in an electrical form, whereas in the accumulator the storing consists of chemical changes in the active materials of the cell. When we speak of the capacity of a cell, we do not mean the same thing as we do when we speak of the capacity of a condenser. The capacity of a condenser is defined as the quantity of electricity which flows into it at the positive terminal when a P.D. of one volt is applied to its terminals. It is important to note that the same quantity flows out at the negative terminal, there being no more electricity in the condenser when charged than when uncharged; its distribution is, however, different in the two conditions. The capacity of an accumulator, on the other hand, is the total quantity of electricity which, starting with the cell fully charged, passes through it before it is discharged. By the cell being fully discharged we do not mean that its P.D. has been reduced to zero, or that all the lead or lead peroxide has been transformed into lead sulphate, for experience has shown that any attempt to do this shortens the life of the cell enormously. The discharge should be carried on until the P.D. has fallen to about 1·8 volts, and the cell should then be considered discharged. We have seen that the action of the cell depends on the production of lead sulphate, which, when the cell is partly discharged, will be distributed throughout the active material of the plates; any attempt, however, to convert all the spongy lead or lead peroxide into lead sulphate would undoubtedly loosen this active material, thus spoiling the electrical connection between it and the grid or body of the plate. In addition to this mechanical separation, the sulphate is very hard and insoluble and is a bad conductor of electricity, so that it is very difficult to recharge a cell which has been badly sulphated.

When an accumulator is said to have a certain discharge capacity in ampere-hours, it is theoretically immaterial whether a large current be taken for a short time or a small current for a long time and also whether the previous charging current was small or large. Like a Daniell cell, for example, an accumulator has a certain electromotive force but no fixed current. The makers, however, specify a certain discharge current as the normal, or as the maximum, for a given size and type of cell. This is not the current which the cell is bound to give, neither is it the largest current the cell is capable of giving, but merely the largest current which the cell should be called upon to give without running the danger of loosening the active material, which may come away from the plates and fall to the bottom of the vessel, or of buckling the plates, which may bend so much under bad treatment as to cause positive and negative plates to come into contact. We can therefore choose any discharging current within practical limits, and will obtain corresponding times of discharge. Theoretically, the capacity should be the same in every case, since, according to Faraday's law, the product $I.t$ is proportional to the quantity of active material decomposed.

In practice, however, it is found that the capacity is considerably decreased when the cell is discharged with a heavy current. This is due to the action being largely superficial in such a case, only a portion of the active material taking part in the chemical changes.

The ampere-hour efficiency is the ratio of the ampere-hours in the discharge to the ampere-hours in the charge. Theoretically, this ratio should be equal to 1, since charge and discharge consist in the chemical conversion and reconversion of the active material, and the quantity of electricity which has passed through the cell is proportional to the quantity of converted active material. Owing to various causes, however, the accumulator, especially if left standing for a long time, discharges itself. The causes of this are as follows: faulty insulation, local currents between parts of the same plate, due to impurities, to unequal conversion of the active material during charge, or to unequal concentration of the acid. Finally, a part of the charging ampere-hours is used in the useless production of gas. This occurs mainly towards the end of the charge when the active material is almost all converted and the hydrogen and oxygen, failing to find any material with which to combine, are given off and lead to so-called "gassing." For these reasons the ampere-hour efficiency is less than 1. It is, however, very high, generally exceeding 0·9, and by suitably arranging the experiment it can be made very nearly equal to 1.

In determining the ampere-hour efficiency it is, of course, very important to cease charging or discharging at the correct time. The charge is generally carried on until the active material on the surface of the plates is nearly all converted, and the cell "gasses," the pressure across the cell then rising to 2·6 or 2·7 volts owing to the gas bubbles which adhere to the electrodes. The discharge, on the other hand, is continued until the pressure drops to 1·8 volts, the drop being to a large extent due to increasing internal resistance as the result of the formation of lead sulphate on the plates. This final value of 1·8 volts should be read on the voltmeter during the discharge with normal current and not after opening the circuit, for the E.M.F. of the cell may then read 2 volts.

The watt-hour efficiency is, however, of greater practical importance than the ampere-hour efficiency. To determine this, we observe the terminal pressure at frequent intervals during charge and discharge, and plot the values observed as ordinates with the time as abscissae (Figs. 26 and 27).

We see that the average pressure during charge is much higher than that during discharge. During charge the terminal pressure is given by the equation

$$V = E_1 + I.R_i \quad \text{........................}(22),$$

while during discharge the equation is

$$V = E_1 - I.R_i \quad \text{........................}(23).$$

The average terminal pressure during charge is therefore greater than that during discharge by twice the internal drop, quite apart from the fact that the electromotive force during discharge is less than that during charge, for in the latter case the E.M.F. is increased by the presence of the gases evolved. The watt-hour efficiency is therefore less than the ampere-hour efficiency, and lies in favourable cases between 0·8 and 0·9, since we have, in addition to the losses already mentioned, the loss through ohmic heating*.

The lead accumulator is heavy, bulky, and yet extremely fragile, and it has proved a very difficult problem to make it light enough for traction purposes without making it incapable of standing the mechanical vibration and unavoidable ill-treatment. The acid is another disadvantage. The Edison cell contains no acid and is mechanically more robust than the lead cell; it is also better able to stand improper electrical operation without permanent injury. This cell is now largely used for electrical vehicles and to some extent for other purposes. The containing vessel is of sheet steel; the grids are also of sheet steel, nickel plated.

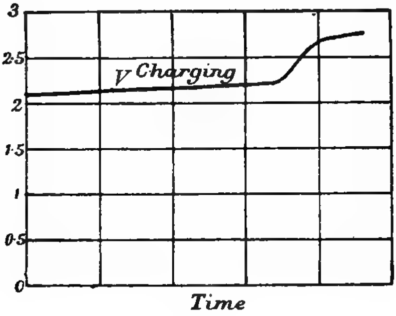


Fig. 26

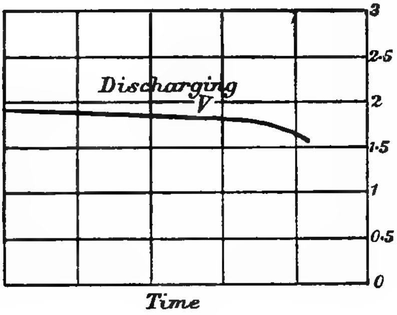


Fig. 27

The active material of the negative plate when charged is finely divided iron, and that of the positive plate nickel dioxide (NiO_2); when discharged these become iron oxide (FeO) and nickel monoxide (NiO). The former is contained in a number of flat rectangular pockets and the latter in a number of vertical tubes, both pockets and tubes being made of thin perforated nickel-plated steel and secured firmly in the frame or grid. The electrolyte is a 20 per cent. solution of potash. The P.D. is about 1·65 volts during charge and about 1·25 during discharge. The long life of the Edison cell is counterbalanced to some extent by its high initial cost and low efficiency.

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* For further information on accumulators see *The Storage Battery* by Treadwell, *Theory of the Lead Accumulator* by Dolezalek, or *Storage Battery Engineering* by Lyndon.

17. Primary Cells.

The simplest voltaic cell consists of plates of copper and zinc in dilute sulphuric acid, the copper terminal exhibiting a positive charge and the zinc a negative one. If the terminals be joined by means of a conductor, a current flows through it from copper to zinc, the current flowing from zinc to copper in the cell. The sulphuric acid is thereby decomposed and the hydrogen, going with the current, adheres to the copper plate in the form of small bubbles. We have, then, a new cell, consisting of hydrogen, sulphuric acid, and zinc, the electromotive force of which opposes that of the original cell. As a consequence the E.M.F. of the cell gradually dies away, and the cell is said to polarise. As the cell, for practical purposes, should have a more or less constant E.M.F., steps must be taken to prevent the polarisation, that is, to hinder the formation of free hydrogen. Either a metal must be deposited on the cathode instead of hydrogen, or the hydrogen must be combined with some other substance at the moment of its evolution.

In **the Daniell cell** the difficulty is overcome by the use of two liquids, copper sulphate and dilute sulphuric acid, separated by a porous earthenware pot. The constituents of the cell may be represented as follows:

$$+\ Cu—CuSO_4 \parallel H_2SO_4—Zn\ - .$$

The positive pole consists of a plate of copper dipping into copper sulphate, the negative pole of a zinc plate in dilute sulphuric acid. The zinc is amalgamated to prevent its consumption by the acid when the cell is not being used. The current flows through the cell from zinc to copper and the hydrogen moving in the direction of the current passes through the pores of the earthenware pot and throws down the copper out of the copper sulphate solution, as shown by the formula

$$H_2 + CuSO_4 = H_2SO_4 + Cu.$$

The zinc is dissolved by the group SO_4, forming zinc sulphate ($ZnSO_4$), and this, being an oxidisation, supplies the energy corresponding to the electrical energy supplied by the cell. This includes the work done in the external circuit, the heating in the cell itself and the energy consumed in precipitating the copper. To put it simply, the total electrical energy developed by the cell is equal to the difference between the energy evolved in the solution of the zinc and the energy consumed in the precipitation of the copper. Since the electrodes remain unchanged, there is no polarisation, and the electromotive force remains constant at 1·07 volts, assuming that we are dealing with pure materials. The current varies, of course, according to the external resistance. The internal resistance depends on the size of cell, but assuming it to be 0·5 ohm the current which would flow through the cell, were it short-circuited, is $\frac{1{\cdot}07}{0{\cdot}5} = 2{\cdot}14$ amperes.

The Bunsen cell consists of carbon in strong nitric acid and zinc in dilute sulphuric acid, the liquids being separated, as before, by a porous diaphragm.

$$+\ C—HNO_3 \parallel H_2SO_4—Zn\ - .$$

The carbon is the positive, the zinc the negative pole. In the sulphuric acid the action is exactly the same as before. The hydrogen passes through the pores and combines with the nitric acid, as shown in the formula:

$$3H_2 + 2HNO_3 = 2NO + 4H_2O.$$

The nitrous oxide (NO) thus liberated is further oxidised by the air to nitric oxide (NO_2) in the form of brown fumes with an objectionable odour. The E.M.F. of a Bunsen cell is 1·8–1·9 volts. Its internal resistance is less than that of a Daniell cell.

The Bichromate cell consists of carbon and zinc in a mixture of dilute sulphuric acid and potassium bichromate (K_2CrO_4, CrO_3):

$$+ \text{C—}H_2SO_4,\ K_2CrO_4,\ CrO_3\text{—Zn} - .$$

There is no porous partition. The carbon is again the positive pole. The radical SO_4 dissolves the zinc as before, while the hydrogen is oxidised by the chromic acid (CrO_3), with the formation of water and chromium oxide (Cr_2O_3), thus:

$$2CrO_3 + 3H_2 = Cr_2O_3 + 3H_2O.$$

The chromium oxide (Cr_2O_3) combines with the sulphuric acid, giving a greenish coloration, which gradually replaces the red tint of the chromic acid (CrO_3). At the same time, the electromotive force, which was primarily 2 volts, falls, so that with continuous heavy use the cell has not a constant E.M.F.

The Leclanché cell consists of carbon and zinc in a solution of sal ammoniac:

$$+ \text{C—}NH_4Cl\text{—Zn} - .$$

With the decomposition of the sal ammoniac, the chlorine goes to the zinc, which it attacks, forming zinc chloride. Although not oxidation, this process is very similar to it, and supplies the energy of the cell. The group NH_4 behaves very like a metal and goes with the current to the carbon, where it splits up into ammonia and hydrogen:

$$2NH_4 = 2NH_3 + H_2.$$

The hydrogen, thus produced, would polarise the cell, were the carbon not surrounded with manganese dioxide, which combines with the hydrogen, giving up a part of its oxygen, in which it is very rich, and producing water and a lower oxide of manganese:

$$2MnO_2 + H_2 = Mn_2O_3 + H_2O.$$

This action is not very rapid and the depolarisation is not instantaneous in its action as in the case of nitric or chromic acid. Although the E.M.F. of the cell is 1·4 volts when carrying no current, it rapidly falls when in use, and the cell is therefore only suitable for intermittent work.

The Weston Standard cadmium cell consists in its best form of an H-shaped glass vessel (Fig. 28) into which two platinum wires are fused. The platinum wires pass through the bottom of each limb into the liquid electrodes, the positive being pure mercury and the negative cadmium, rendered liquid by being amalgamated with mercury (12 to 13 per cent. of cadmium). The electrolyte is a saturated solution of cadmium sulphate ($CdSO_4$), its saturation being assured by the addition of some crystals of the salt. When current passes, the cadmium in the electrolyte moves towards the mercury electrode and would alloy itself with it, thus making the two electrodes more and more alike, and gradually reducing the E.M.F. of the cell. This is prevented by placing above the mercury a paste consisting of mercurous sulphate (Hg_2SO_4), cadmium sulphate ($CdSO_4$), and metallic mercury. The cadmium ions displace the mercury from the mercurous sulphate forming cadmium sulphate and metallic mercury, thus:

$$Cd + Hg_2SO_4 = CdSO_4 + 2Hg.$$

The positive electrode is unaffected and remains pure metallic mercury, polarisation being thus avoided. At the negative electrode cadmium is, of course, dissolved by the SO_4 group.

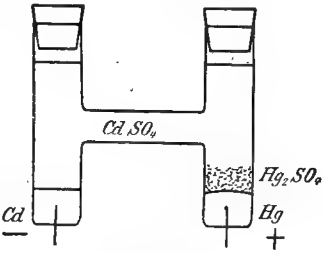

Fig. 28

The electromotive force is very constant at 1·019 volts, and is hardly affected by small changes of temperature. It must only be allowed, however, to give very small currents and is therefore only used in the potentiometer method. It is now largely used for the calibration of instruments by this method, and has, to a large extent, replaced the Clark cell, which was formerly used for the same purpose, but which is much more sensitive to changes of temperature*.

18. The Voltameter.

The legal definition of the ampere is based on the silver voltameter. A small platinum bowl is used as the cathode, and contains a solution of silver nitrate ($AgNO_3$) (20 to 40 parts by weight to 100 parts of water). The anode consists of pure silver. The solution is only used until 3 grammes of silver have been deposited per 100 c.cm of solution, and the deposit on the bowl must not exceed 0·1 gramme of silver per square centimetre. The current density on the anode surface must not exceed 0·2 ampere per sq. cm while on the cathode it must not exceed 0·02 ampere. The bowl is weighed before the experiment, and after the experiment it is washed with distilled water, free from chlorine, until the washings give no cloudiness on the addition of hydrochloric acid. This proves the perfect removal of every trace of silver nitrate, which would otherwise form a precipitate of silver chloride when the acid was added, in accordance with the formula:

$$HCl + AgNO_3 = AgCl + HNO_3.$$

The bowl is then washed for 10 minutes with hot distilled water (70°–90° C.) and again rinsed with cold water until the washings remain perfectly clear on the addition of hydrochloric acid. It is then dried in warm air, cooled and weighed.

Great care must be taken not to touch the inside of the bowl with the fingers either before or during the experiment. It is a good thing to heat the bowl to redness in the tip of a Bunsen flame to destroy any organic matter, before commencing the experiment. This should not be done if the bowl contains any silver, as an easily fusible alloy of silver and platinum may be formed. The cold or luminous part of the Bunsen flame must be avoided or the bowl may become very brittle owing to the formation of platinum carbide.

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* For further information on primary batteries see *Primary Batteries* by W. R. Cooper or *Practical Electricity* by W. E. Ayrton. For cadmium cells see F. E. Smith, *Phil. Trans.* series A, vol. CCVII. pp. 393–420.

If we represent by

w_1, the weight of bowl before experiment,
w_2, ,, ,, after experiment,
t, the time in seconds,

then, since 1 ampere deposits 1·118 mg of silver in 1 second, we have

$$I = \frac{w_2 - w_1}{1{\cdot}118 \, . \, t} \text{ amperes} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

Copper voltameters and water voltameters are suitable for heavier currents, but do not give such reliable results as the silver voltameter. They are very useful, however, for general laboratory work, from an educational point of view.

CHAPTER III

19. Strength of magnet pole.—20. Strength of magnetic field.—21. Lines of force.—22. Magnetic potential.—23. Iron in magnetic field.—24. The earth's field.

19. Strength of Magnet Pole.

A magnet is a piece of steel which possesses the property of attracting pieces of iron to itself. The name is supposed to have originated in the discovery near the town of Magnesia of pieces of iron ore possessing this property. If a bar of steel possessing this property be plunged into iron filings, they are found to cling to it, especially at its ends (Fig. 29). These places, where the force of attraction seems strongest, are called the poles of the magnet. In very long and thin magnets it can be assumed that the attractive force is directed towards a single point near each end. These points are not at the extreme ends, but at the points N and S in Fig. 29. The line joining these points is called the **magnetic axis.**

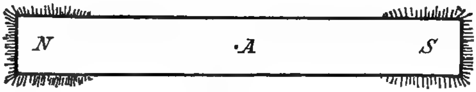

Fig. 29

If the magnet is free to rotate about a vertical axis A, one of its poles will point approximately towards the geographical north. This pole is called the north pole of the magnet, the other being called the south pole. If two magnets be brought near together, it is observed that their like poles repel each other, while their unlike poles attract each other. It follows from this that the geographical north pole of the earth possesses physical south polarity, and *vice versa.*

Coulomb was the first to measure the force between two magnet poles. He suspended a long thin horizontal magnet by means of a fine metallic wire. By turning the upper end of this wire he moved the north pole of the suspended magnet away from the south pole of another vertically placed magnet. Since the turning-moment or torque exerted by a twisted wire is proportional to the angle of twist, he was able to determine the relation between the attractive force between the poles and the distance between them. He found the force to be inversely proportional to the square of the distance. He found, moreover, that the force was doubled by putting another similar magnet with one of the two already in use, that is, by doubling the strength of one of the poles. If then

$f =$ the force exerted on each pole by the other,

$m_1 =$ the strength of one pole,

$m_2 =$ the strength of the other pole,

and $r =$ the distance in centimetres,

we have $$f = \frac{m_1 \cdot m_2}{r^2} \qquad \text{.............................(25).}$$

If we choose the units of force and length, this equation gives us the definition of unit pole strength. **As the unit of length we take the centimetre, and as the unit of force, that force which, acting on the mass of 1 cubic centimetre of water, produces an acceleration of 1 cm per second per second.** This force is called **a dyne.** Since the kilogramme weight is a force which produces an acceleration of 981 cm per second per second in the mass of a kilogramme or 1000 c.cm of water, a kilogramme weight is equal to 981,000 dynes. Hence

$$1 \text{ dyne} = \frac{1}{981{,}000} \text{kg}^* \doteq 1{\cdot}02 \text{ milligrammes}^*.$$

If we wish to express Coulomb's law in the simple form of equation (25) we cannot now choose arbitrarily the unit of pole strength, but must obtain the unit from the equation. If in equation (25) we put $m_1 = 1$, $m_2 = 1$ and $r = 1$, then we have $f = 1$. **Hence that pole has unit strength which exerts a force of 1 dyne on a similar pole at a distance of 1 cm.** The strength of a pole or a quantity of magnetism is therefore measured by the force it exerts under certain conditions, and we define the pole strength m as **the force exerted by the pole on a pole of unit strength at a distance of 1 centimetre.**

20. Strength of Magnetic Field.

The space surrounding a magnet, or any space where magnetic effects can be observed, say, by means of a compass needle, is called a magnetic field. The magnetic effect at any point near a magnet can be determined from a consideration of the combined effects of the two poles. At the point A (Fig. 30), in the neighbourhood of a magnet, imagine a freely movable north pole, the corresponding south pole being so far away as to be negligible. This north pole will be repelled in the direction AB by the north pole N, and attracted in the direction AC by the south pole S. The forces are inversely proportional to the squares of the distances. In Fig. 30, for example, the distances are 2 : 1 and the forces 1 : 4. By the parallelogram of forces the resultant of AB and AC is AD. This is therefore the direction of the magnetic force at this point. If a compass needle be placed at A the magnetic forces due to the bar magnet will act on its poles and turn the needle until its magnetic axis lies along AD.

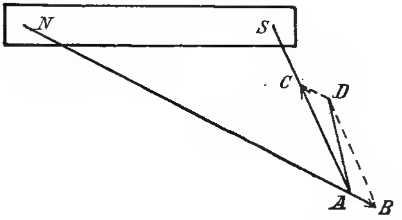

Fig. 30

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* To indicate that forces and not masses are referred to, a star will be added to the units of mass when used as forces.

Thus the magnetic axis of a magnetic needle, free to move in any direction, indicates the direction of the magnetic field at any point.

To find the strength of the magnetic field at any point, we place a pole of strength m at the point and determine the force acting on it. This force is proportional both to the strength of the pole m on which the field is acting and to the strength of the field. If H represents the strength of the field, we have

$$f = m.H \quad \text{.............................(26).}$$

If, in this equation, we put $m = 1$, then $f = H$. **The field strength H is therefore equal to the force in dynes exerted on unit pole at the given point.** Hence that field has unit strength which exerts a force of 1 dyne on unit pole.

The force on a pole at A in Fig. 30 was due to the two poles N and S. We shall consider now the case represented in Fig. 31, where one pole of the magnet is very far away. As its effect decreases as the square of the distance, this pole can be neglected. If now the pole strength of the magnet be m and a unit pole be placed at the point A, r cm from the pole m, the force on the unit pole, by Coulomb's law, is

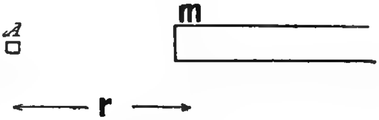

Fig. 31

$$H = \frac{m.1}{r^2} = \frac{m}{r^2} \quad \text{..........................(27).}$$

Since we are dealing with the force on unit pole, i.e. with strength of field, we have put H in the above equation in the place of f.

We imagine now a magnet needle of pole strength m and having a magnetic axis of length l, placed in a uniform field of strength H. The axis of the needle is at right angles to the direction of the magnetic force, which in Fig. 32 is supposed to act downwards. On the north pole there acts a force of $f_1 = m.H$ with an arm or leverage of $\frac{l}{2}$, and on the south pole is a force $f_2 = m.H$ with an equal arm. The total turning-moment is therefore

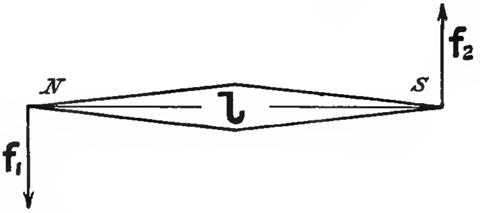

Fig. 32

$$\frac{l}{2}(f_1 + f_2) = 2mH\frac{l}{2} = H.ml.$$

Since the force is in dynes and the arm or length in cm, the turning-moment is in centimetre-dynes. The turning-moment is therefore the product of two quantities, of which the first is the field strength and the second a magnetic characteristic of the needle, viz. the product of its pole strength and length. This product $m.l$ is called the magnetic moment of the needle.

21. Lines of Force.

So far, we have based our assumptions on the now obsolete view according to which a certain quantity of magnetism is concentrated at the polar points, from which effects are caused at a distance without reference to the intervening space. As a matter of fact, poles are never concentrated at points, and modern physics no longer recognises action at a distance. The older view is, however, of great value inasmuch as it explains in a simple manner the results of many magnetic experiments, and enables us to express them mathematically. For a deeper insight into magnetic phenomena we must turn to Faraday's ingenious conception of lines of force, which enables us to form a mental picture of the processes underlying the various phenomena. Faraday did not believe in action at a distance, and to him the force in the neighbourhood of a magnet did not come into existence only when another pole was placed there on which the force could act. The space around a magnet is in a peculiar state. It is filled with magnetic forces which flow, as it were, from pole to pole. This view explains magnetic phenomena as well as, or even better than, the old theory, and the agreement between the results obtained mathematically from the one and the results obtained by a consideration of the other strengthens our belief in the correctness of both theories, or, at least, of the results obtained from them.

Faraday's conception is based on a simple experiment which every student should repeat for himself. If iron filings are shaken through a sieve on to a magnet and the board or table on which it lies is lightly tapped, the filings arrange themselves in characteristic curves (Fig. 33). The iron filings become magnetised by their proximity to the magnet, and acting as magnetic needles arrange themselves with their axes along the direction of magnetic force. The north pole of one filing attracts the south pole of the next with the result that curves are produced which indicate the direction of the magnetic forces much more plainly than the geometrical construction of Fig. 30. These curves are called "lines of force" and are conventionally assumed to proceed from the north pole, through the surrounding medium, to the south pole. A compass needle sets itself with its axis tangential to the line of force, and its north pole pointing along the positive direction of the line. A freely moving north pole, whose corresponding south pole is far removed, will move along the line of force from N to S (Fig. 33). **Hence we can define a line of force as the path of a freely moving north pole.** It is essentially nothing more than the direction of the magnetic force. When dealing

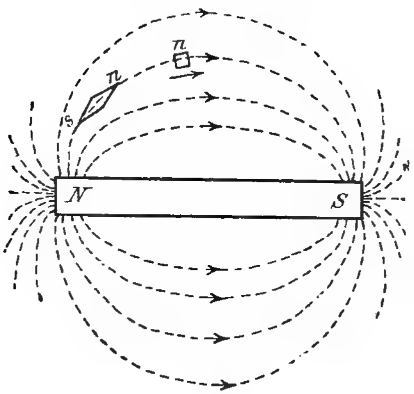

Fig. 33

with magnetic problems it is usual to refer to the lines of force as if they had a real existence, somewhat similar to the visible lines of the iron filings experiment.

The line of force is, however, far more than a convenient indication of the direction of the magnetic force; it enables us to express, in a very convenient way, the magnitude of the force, that is, the strength of the magnetic field. To this end we observe that the lines of force radiating from a pole do not lie in one plane, but radiate out into space in all directions. Imagine a number of concentric spheres with the pole as centre; radiating from the pole the lines of force will pass through the surfaces of successive spheres. Since the area of a sphere increases as the square of its radius, the number of lines of force per unit area must decrease in the same ratio. We are now able to see why, in Coulomb's law, the force f varied inversely as the square of the distance, and why, in equation (27) on page 45, the strength of the field in the neighbourhood of a pole decreased as the square of the distance. **We can now define the strength of a magnetic field very simply as the number of lines of force per square centimetre.**

If now the strength of field is to be measured both by the force on unit pole and by the number of lines per square centimetre, both methods giving the same result, the unit for the line of force cannot be arbitrarily chosen. In a field of unit strength there must be one line per sq. cm. To make this clearer imagine a unit pole at the centre of a sphere of 1 cm radius (Fig. 34). If now another unit pole be placed on the surface of the sphere, the force acting on it, according to Coulomb's law, is

$$H = \frac{1 . 1}{1^2}.$$

On the above assumption the number of lines per sq. cm must be the same figure, viz. 1. Hence a line of force represents the total flux of force, so to say, which passes through 1 sq. cm of the surface of a sphere of 1 cm radius, described about a unit pole. The surface of a sphere is $4\pi r^2$, that is, in our case, 4π sq. cm. Since there is one line per sq. cm the total number of lines radiating from unit pole is 4π. A line of force is thus $1/4\pi$ of the total flux of force radiating from unit pole.

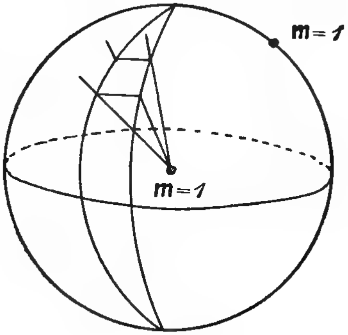

Fig. 34

The total number of lines Φ radiating from a pole m can be found from the formula:

$$\Phi = 4\pi m \quad \dots\dots\dots\dots\dots\dots\dots\dots(28),$$

and the total number of lines crossing a surface of A sq. cm over which there is a normal uniform field strength is

$$\Phi = H.A \quad \dots\dots\dots\dots\dots\dots\dots\dots(29).$$

If, for example, the field strength in the air-gap between the poles and armature of a dynamo is 7000, this means that 7000 lines of force leave every sq. cm of the north polar surface. If the surface of the pole facing the armature is 400 sq. cm the total number of lines entering the armature from the north pole is

$$\Phi = H.A = 7000.400 = 2{\cdot}8.10^6 \text{ lines.}$$

22. Magnetic Potential.

We have already seen that a freely moving north pole in a magnetic field moves along the lines of force. If now we form the product from point to point of the force acting on the pole and the distance moved through by it, and take the sum of the products so obtained, we find the work done by the magnetic field in moving the pole. This work is either used in overcoming mechanical resistance, or in storing kinetic energy in the moving body. If we, on the other hand, move a north pole *n* (Fig. 35) against the direction of the line of force from *S* to *N*, we have to overcome the opposing force of the magnetic field, which tends to drive the north pole from *N* to *S*. We must therefore do an amount of work equal to the summation of the product of force and distance along the whole path traversed. This work is stored as potential energy, to be liberated once more when the north pole, under the influence of the field, is driven back. This is quite analogous to the case in which we lift a weight, thus overcoming the opposing force of gravity. The work done is stored as potential energy in the raised weight, to be liberated when the weight falls from the higher to the lower level. We see, therefore, that there is a **difference of magnetic level or magnetic potential** between the different points on a line of force. **As in Section 11, we define this difference of magnetic potential as the work done in moving the positive unit quantity of magnetism, that is, a unit north pole, from the lower to the higher potential.** Inversely, it could be defined as the work done by the field when a unit north pole is moved from the higher to the lower potential. We have here assumed that the potential of the north pole is positive and that of the south pole negative. The potential at the middle point of a line of force will be zero.

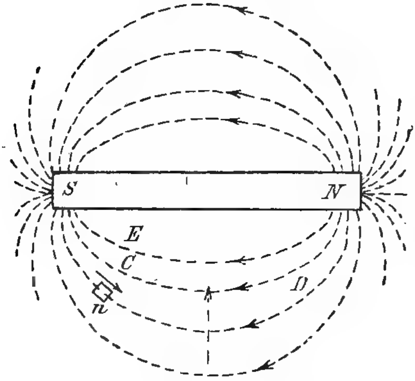

Fig. 35

In moving the north pole *n* from *C* to *D* in Fig. 35 it is immaterial whether we follow the line of force or go any other way. This follows from the conservation of energy. The energy used in moving the north pole *n* from *C* to *D* is constant, by whatever path we go, for it is stored as potential energy, its amount depending only on the initial and final positions *C* and *D*. The same amount of energy will be restored in every case when, under the action of the field, the pole *n* is driven back from *D* to *C*.

On the other hand, to move the pole *n* from *C* to *E* requires no expenditure of energy, since the motion is at right angles to the lines of force. There is no difference of potential between the points C and E. Such points do not, of course, lie wholly in one plane, but exist in the surrounding space. They lie on a so-called equipotential surface or plane, which is normal to the direction of the field at every point, and on which a pole can be moved without work being done.

We will now consider a homogeneous field, in which all the lines of force are parallel. If a pole be moved against the direction of the field (Fig. 36) the force on the pole is constant over the whole distance l. If the pole have unit strength, the potential difference between the points A and B separated by a distance l will be equal to the product of the force and the distance, i.e. $H.l$.

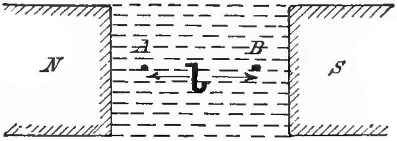

Fig. 36

Up to the present we have spoken only of potential differences, but we will now go a step further and define magnetic potential itself. We will imagine a north pole m, concentrated at a point from which the south pole is so far removed as to be negligible. The lines of force radiate into space, and under their influence a north pole at A (Fig. 37) would be driven away to an infinite distance. The force acting on it would decrease as the square of the distance, and would finally vanish. The total amount of work done in driving unit pole from A to infinity is called the potential at A, or the potential of m at A, and we thus assume that at an infinite distance the potential is zero.

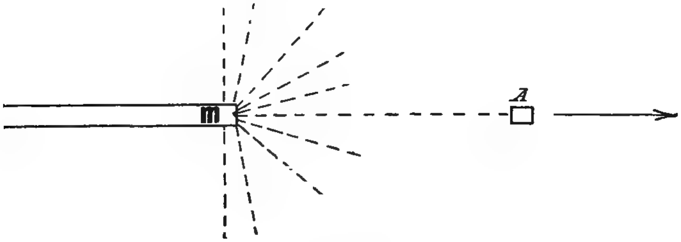

Fig. 37

It may appear strange that the potential at an infinitely distant point is the same as that at the middle point of a line of force, which we found above to be also zero. It would, however, be possible to bring a north pole from an infinite distance, along the path indicated by the dotted arrow in Fig. 35, to the centre of any line of force without any expenditure of energy, for the north pole would be equally attracted by the south pole of the magnet and repelled by its north pole, and we see that the motion is always at right angles to the lines of force.

If the idea of potential proves somewhat of a stumbling-block to the beginner, he will do well to think always of a difference of level and remember that, as a rule, we are concerned with differences of potential, and not with the absolute value of potential.

23. Iron in a Magnetic Field.

If iron filings be sprinkled on a magnet near which is a piece of iron, we notice that the iron draws the lines of force within itself (Fig. 38). As a result, the lines of force entering the end of the piece of iron nearest the magnet pole are very numerous, while, in other places, the field is weakened. This was formerly explained as a case of induction, the magnet pole inducing a pole of opposite sign to itself on the side of the iron nearest to it. This difference in sign of the poles of magnet and iron explained the well-known fact that the iron is attracted by the magnet.

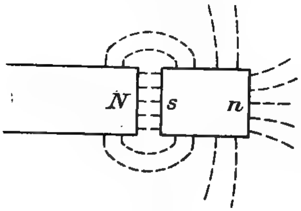

Fig. 38

The theory of the lines of force leads to the same conclusion. We suppose the iron to have a special conductivity or permeability for the lines of force, so that they exhibit a great tendency to pass through iron, if possible. The attraction of the iron by the magnet pole leads us then to the assumption that the lines of force strive to shorten themselves as much as possible, in much the same way as a stretched elastic band tends to shorten itself. We must also imagine the lines to exert a lateral pressure tending to force themselves apart, since they would otherwise collect in a dense mass along the shortest possible path. They must therefore repel each other mutually in a direction at right angles to their own direction, as do two parallel bar magnets, placed side by side with their similar poles together.

If the lines of force have a great tendency to pass through iron, owing to its superior conductivity compared with the surrounding air, it is evident that in Fig. 38 a great number of lines will enter the side of the iron nearest the pole *N* of the magnet*. This side of the iron will therefore be a south pole, since, on our former assumptions, south polarity is exhibited where lines of force enter the material. The further extremity of the iron will be a north pole, as the lines leave the iron at that end. The magnetisation of the iron is stronger, the better its conductivity for the lines of force. It is stronger, for example, in wrought iron or mild steel than in hard cast iron or hardened steel under exactly similar conditions.

This fact leads us to the assumption that the process involved in the magnetisation of a piece of iron is similar to the action of a magnetic field on a number of iron filings. We assume that the smallest particles, the so-called molecules of the iron or steel, are naturally magnetic, but point in all directions without law or order (Fig. 39). They have therefore no external effect and the iron appears unmagnetised. When a magnet is brought near, the molecules are turned, like compass needles, to point in the same direction (Fig. 40). The ends of the piece of iron will now exhibit "free" magnetism, while towards the middle of the piece the molecules neutralise each other. It is evident that the molecules of wrought iron are more readily turned than those of hard steel.

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* When lines of force pass from one medium to another, the tangents of the angles between the lines of force and the normal to the surface are proportional to the permeabilities of the media. Because of the great permeability of iron, lines of force passing from iron to air must emerge almost perpendicular to the surface.

This molecular theory of magnetism is supported by the fact that wrought iron is only temporarily magnetised, while steel becomes a permanent magnet. When the magnetising force is removed, the molecules of the elastic wrought iron return almost entirely to their former chaotic arrangement, leaving only a small trace of remanent magnetism. With hard steel, on the other hand, the molecules, when once turned, remain in their new position and we have a permanent magnet. It is now clear why a steel magnet exhibits no magnetism at the middle of its axis, and yet, when broken in two, the two rough ends are found to be poles of opposite sign.

The strongest point in favour of the molecular theory is, however, the development of heat in a piece of iron subjected to rapidly reversing magnetism. A reversal of magnetism means a rotation of the molecules, and this is opposed by their friction, the molecules tending to remain in their previous condition and opposing any change. This phenomenon is therefore called hysteresis, i.e. a lagging behind. The work done in reversing the magnetism appears as heat, the energy expended per second being proportional to the frequency of the reversal and to the volume of the iron. It is also, of course, dependent on the hardness of the iron.

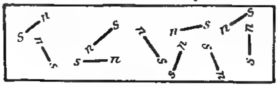

Fig. 39

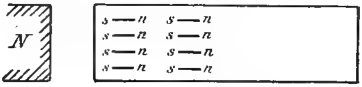

Fig. 40

24. The Earth's Field.

As the magnetic poles of the earth do not correspond with the geographical poles, the axis of a magnetic needle makes an angle with the geographical meridian, which is known as the angle of declination. In England the declination is to the west, and in London at the present time is 16°. If the needle be freely movable in the vertical as well as the horizontal plane, it will form an angle with the horizontal, known as the inclination. In England the north pole will be below the centre, and the inclination in London is 67°. With the ordinary magnetic needle, moving about a vertical axis, we are only concerned with the horizontal component of the earth's field. In London, at present, this is equal to 0·185, that is, the horizontal force on a unit pole is 0·185 dyne.

CHAPTER IV

25. Magnetic effect of a straight conductor.—26. Magnetic effect of a single-turn coil.—27. Solenoid.—28. Magnetisation curves.—29. The magnetic circuit.—30. Lifting power of an electromagnet.—31. Hysteresis.—32. Dynamic effect of parallel currents.—33. Induced electromotive force.—34. Laws of mutual induction.—35. Self-induction.—36. Eddy currents.

25. Magnetic Effect of a Straight Conductor.

It is found by experiment that a magnetic needle is deflected out of its north-and-south direction when near a conductor carrying a current. Hence the electric current produces a field in its neighbourhood. The direction of the lines of force of this field can be found by moving a compass needle continually in the direction indicated by it, or by sprinkling iron filings on a plane through which the conductor passes normally (Fig. 41).

The filings arrange themselves in concentric circles around the conductor as a centre. The lines of force produced by the current are therefore closed circles and a freely moving north pole—its corresponding south pole being far removed—would rotate about the conductor in the direction of the lines of force. This direction is always normal to the plane through conductor and pole. It can be found by Ampère's swimming rule: **If a person swims with the current and looks to the magnet needle, its north pole will be deflected to his left hand.** Another rule, requiring less imaginative effort, is the corkscrew rule: **The directions of current and field are related in the same way as the directions of translation and rotation of a right-handed screw.**

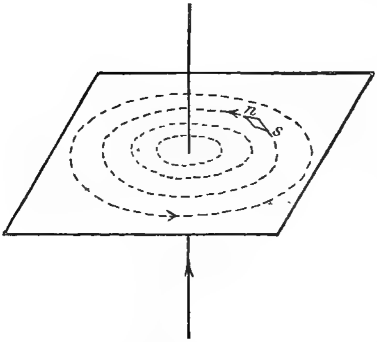

Fig. 41

The action between conductor and pole is naturally mutual. If the pole is fixed and the conductor is movable, the latter will move in a direction at right angles to a plane through conductor and pole. The motion will thus be at right angles to the plane through the conductor and the lines of force meeting it, due to the pole alone. The direction of motion can be found by Ampère's rule, in this case also, if the rule be expressed as follows: **If a person swims with the current and looks toward the north pole, whence the lines of force proceed, the north pole tends to move to the left. If the pole is fixed, the conductor tends to move to the right.** In this form, the rule is applicable to motors, in which a uniform field proceeds from a large pole face. In Fig. 42 a conductor, carrying a current in the direction indicated, is in front of a north pole, from which the lines of force come out normally to the paper. If we imagine ourselves swimming with the current and facing the north pole, we see that the conductor will move to the right, as shown by the dotted arrow. Similarly, in Fig. 43 a current-carrying conductor is in front of a south pole into which the lines of force enter. If we imagine a swimmer with his head to the right and feet to the left, looking out from the surface of the paper, his right hand will point in the direction of the dotted arrow and the conductor will therefore move towards the top of the page.

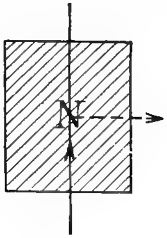

Fig. 42

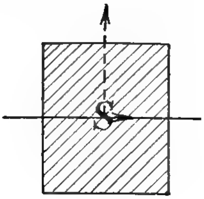

Fig. 43

As a further example, imagine a horizontal conductor carrying a current in a magnetic field, the direction of which is from right to left, perpendicular to the vertical plane through the conductor (Fig. 44). Swimming with the current and looking to the right, where we may imagine a north pole to be situated, our left hand points downwards, and the north pole would, if

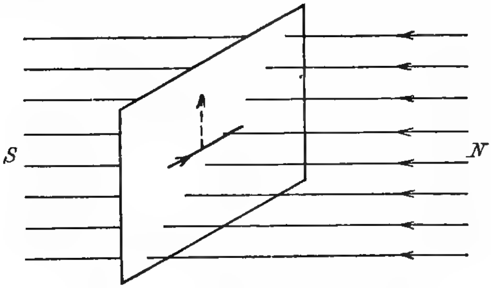

Fig. 44

possible, move in this direction. If the magnetic field is stationary the conductor will, if possible, move in the direction of our right hand, that is, vertically upwards as shown by the dotted arrow.

We must now determine the force exerted on each other by a pole and a small element of current-carrying conductor. In Fig. 45 let

I' = the current, in units which we have yet to determine,
dl = the length of the element of conductor in centimetres,
m = the strength of the pole, in units already defined,
r = the distance between pole and element of conductor in cm,
ϕ = the angle between r and dl,
df = the force in dynes between pole and element of current

It is found by experiment that the force is proportional to the current, to the pole strength, and to the length of the element of conductor. If ϕ is not a right angle, the length dl must be replaced by $dl \sin \phi$, as the number of the lines radiating from the pole m which meet dl is equal to the number crossing $AB = dl \sin \phi$. Finally, the force is inversely proportional to the square of the distance, since the pole m radiates lines of force equally in all directions and their density must therefore decrease with the square of the distance. We arrive, therefore, at the experimental result, that

$$df = \frac{m \,.\, I' \,.\, dl \,.\, \sin \phi}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

Although sometimes called Biot-Savart's law, this equation was originally established by Laplace. From equation (27) on page 45 we see that $\frac{m}{r^2}$ is equal to the field strength H produced by the pole m at the point occupied by the element of conductor. Equation (30) can therefore be written thus:

$$df = H \,.\, I' \,.\, dl \sin \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

where ϕ represents the angle between the conductor and the lines of force cutting it. If now the field is uniform and the conductor is straight and l cm long and perpendicular to the lines of force, the total force acting on the conductor is

$$f = H \,.\, I' \,.\, l \text{ dynes} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

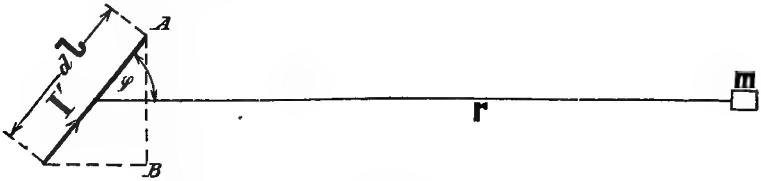

Fig. 45

This equation is applicable to the cases represented in Figs. 42, 43 and 44. We must, however, determine the unit in which I' must be expressed, for, having chosen the units for f, H and l, the current cannot be expressed arbitrarily in any unit, such as the ampere.

If in equation (32) we put $f = 1$, $H = 1$, and $l = 1$, then $I' = 1$. **The unit of current in equation (32) is therefore that current which exerts a force of 1 dyne on 1 cm. of conductor through which it flows, in a field of unit strength, the conductor and field being at right angles.** This unit is called the absolute unit of current. It appeared, formerly, to be too large for practical purposes and a tenth part of it was therefore adopted as the practical unit and called an ampere. Very accurate experiments were then made to determine the amount of silver deposited in a given time by this practical unit of current, the result being 1·118 milligrams per second.

If now

I' represents the strength of current in absolute units,
while I ,, ,, ,, ,, amperes,

we see that the number of amperes will always be ten times as large as the number of absolute units, i.e.

$$I = 10I',$$

or

$$I' = \frac{I}{10} \quad \text{.................................(33).}$$

Substituting in equation (32), we have

$$f = H \cdot \frac{I}{10} \cdot l \quad \text{...............................(34).}$$

The importance of this result is evident at once, if we apply it to electric motors. Suppose, for example, that 944 wires each 24 cm long lie upon the periphery of an iron drum (Fig. 46). Suppose, further, that two-thirds of the total number of wires lie under the poles and that each wire carries a current of 10 amperes. We assume that the wires are so connected that the turning-moments produced under each pole are in the same direction and can therefore be added. Let the strength of the field or the number of lines of force per sq. cm be 7000, and the diameter of the armature 40 cm. We are required to find the turning-moment or torque of the motor in metre-kilogrammes.

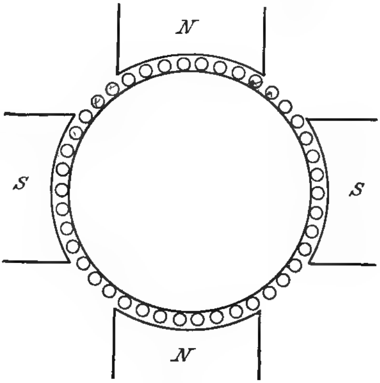

Fig. 46

The total length of wire under the poles is

$$l = \frac{2}{3} \cdot 944 \cdot 24 = 15{,}000 \text{ cm}.$$

From equation (34) we have

$$f = H \cdot \frac{I}{10} l = 7000 \cdot \frac{10}{10} \cdot 15{,}000 = 105 \cdot 10^6 \text{ dynes}.$$

Now from page 44 we know that 1 dyne $= \frac{1}{981{,}000}$ kg*.

The force acting on the periphery is therefore

$$F = \frac{105 \cdot 10^6}{981{,}000} = 107 \text{ kg*}.$$

This force acts at a radius of 20 cm = 0·2 metre, giving a turning-moment of

$$M_t = 107 \cdot 0{\cdot}2 = 21{\cdot}4 \text{ metre-kilogrammes}.$$

26. Magnetic Effect of a Single-turn Coil.

If either the swimming rule or the corkscrew rule be applied to a single-turn coil (Fig. 47), we find that all the lines of force produced by the current enter the coil from one side and come out at the other, closing on themselves by passing round outside the coil. The coil is thus a magnetic shell or disc, having a south pole on one face, where the lines of force enter, and a north pole on the other, where they leave. If then we face the disc or coil, and the current is flowing in a clockwise direction, we are facing a south pole; whereas, if the current is flowing in an anti-clockwise direction, we are facing a north pole.

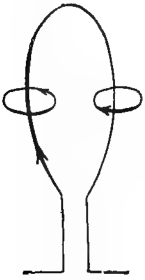

Fig. 47

In order to determine the magnitude of the force exerted on a pole m, situated at the centre of the coil, we shall consider, in the first place, a small element dl of the coil. The length of the element is at right angles to the radius, that is, to the line joining the element of conductor to the pole at the centre. The value of $\sin\phi$ is therefore 1. If the radius of the coil be r cm, we have from equation (30) on page 54,

$$df = \frac{m \,.\, I' \,.\, dl}{r^2}.$$

The total length of all such elements of conductor is $2\pi r$. The total force is therefore

$$f = \frac{m \,.\, I' \,.\, 2\pi r}{r^2} = \frac{m \,.\, I' \,.\, 2\pi}{r} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Now the strength of field H is defined as the number of lines of force per sq. cm or as the force on unit pole. Putting $m = 1$ in equation (35) we obtain the field strength at the centre as

$$H = \frac{2\pi \,.\, I'}{r} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

We will now make use of this result to explain the principle of a certain class of practical measuring instruments. A large number of such instruments depend for their action on the effect of a current-carrying coil or bobbin on a magnet needle. The movement, however, by which the effect is measured, alters the relative position of coil and needle, thus destroying the proportionality between the current in the coil and its force on the needle. Such instruments must therefore be calibrated empirically.

In other instruments, however, a definite relation between the current and the reading of the instrument is maintained. If, for example, the deflected needle is always brought back to its neutral or zero position by twisting one end of a spiral spring, the other end of which is attached to the needle, the turning-moment exerted on the needle by the current is exactly equal to that exerted by the spring, which, we know, is proportional to the angular twist applied to it. The current in the coil is therefore proportional to the angle through which the torsion head has to be turned.

Another method, by which proportionality may be maintained within certain limits, is to work with very small deflections which can be read by means of a beam of light, reflected on to a scale from a small mirror attached to the magnet needle. In such mirror galvanometers the relative movement between coil and needle is so small that proportionality is maintained over a certain range.

Finally, the diameter of the coil can be made so large, compared with the length of the needle at its centre, that, whether deflected or not, the poles of the needle can be assumed to be at the centre of the coil. The instrument is then known as a tangent galvanometer. For the sake of clearness, the needle in Fig. 48 is shown much larger, compared with the diameter of the coil, than it actually is. The tangent galvanometer is set up so that the plane of the coil is vertical and in the magnetic meridian. When no current is flowing, the axis of the needle will lie in the plane of the coil. In the plan of the galvanometer (Fig. 48) the coil will be represented by a straight line parallel to the lines of force of the horizontal component h of the earth's magnetism. When a current flows in the coil, each pole m of the needle will be subjected to a force f perpendicular to the plane of the coil; from equation (35) we have

$$f = \frac{m \,.\, I' \,.\, 2\pi}{r}.$$

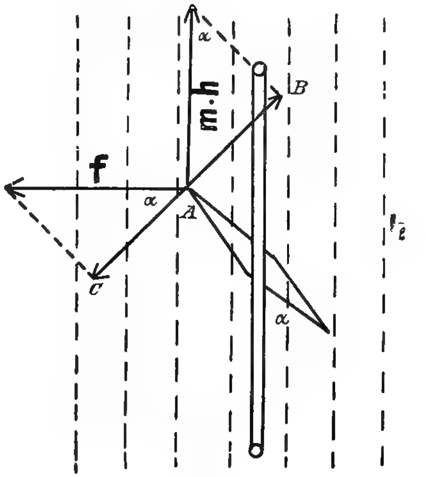

Fig. 48

The force on each pole due to the earth's field is equal to $m.h$ and acts parallel to the plane of the coil. Under the action of these two forces the needle comes to rest at an angle α to the plane of the coil. The component of the force due to the earth, acting at right angles to the needle, is

$$AB = m.h.\ \sin\alpha,$$

while the opposing component of the force due to the current in the coil is

$$AC = f \,.\, \cos\alpha = \frac{m \,.\, I' \,.\, 2\pi}{r} \cos\alpha.$$

Since the needle is in equilibrium, $AB = AC$, or

$$m \,.\, h \,.\, \sin\alpha = \frac{m \,.\, I' \,.\, 2\pi}{r} \cos\alpha.$$

Cancelling m out from each side of the equation, we see that the reading of the galvanometer is quite independent of the strength or size of the magnetic needle. Solving the above equation for I', we have

$$I' = \frac{h \,.\, r}{2\pi} \tan\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots (37).$$

Since the number of amperes is 10 times the number of absolute units we have

$$I = \frac{10 \,.\, h \,.\, r}{2\pi} \tan\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

If the horizontal component of the earth's field at any place is known, the strength of current can thus be calculated from the values of r and α. Conversely, a standardised tangent galvanometer can be used to determine the horizontal component.

We will now consider the case in which the pole m lies on the axis AA of the coil, but at a distance from it (Fig. 49). If the line a joining the pole to the circumference of the coil makes an angle α with the axis, then

$$a = \frac{r}{\sin\alpha}.$$

Now the element of conductor at B is perpendicular to the plane of the paper and is therefore at right angles to the line a which lies in that plane. Putting $\sin\phi = 1$ in equation (30) on page 54, we get

$$df = \frac{m\,.\,I'\,.\,dl}{a^2} = \frac{m\,.\,I'\,.\,dl\,.\,\sin^2\alpha}{r^2}.$$

This force acts perpendicularly to the plane containing the pole and the element of conductor, that is, in the direction CD. The component acting along the axis is $CE = df.\sin\alpha$. Adding up the forces due to all the elements of conductor constituting the coil, we find that all the components such as ED, at right angles to the axis, neutralise one another, while the components along the axis must be added. The total force acting on the pole m is therefore along the axis AA and has a magnitude

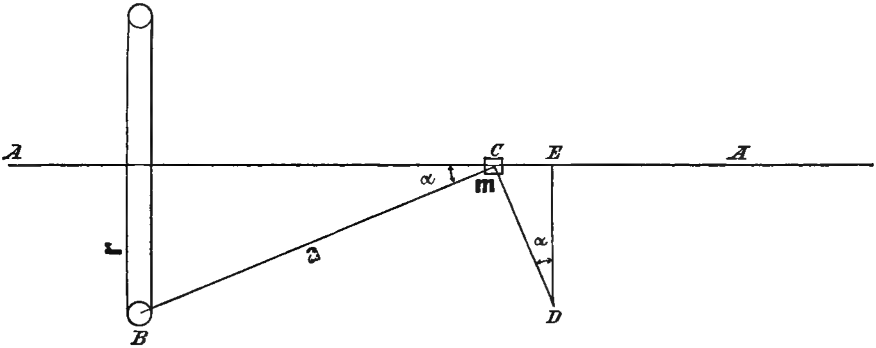

Fig. 49

$$f = \Sigma df\,.\,\sin\alpha = \frac{m\,.\,I'\,.\,\sin^3\alpha}{r^2}\,.\,\Sigma dl.$$

Since Σdl is equal to $2\pi r$, this reduces to

$$f = \frac{2\pi m\,.\,I'}{r}\,.\,\sin^3\alpha \quad\text{..........................}(39).$$

To find the field strength at point C, that is, the force on a unit pole at this point, we must put $m = 1$ in equation (39), giving

$$H = \frac{2\pi\,.\,I'}{r}\,.\,\sin^3\alpha \quad\text{..........................}(40).$$

This equation will prove of great assistance to us in the next section in calculating the field of a solenoid.

27. Magnetic Field of a Solenoid.

The lines of force due to two parallel conductors carrying currents in the same direction combine to form lines of force which encircle both conductors. Two conductors are shown, for example, in Fig. 50, passing vertically through the paper, and carrying currents which flow towards us. The points in the cross-sections of the wires may be considered as representing the points of arrows indicating the direction of the current. If now we draw the lines of force due to each individual conductor, we find that they are in opposite directions in the space between the conductors, and therefore mutually neutralise one another more or less. At points such as *A* and *B*, on the other hand, the forces due to both conductors are in the same direction, and the density of the lines of force is greater than that due to either conductor alone. We could calculate in this way the strength and direction of the field at every point, but it is simpler to make use of iron filings and obtain a diagram such as Fig. 51. We shall thus see clearly that the lines embrace both conductors.

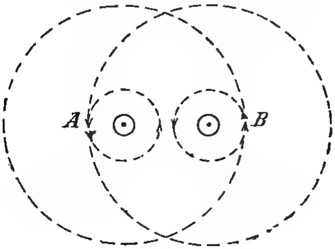

Fig. 50

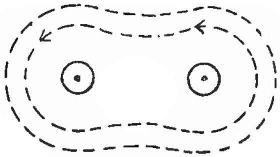

Fig. 51

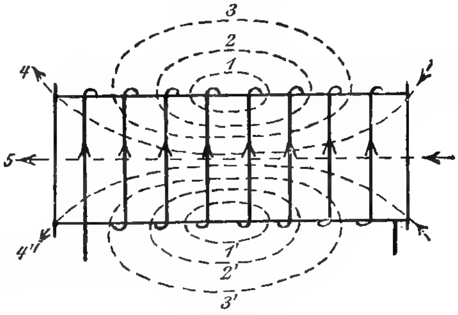

Fig. 52

If the wire be wound into a solenoid, that is, a long cylindrical coil of small diameter, the lines of force pass through each individual turn or coil as seen in the previous section. They combine now to form much longer lines threading more or less through the whole solenoid, inside which they run approximately parallel to the axis. Some of the lines such as 1 and 1′ in Fig. 52 are produced by the two middle turns only, others such as 2 and 2′ by the four middle turns and so on, while others such as 4, 4′ and 5 are due to the combined action of all the turns. We see, therefore, that a current-carrying solenoid is very similar to a bar magnet, for its lines of force do not enter and leave entirely at the ends, but, to a large extent, emerge from the sides of the north polar half of the magnet and doubling back re-enter the sides of the south polar half.

The polarity of the solenoid can be determined by either of the rules already given. Swimming with the current and looking into the solenoid, the left hand indicates the north polar end, and also the positive direction of the lines of force inside the solenoid.

It can be seen that every line of force goes through the solenoid at its mid-point. The field strength, that is, the number of lines of force per square centimetre inside the solenoid, is a maximum at this mid-section, and falls off on either side towards the ends. This is also the reason why an iron core is drawn into such a solenoid. Iron always tends to move into the strongest part of the magnetic field so that the number of lines passing through it from end to end is a maximum. If we imagine a rod of iron half in the solenoid and half out, we see that the lines would have a more convenient path and therefore increase in number in the iron rod, were it further inside the solenoid; it will therefore be drawn in. The seat of the force on the rod can be seen at once when we consider that the lines enter or leave that end of the rod which is inside the solenoid in parallel paths, whereas these same lines leave or enter that part of the rod which is outside the solenoid in all directions. A great number of galvanometers, ammeters and voltmeters depend for their action on this principle; they are generally known as soft-iron instruments.

We will now investigate the field strength at the centre of the solenoid. We will represent by

I' the strength of current in absolute units,
T the number of turns,
l the length of solenoid in centimetres,
r the radius of solenoid in centimetres.

The T turns carrying the current I' are equivalent to a belt l cm wide carrying a current $T.I'$. The current which in Fig. 53 flows in the strip of width dx is therefore equal to

$$T\,.\,I'\,.\,\frac{dx}{l}\,.$$

Imagine a unit pole to be placed at the centre of the solenoid. The force exerted on it by the band dx can be found by equation (40) on page 58.

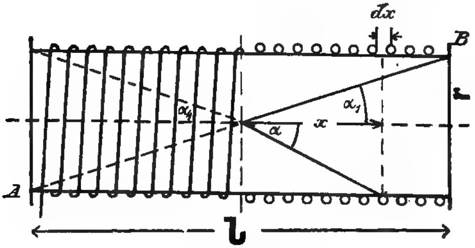

Fig. 53

For I' in the equation we must substitute $T\,.\,I'\,.\,\frac{dx}{l}$; this gives us

$$dH = \frac{2\pi\,.\,T\,.\,I'}{r}\,.\,\frac{dx}{l}\,.\,\sin^3\alpha.$$

Now $x = r\,.\,\text{cotan}\,\alpha$ and $dx = -\frac{r}{\sin^2\alpha}\,.\,d\alpha.$

Putting this value for dx in the equation for dH and integrating between the limits $\pi - \alpha_1$ and $+ \alpha_1$, we get

$$H = -\int_{\pi-\alpha_1}^{+\alpha_1} \frac{2\pi . T . I'}{l} . \sin\alpha . d\alpha = \frac{2\pi . T . I'}{l}\Big[\cos\alpha\Big]_{\pi-\alpha_1}^{+\alpha_1} = \frac{4\pi . T . I'}{l} . \cos\alpha_1 \quad\text{.................(41).}$$

For very long solenoids $\cos\alpha_1$ is practically equal to 1, so that

$$H = \frac{4\pi . T . I'}{l} \quad\text{..............................(42).}$$

If the current be expressed in amperes, this formula becomes

$$H = \frac{0{\cdot}4\pi . TI}{l} \quad\text{..............................(43).}$$

If the unit pole be placed at the centre of one of the ends of the solenoid, the integration must be carried out between the limits $\frac{\pi}{2}$ and $\frac{\alpha_1}{2}$ (Fig. 54). We obtain then

$$H = -\int_{\pi/2}^{\alpha_1} \frac{2\pi . T . I'}{l} . \sin\alpha . d\alpha = \frac{2\pi . T . I'}{l}\Big[\cos\alpha\Big]_{\pi/2}^{\alpha_1} = \frac{2\pi . T . I'}{l} . \cos\alpha_1 \quad\text{.................(44).}$$

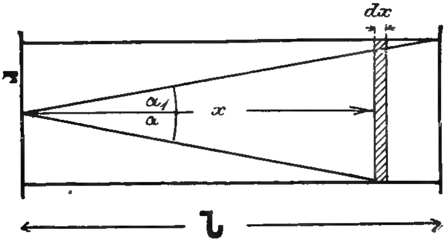

Fig. 54

As before, if the solenoid is very long, α_1 is very small and $\cos\alpha_1$ can be taken as unity. The field strength at the ends is thus half that at the middle, and of the total number of lines passing through the middle of the solenoid a half passes through the ends while the other half leaves by the cylindrical surface.

28. Magnetisation Curves.

Under similar conditions, the number of lines of force per square centimetre is greatly increased by the presence of iron within the coil. The solenoid then becomes an electromagnet. It is as if the lines of force which would be present were the space occupied by air produced or induced a much greater number of lines in the iron. **The field strength or number of lines of force per square centimetre in the air, which is represented by the letter H, is therefore also called the magnetising force. The number of lines per square centimetre in the iron is called the magnetic induction and is represented by the letter B.** It is to be noticed that the induction B includes the lines which were present before the iron was introduced as well as the new lines due to its introduction.

We account for the greatly increased number of lines in iron by assuming that its magnetic conductivity or permeability is much greater than that of air. The ratio of this permeability to that of the air is represented by the Greek letter μ. **The permeability μ gives, therefore, the magnetic conductivity of the iron compared with air; other things being equal, the number of lines produced in the iron is μ times the number which would have been produced in the air.** From this it follows that

$$B = \mu . H \qquad \ldots\ldots (45),$$

or

$$\mu = \frac{B}{H} \qquad \ldots\ldots (46).$$

For air the permeability is evidently 1, and we have $B = H$.

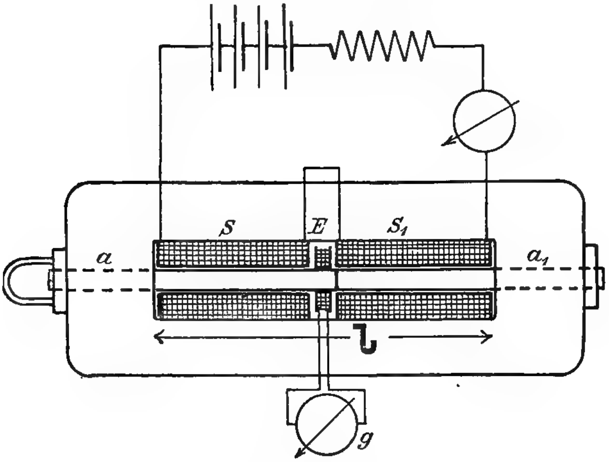

Fig. 55

The value of μ depends above all on the quality of the iron. The permeability of annealed armature stampings or of mild cast magnet steel may sometimes exceed 3000. The permeability of a given piece of iron or steel varies moreover with the degree to which it is magnetised.

The experimental determination of the permeability may be carried out in the way indicated in Fig. 55. The rod of iron to be tested is made in two parts, and passes through holes a and a_1 in a massive iron yoke and also through the magnetising coils S and S_1, and the small secondary coil E. A current I is passed through the coils S and S_1, and from a knowledge of the number of turns in these coils, together with the length l of the iron rod, the value of H can be found from the equation

$$H = \frac{0{\cdot}4\pi . I . T}{l}.$$

The length l does not include the parts of the rod inside the holes a and a_1, as these carry very little of the magnetic flux, which leaves the rod in all directions as soon as the rod enters the yoke. The latter being very massive offers very little resistance to the magnetic lines and can therefore be neglected. The left-hand half of the rod is now pulled out very suddenly, whereupon the secondary coil E is ejected by means of a spring. The magnetic flux through the coil is thus suddenly removed, with the result that an electromotive force is induced in the coil, causing a momentary current to flow through the so-called ballistic galvanometer g, which is connected up in series with the coil E. From the deflection or swing of the ballistic galvanometer, the number of lines passing through the iron rod, and therefore cut by the coil E, can be calculated. By dividing this number of lines by the cross-section of the rod we obtain the induction density B. The experiment can then be repeated with another value of the magnetising current, and in this way a set of corresponding values of H and B obtained.

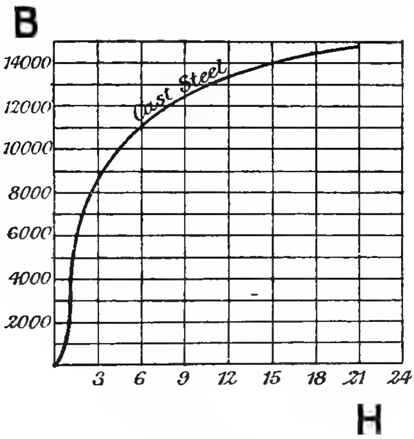

Fig. 56

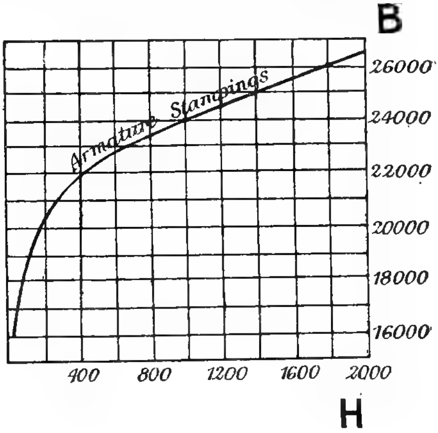

Fig. 57

Tests made on a sample of cast steel gave the following results (compare Fig. 56):

$H =$	$B =$	$\mu = \frac{B}{H} =$
0·9	1130	1260
1·55	5200	3350
2·7	8160	3020
3·75	9480	2530
8·55	12440	1460
18·1	14510	800
34·5	15710	460
82·7	17150	210
145·3	18200	130

A peculiar point in the above table is the low permeability possessed by the steel when weakly magnetised. The molecules are evidently but slightly affected by small values of the magnetising force. With stronger fields, covering the range of inductions from 5000 to 9000, B is roughly proportional to H. Finally, we see that beyond a certain point a continual increase in the magnetising force has little effect on the magnetic induction. The iron is then said to be saturated. It is impossible, however, to say at what point saturation commenced.

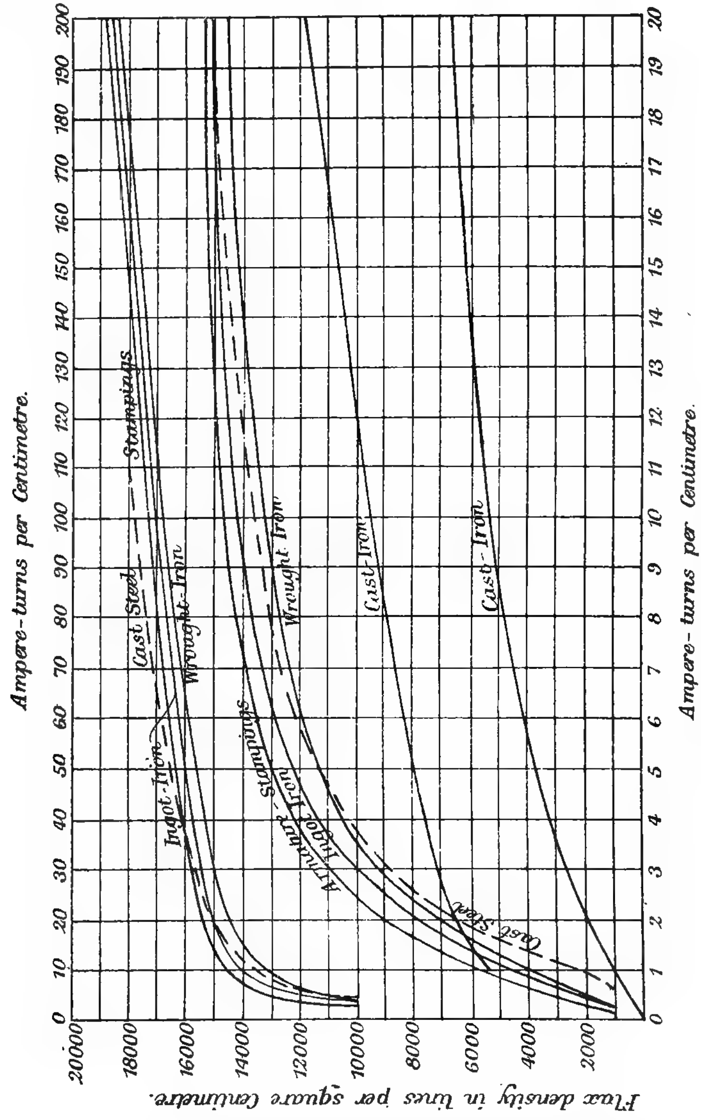

Fig. 58

(From Kapp's *Dynamo Machines.*)

If now we plot these results on squared paper, with the magnetising force H as abscissae and the induction B as ordinates, we obtain the so-called magnetisation curve. This is shown for the specimen of steel in Fig. 56. Beside it in Fig. 57 the upper part of the magnetisation curve for armature stampings is given, which is of special interest on account of the fact that the teeth of modern armatures are often very highly saturated. This curve gives the following values*:

H	B	μ
$H = 100$	$B = 18,700$	$\mu = 187$
300	21,300	71
500	22,500	45
700	23,100	33

If the magnetising force be still further increased, the value of B increases uniformly by an amount 2·5 times the increase of H. The upper part of the curve in Fig. 57 is thus a straight line. If this line be produced backwards it cuts the ordinate axis in the point $B = 21,350$. For very high saturation we have therefore

$$B = 21,350 + 2{\cdot}5H.$$

For practical purposes the magnetisation curves as drawn above are not convenient, as we generally require to know the necessary number of ampere-turns per centimetre length of path for a given value of B. This cannot be read off directly from the curve, but must be obtained from equation (43) on page 61 by dividing the corresponding value of H by $0{\cdot}4\pi$. Thus, if the number of ampere-turns be X, the ampere-turns per cm will be

$$\frac{X}{l} = \frac{I\,.\,T}{l} = \frac{H}{0{\cdot}4\pi} = 0{\cdot}8H \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47).$$

The calculations are simplified if the magnetisation curves are drawn with $\frac{X}{l}$ for abscissae instead of H, as is done in Fig. 58. For any value of the flux density B we can then read off exactly the necessary number of ampere-turns per centimetre of path.

From Fig. 58 we see that wrought iron, cast steel, and ingot iron require very small magnetising forces up to $B = 14,000$. Cast iron, on the other hand, is much less satisfactory, and it is for this reason that the magnet frames of large machines are now very often made of cast steel notwithstanding its higher price.

29. The Magnetic Circuit.

For a closed magnetic circuit such as, for example, an iron anchor ring, it is practically immaterial whether the ampere-turns are distributed over the whole circumference or are wound all together at one part, as shown in Fig. 59. In either case we can apply equation (43) on page 61 and get

$$B = \mu H = 0{\cdot}4\pi\,.\,\frac{X}{l} \quad \ldots\ldots\ldots\ldots\ldots(48),$$

where l applies no longer to the length of the coil, but to the mean length of the path of the magnetic lines.

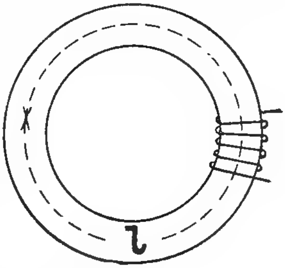

Fig. 59

If now A represents the cross-sectional area of the iron measured perpendicularly to the lines, then the total number of lines or the total magnetic flux is given by the equation

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* From the *E.T.Z.* (*Elektrotechnische Zeitschrift*), 1901, p. 769.

$$\Phi = B \,.\, A = \mu \,.\, H \,.\, A = \frac{\mu \,.\, 0{\cdot}4\pi \,.\, X \,.\, A}{l},$$

or

$$\Phi = \frac{0{\cdot}4\pi \,.\, X}{\dfrac{l}{\mu \,.\, A}} \quad \dots\dots\dots\dots\dots\dots\dots\dots (49).$$

The denominator of the right-hand side of this equation has a similar form to the expression for electrical resistance in the equation

$$R = \rho \,.\, \frac{l}{a}.$$

The similarity becomes plainer when we remember that the conductivity is the reciprocal of the specific resistance. The permeability μ in equation (49) can be looked upon as the magnetic conductivity, and its reciprocal $\frac{1}{\mu}$ then corresponds to the specific resistance ρ.

From this point of view the expression $\frac{l}{\mu \,.\, A}$ represents a magnetic resistance. It is proportional to the length and inversely proportional to the conductivity and to the cross-section of the magnetic path. Through this magnetic resistance we may imagine the lines of force to be driven in much the same way that the electric current is driven through an electrical resistance. It is because of this analogy that we speak of the total number of lines as the magnetic flux. The flux round the magnetic circuit is produced by the ampere-turns $I \,.\, T$, which may therefore be compared to the electromotive force which drives the electric current through the resistance of the electric circuit. This is, perhaps, made more evident by a consideration of the equation

$$H \,.\, l = 0{\cdot}4\pi \,.\, I \,.\, T,$$

which follows from equation (43) on page 61. Since H is the force on a unit pole, the product $H \,.\, l$ represents the work done in moving the unit pole in air against the lines of force over the whole distance l. This is, however, what we considered in Section 22 under the name of magnetic potential difference. The magnitude $0{\cdot}4\pi IT$ is evidently the magnetic analogy of the electromotive force of an electric circuit; it is therefore called the *magnetomotive force*. Equation (49) can now be written in the form:

$$\text{magnetic flux} = \frac{\text{magnetomotive force}}{\text{magnetic resistance}}.$$

The *magnetic resistance* of the whole or any part of a magnetic circuit is generally referred to as its *reluctance*.

This law of the magnetic circuit is of enormous importance in the design of electrical machinery. It is somewhat analogous to the law of the electrical circuit, i.e. Ohm's law, but we must be very careful lest we be misled by this electrical analogy. The similarities between the electric and the magnetic circuit are, after all, mainly only apparent. Although we speak of the total number of lines N as a magnetic flux, it is merely a name which enables us to follow the electrical analogy more readily; once the lines of force are produced they remain at rest and there is no flow of magnetism around the circuit. A far more accurate analogy to the magnetic flux, but one which is not so likely to appeal to an engineer, is found in the flux of electrostatic lines which connects two oppositely charged bodies, such as the plates of a condenser; permeability is here represented by the dielectric constant.

A great and fundamentally important difference between an electric current and a magnetic flux lies in the fact that work must be done to drive a current through a resistance, the energy thus expended appearing as heat in the resistance. The maintenance of a magnetic field or of magnetic induction involves, on the other hand, no expenditure of energy. None of the energy expended in the field windings of a dynamo, for example, is transformed into magnetism, but every trace of it is transformed into heat in the resistance of the field winding, and can be calculated from the product $I_m{}^2R_m$. The consumption of energy would be just the same if the current remained the same, although the space within the coil were filled with air instead of iron, thus reducing the number of lines to a very small value. For this reason there is no direct loss of energy in a dynamo due to a part of the magnetic flux leaking from pole to pole without passing through the armature.

Above all, however, attention must be drawn to the fact that electrical resistance depends simply on the length and cross-section of the conductor, whereas magnetic reluctance is also dependent on the degree of saturation of the iron.

We will now illustrate, by means of an example, the application of this law of the magnetic circuit to the calculation of a dynamo. The magnetic circuit consists of several parts differing in length, cross-section and material (air, wrought iron, cast iron, etc.). A part of the total magnetic flux is lost, so far as the armature is concerned, by leakage through the air, so that some parts of the magnetic circuit carry a larger number of lines of force than other parts. We have then to calculate the number of ampere-turns required on the poles to produce a given magnetic flux through the armature. To do this we divide the flux or total number of lines in each part of the magnetic circuit by the cross-section of that part. In this way we obtain the induction or flux density in lines per square centimetre at each point, thus

$$B_1 = \frac{\Phi_1}{A_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50).$$

We then find from the magnetisation curve of the material of which any part is constructed, the value of H_1, that is, the magnetising force necessary to produce the induction B_1 in the material concerned. Unfortunately the relation between B and H is no simple one, but depends in a complicated way on the saturation of the iron; we are therefore compelled to use magnetisation curves, which have been found experimentally for the materials which we are using. From H_1 and equation (47) on page 65 we have

$$X_1 = 0{\cdot}8H_1 . l_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51),$$

where l_1 is the length of path in the part under consideration. This calculation must be made for each portion of the magnetic circuit, and the results added together, giving

$$\Sigma X = X_1 + X_2 + X_3 \ldots = 0{\cdot}8H_1 . l_1 + 0{\cdot}8H_2 . l_2 + 0{\cdot}8H_3 . l_3 \ldots\ldots (52).$$

If the magnetisation curves are drawn with $\frac{X}{l}$, i.e. ampere-turns per cm, for abscissae, instead of H, the calculation for the path in iron is simpler, while for the air-gap we still have the equation

$$X_g = 0{\cdot}8B_g . l_g = 0{\cdot}8H_g . l_g.$$

We then have

$$\Sigma X = \left(\frac{X}{l}\right)_1 . l_1 + \left(\frac{X}{l}\right)_2 . l_2 + \ldots + 0{\cdot}8H_g . l_g,$$

where the numbers 1, 2, etc., refer to various parts of the iron circuit and g refers to the air-gaps.

As an example we take a machine with the following main dimensions:

Diameter of armature	D	= 20 cm,
,, ,, shaft	d_s	= 3 ,,
Axial length of armature ...	L	= 20 ,,
Angle subtended by pole ...	β	= 120°,
Length of each air-gap		0·4 cm,
Cross-section of poles and yoke...	A_m	= 400 sq. cm,
Length of path in poles and yoke	l_m	= 110 cm.

The armature is built up of laminations stamped out of wrought iron sheet and separated from each other by thin paper, the space lost in this way being 15 per cent. of the whole. We shall assume that the whole magnet is of cast iron. As indicated in Fig. 60, a part of the magnetic flux produced in the field magnets is lost by leakage. We shall assume that a sixth part of the flux produced is lost in this way, so that, if Φ be the flux in the armature and Φ_m that in the magnet,

$$\Phi_m = 1{\cdot}2 . \Phi.$$

We will now determine the ampere-turns required to produce a flux Φ in the armature of $2{\cdot}5 . 10^6$ lines.

We must first find the length and cross-section of each part of the magnetic circuit. The cross-section of the armature perpendicular to the lines of force is found by multiplying the difference $D - d_s$ by the axial length L, and also by 0·85 to allow for the paper insulation. Hence,

$$A_a = L\,(D - d_s) . 0{\cdot}85 = 290 \text{ sq. cm.}$$

The mean value of l_a is evidently about 20 cm.

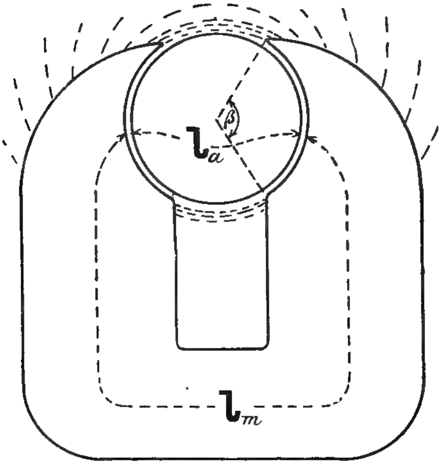

Fig. 60

To find the cross-section of the path in the air-gaps we must reduce the cylindrical surface of the armature in the ratio $\beta : 360$. We have then

$$A_g = D \,.\, \pi \,.\, L \,.\, \frac{\beta}{360} = 420 \text{ sq. cm.}$$

To find the length of the path in air we must multiply the length of the gap between armature and pole by 2, since the lines have to cross the gap both on entering and on leaving the armature. We have, therefore,

$$l_g = 2 \,.\, 0{\cdot}4 = 0{\cdot}8 \text{ cm.}$$

For the field magnet we have given that

$$A_m = 400 \text{ sq. cm,}$$
$$l_m = 110 \text{ cm.}$$

We know also that

$$\Phi_m = 1{\cdot}2 \,.\, \Phi = 1{\cdot}2 \,.\, 2{\cdot}5 \,.\, 10^6 = 3 \,.\, 10^6.$$

Bringing all the above results together, we have

$$\begin{array}{lll} \Phi = 2{\cdot}5 \,.\, 10^6, & & \Phi = 3 \,.\, 10^6, \\ A_a = 290, & A_g = 420, & A_m = 400, \\ l_a = 20, & l_g = 0{\cdot}8, & l_m = 110. \end{array}$$

From these figures it follows that

$$B_a = \frac{\Phi}{A} = 8600, \qquad B_g = \frac{\Phi}{A_g} = 5960, \qquad B_m = \frac{\Phi_m}{A_m} = 7500.$$

From the magnetisation curve for armature stampings in Fig. 58 on page 64 we see that the ampere-turns per cm required to produce an induction B of 8600 are

$$\left(\frac{X}{l}\right)_a = 2.$$

This is the number of ampere-turns required to drive the induction B_a through one centimetre of armature iron. The total number of ampere-turns necessary to drive the flux through the armature is therefore

$$X_a = \left(\frac{X}{l}\right)_a \,.\, l_a = 40.$$

Similarly we find that the ampere-turns per cm for an induction B_m of 7500 in cast iron are

$$\left(\frac{X}{l}\right)_m = 40.$$

From this it follows that to drive the flux at an induction B_m of 7500 through a length $l_m = 110$ cm of cast iron requires a number of ampere-turns

$$X_m = \left(\frac{X}{l}\right)_m \,.\, l_m = 40 \,.\, 110 = 4400.$$

Finally, the number of ampere-turns X_g required to drive the flux across the air-gap is found from the equation

$$X_g = 0{\cdot}8 B_g \,.\, l_g = 3820.$$

We have now $\Sigma X = X_a + X_m + X_g = 8260.$

It is immaterial whether, for example, we use 8260 turns carrying 1 ampere, or 4130 turns carrying 2 amperes, neglecting, for the present, all considerations of efficiency, etc. In many cases the magnetic circuit is more complicated than the above, including armature teeth and pole cores. The necessary ampere-turns for these parts are, however, calculated in exactly the same way as we have already indicated.

By means of this law of the magnetic circuit it is possible, under certain conditions, to calculate the leakage from the dimensions of the machine. This calculation makes the analogy between magnetic and electrical potential difference very plain. As electrical potential differences are expressed in volts, so can magnetic potential differences be expressed in ampere-turns. If a resistance lies in parallel with a lamp across which a certain pressure is maintained, a part of the total current flows through the resistance. If this parallel resistance is due to an earth, that is, to a fault in the cable, the current flowing by this by-path is a loss. Similarly, there is a magnetic potential difference or pressure between the pole-shoes of a machine, which drives the magnetic flux along two parallel paths, the useful flux going through the armature and the leakage flux through the air. This magnetic pressure is due to the ampere-turns $X_a + X_t + X_g$, which are necessary to drive the useful flux through the armature core, the teeth and the air-gaps.

In the multipolar alternator shown in Fig. 61, assume that the sum $X_a + X_t + X_g = 6200$. This same magnetic pressure acts across the air space between two adjacent pole-shoes. Assume that the axial length of the armature is 30 cm, the radial depth of the pole-shoe 2·5 cm, and the distance l_{l_1} between the adjacent pole-shoes 6·5 cm. The cross-section of the leakage path between the pole-shoes is, therefore, $30 . 2{\cdot}5 = 75$ sq. cm. Now, the lines of force pass from the north pole to the south pole on each side of it, so that the total cross-section is doubled, making it 150 sq. cm. Assuming that the lines of force spread out as shown in the figure, and that their maximum cross-section is double that at the ends, we get for a mean value of the cross-section

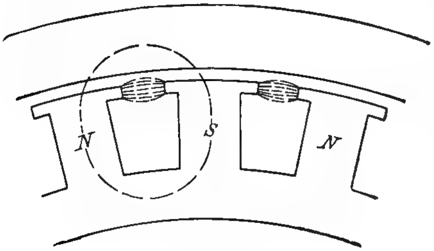

Fig. 61

$$A_{l_1} = \frac{150 + 300}{2} = 225 \text{ sq. cm.}$$

Putting these values of l_{l_1} and A_{l_1} in equation (49) on page 66, and remembering that μ for air is 1, we get for the leakage from the pole-shoe

$$\Phi_{l_1} = \frac{0{\cdot}4\pi . 6200}{\frac{6{\cdot}5}{225}} = 0{\cdot}27 . 10^6 \text{ lines.}$$

The fundamental formula $H = \frac{0{\cdot}4\pi . I . T}{l}$ would have led to the same result.

To this leakage from the pole-shoe we must add the leakage from the pole-core. We shall assume that the mean distance l_{l_2} between two adjacent pole-core faces is 10 cm, and the radial length of the pole-core 16 cm. With an axial length of 30 cm the cross-section of this leakage flux taken on each side of the pole is

$$A_{l_2} = 2.30.16 = 960 \text{ sq. cm.}$$

At the top of the pole the magnet potential difference is due to 6200 ampere-turns while at the root it is 0. As a mean we can therefore take 3100 ampere-turns, which gives a leakage flux of

$$\Phi_{l_2} = \frac{0{\cdot}4\pi X}{\dfrac{l_{l_2}}{A_{l_2}}} = \frac{0{\cdot}4\pi \,.\, 3100}{\dfrac{10}{960}} = 0{\cdot}37 \,.\, 10^6.$$

The total leakage flux from the pole is therefore

$$\Phi_l = \Phi_{l_1} + \Phi_{l_2} = 0{\cdot}27 \,.\, 10^6 + 0{\cdot}37 \,.\, 10^6 = 0{\cdot}64 \,.\, 10^6.$$

We have neglected the leakage between the flanks or end surfaces of the poles, which would make the total leakage slightly more than the above value.

If we assume that the useful flux through the armature is $3 \,.\, 10^6$ lines per pole, the flux at the root of the pole will be

$$\Phi_m = \Phi + \Phi_l = 3{\cdot}64 \,.\, 10^6,$$

which gives a leakage coefficient of

$$\nu = \frac{\Phi_m}{\Phi} = \frac{3{\cdot}64 \,.\, 10^6}{3 \,.\, 10^6} = 1{\cdot}2.$$

30. The Lifting Power of an Electromagnet.

Although, as pointed out in the last section, no work is done in maintaining a magnetic field, work must be done to establish the field. In Fig. 62 is shown an iron ring which we shall suppose to be uniformly wound over its whole periphery with T turns. Some of these turns are shown, on the right in elevation, on the left in section. By the dotted lines of force we do not mean to suggest that a part of the flux produced leaks across the interior space, although, were the ring wound as shown in the figure, this would be the case. The dotted lines show how the lines of force, due to the establishment of a current in the turns shown in section, spread out, commencing

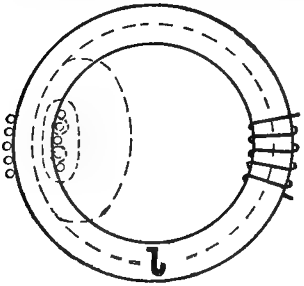

Fig. 62

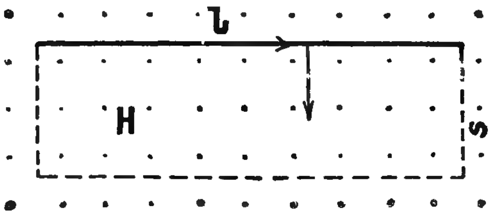

Fig. 63

as small circles round the individual wires and then combining with the lines due to other wires to form lines encircling several wires. This spreading out goes on until each line of force lies entirely in the iron, it having cut all the T wires passing through the interior of the ring. As soon as the current ceases to grow and becomes steady, the lines of force become stationary.

In order to calculate the work done in thus establishing the magnetic field, we will consider a straight wire of length l cm carrying a current of I' absolute units, in a field of strength H perpendicular to the wire (Fig. 63). The direction of the field is assumed to be away from the reader and perpendicular to the paper, as indicated by the dots. We will assume that the conductor is moved a distance s in a direction at right angles both to its own length and to the direction of the field, the direction of motion being such that the force exercised by the field on the conductor has to be overcome. The work done is found by multiplying the force in dynes by the distance in centimetres, thus

$$W = f.s = H.I'.l.s \text{ ergs},$$

where I' is the current in absolute units (see p. 54).

Now, $l.s$ is the area swept out by the motion of the wire, and as H is the number of lines of force per sq. cm, $H.l.s$ is equal to the total number of lines Φ cut by the wire; hence

$$W = H.I'.l.s = \Phi.I' \text{ ergs} \quad \ldots\ldots\ldots\ldots(53).$$

Now, if the current I' in the case represented in Fig. 62 is increased by an amount dI', the induction or flux density increases by an amount dB, no matter whether the ring contains iron or air. If the cross-sectional area of the ring be A sq. cm, the number of newly created lines will be $A.dB = d\Phi$. These newly created lines cut, as we have seen, the T turns composing the ring winding, which carries a current I'. The work done in this change, according to equation (53), is

$$dW = T.d\Phi.I' = T.dB.A.I' \text{ ergs}.$$

From equation (42) on page 61, we have

$$T.I' = \frac{H.l}{4\pi},$$

and substituting this in the above equation, we have

$$dW = \frac{A.l}{4\pi}.H.dB \text{ ergs} \quad \ldots\ldots\ldots\ldots(54).$$

If the coil be wound on a ring of wood or other non-magnetic material, $B = H$, and the total amount of work done in raising the magnetic induction from 0 to B will be

$$W = \int_0^B \frac{A.l}{4\pi}.B.dB = \frac{B^2.A.l}{8\pi} \text{ ergs} \quad \ldots\ldots\ldots\ldots(55).$$

This amount of energy has to be given to the coil, as we shall see in Section 35, in the form of electrical energy. The volume of the magnetic field is $A.l$, so that each cubic centimetre of the magnetic field contains a store of potential energy equal to $\frac{B^2}{8\pi}$ ergs. We may picture the ether in a state of strain, somewhat like a twisted steel spring.

We will now make use of this result, in order to find the lifting power of an electromagnet. Fig. 64 represents an iron ring, uniformly wound so as to obviate magnetic leakage, and divided into two similar halves. We have seen that if the magnetic flux round the ring decreases by an amount $d\Phi = A.dB$ an electromotive force is induced in the electric circuit in the same direction as the magnetising current. As the induced E.M.F. and the current are in the same direction, work is done on the electric circuit at the expense of the energy stored in the iron of the ring. If, now, we assume that the small decrease of flux is caused by separating the two halves of the ring by a small amount dl, the above reasoning still holds good and the iron loses an amount of energy which is given up to the electric circuit. To separate the two halves of the ring, however, we exerted a force of f dynes and did $f.dl$ ergs of work, which must evidently be stored in the magnetic field in the air-gaps. Now, let the total cross-section of the two air-gaps be A sq. cm, so that the volume of the magnetic field in air is equal to $A.dl$ cubic centimetres. The flux-density B is the same in the air-gap as in the iron, and we have found above that the energy stored in each cubic centimetre of the field in air is equal to $\frac{B^2}{8\pi}$ ergs. Hence

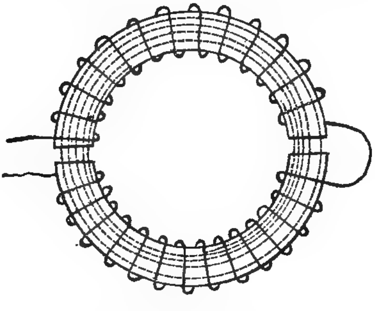

Fig. 64

$$\text{energy stored in air-gaps} = \frac{B^2}{8\pi}.A.dl \text{ ergs,}$$

therefore
$$f.dl = \frac{B^2.A}{8\pi}.dl,$$

and
$$f = \frac{B^2.A}{8\pi} \text{ dynes} \quad \ldots\ldots(56).$$

If F be the force in kilogrammes,

$$F = \frac{B^2.A}{8\pi.981{,}000} = 4B^2.A.10^{-8} \text{ kgs*} \quad \ldots\ldots(57).$$

In a horse-shoe magnet both poles are effective, and the double cross-section must therefore be included in A. If, for example, the cross-section of each pole is 10 sq. cm and the induction $B = 18{,}000$, we have

$$F = 4.18{,}000^2.2.10.10^{-8} = 260 \text{ kilogrammes.}$$

The lifting power of a magnet is often largely affected by magnetic leakage, which we have here neglected.

We shall now consider what happens when an electromagnet attracts its armature. We shall assume that the magnetisation curves of the magnetic circuit, both before and after the attraction of the armature, are represented by straight lines through the origin. In Fig. 65 OD and OC represent the magnetisation curves before and after attraction, the length of each air-gap being $l_1/2$ before and $l_2/2$ after attraction. The travel of the armature is then $(l_1 - l_2)/2$. If the winding is connected to mains of constant P.D. the current-turns will have the same value after as before. In Fig. 65 OE represents the current-turns and OA and OB the original and subsequent values of the magnetic flux.

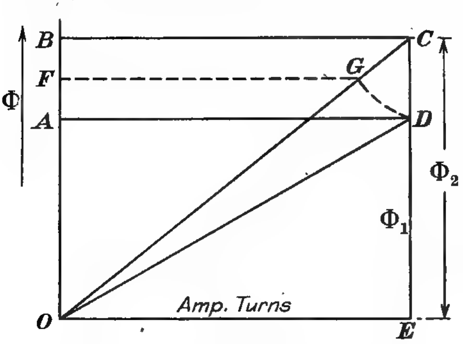

Fig. 65

It has been shown that the additional energy stored in a magnetic field when the flux increases is given by the formula

$$dW = TI'.\,d\Phi \text{ ergs,}$$

where I' is the current in absolute units.

Hence the total energy stored in the magnetic field when $T.I' = OE$ and the flux $\Phi_1 = OD$ will be represented by the area of the triangle ODA which is equal to that of the triangle OED. When the armature is attracted and the flux has increased to $\Phi_2 = OC$ the store of energy is increased to OEC. Hence the energy stored in the magnetic field has increased by an amount represented by the area ODC.

Hence the mechanical energy required to raise the armature is not supplied from any stored magnetic energy, but sufficient energy is drawn from the mains to raise the armature and, at the same time, increase the magnetic energy. This energy is supplied by the current flowing against a back E.M.F. induced in the winding by the increasing flux. This back E.M.F. causes the current to fall temporarily during the operation, the current-turns following some such curve as DG; the flux is assumed to have reached the value OF at the moment when the armature reaches the end of its travel; the current-turns then increase from FG to BC following the line OGC. From the formula $dW = T.\,I'.\,d\Phi$ it follows that the total energy supplied from the mains is represented by the area $ABCGDA$. If the area DGC be small, or if, by keeping the current constant during the operation, this area be reduced to zero, the total energy supplied is represented by the rectangle $ABCD$ which has twice the area of the triangle DOC.

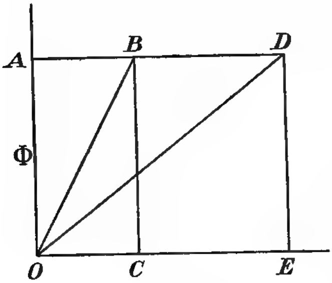

Fig. 66

Hence, we obtain the interesting result that, of the total energy drawn from the mains (neglecting that dissipated due to resistance), a half goes to increase the store of magnetic energy and a half is converted into mechanical energy. It must not be forgotten, however, that this result depends on the assumption of linear magnetisation curves.

If it be assumed that the magnetic flux is kept constant during the movement of the armature, there can be no induced E.M.F. and therefore no supply of energy from the mains. In this case the mechanical energy must be withdrawn from the stored magnetic energy. In Fig. 66 OD and OB are the magnetisation curves before and after the operation. The magnetic energy has decreased from OED to OCB, that converted into mechanical energy being represented by the triangle OBD, the area of which is equal to $\Phi . BD/2$, BD being the current-turns required to drive the flux across the difference between the original and subsequent air-gaps, i.e.

$$BD = B\,(l_2 - l_1)/4\pi.$$

Since, if each air-gap has an area $A/2$, $\Phi = BA/2$ the magnetic energy converted into mechanical energy

$$= \frac{\Phi . BD}{2} = \frac{B^2 A}{16\pi}\,(l_2 - l_1) = f\,\frac{l_2 - l_1}{2}.$$

Hence $f = \dfrac{B^2 A}{8\pi}$ dynes, as found above at equation (56).

31. Hysteresis.

In determining the magnetisation curve, we start with the iron in an unmagnetic state and gradually increase the magnetising current, thus increasing the value of H, and with it the value of the induction B. Plotting the values thus obtained with the magnetising forces $H = \dfrac{0{\cdot}4\pi TI}{l}$ as abscissae and the inductions B as ordinates, we obtain the curve OA in Fig. 67.

If, after reaching any arbitrary value of the induction, such as $AG = B_{\max}$, we decrease the magnetising current, we find that, for any given value of the magnetising force H, the induction is greater than it was with increasing magnetisation. Plotting these values of B and H for decreasing values of H we obtain the curve AB. It appears as if a certain amount of the magnetic induction still remains in the iron after the magnetising force has been removed. This phenomenon is therefore known as hysteresis, i.e. a lagging behind. For a magnetising force $H = 0$ the value of the induction B is OB. This is therefore the induction or flux-density of the remanent magnetism. Hysteresis and remanent magnetism are therefore, in a sense, one and the same thing. We explain both phenomena by the tendency of the molecules, in virtue of a kind of molecular friction, to remain in any position they have once taken up.

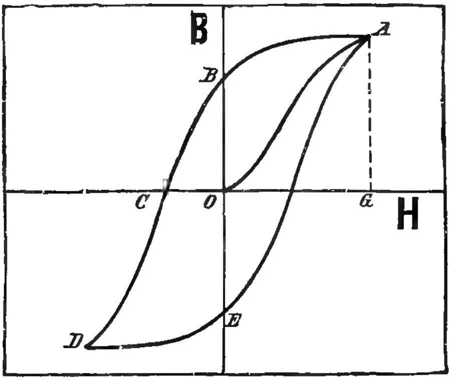

Fig. 67

If, now, the direction of the current be reversed, a certain value of the magnetising force will be required before the remanent magnetism is destroyed. This force is represented in Fig. 67 by the length OC, and is called the coercive force. It is evident that the iron molecules oppose the reversal of magnetism, and it is only when the magnetising force exceeds *OC* that an induction is obtained in the reverse direction. A further increase in the magnetising current gives the curve *CD*, and the succeeding decrease in the current, the curve *DE*. The ordinate *OE* is, as before, the remanent magnetism, etc.

An experiment with the specimen of cast steel already mentioned (Fig. 56) gave the following results:

Decreasing Induction		Increasing Induction	
H	*B*	*H*	*B*
145·3	18,250	− 2·2	− 6,240
62·7	16,800	− 5·9	− 11,060
24·2	15,590	− 11·9	− 13,460
3·2	13,080	− 34·2	− 15,710
0	10,200	− 61·6	− 16,680
− 1·25	6,110	− 145·3	− 18,250
− 1·5	0		

We see from this table that the remanent magnetism has a value $B = 10{,}200$, while the coercive force is 1·5. Both of these figures are dependent on the value of $B_{\max}$ reached in the experiment, in the above case 18,250; for high values of the induction, however, the variation is very small.

If the above values be plotted on squared paper we get a curve like *ABCD* in Fig. 67. The other side of the loop *DEA* can be drawn symmetrical with the first side. If the area of the loop is now measured, the abscissae being taken to the scale of *H*, and the ordinates to the scale of *B*, it is found to be 170,000. It can now be shown that this area is closely related to the amount of energy dissipated during the complete cycle through the intermolecular friction. From equation (54) on page 72 we know that the work done when the induction *B* is increased by the amount *dB* is

$$dW = \frac{A\,.\,l}{4\pi}\,.\,H\,.\,dB.$$

$A\,.\,l$ is equal to the volume *V* of the iron ring in c.cm. If we divide by *V* and integrate between the limits 0 and *B* we obtain the work done per cubic centimetre, thus

$$\frac{W}{V} = \frac{1}{4\pi}\sum_{0}^{B} H\,.\,dB \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58).$$

Now, $H.dB$ is represented by one of the narrow strips in Fig. 68, and $\Sigma H.dB$ by the whole shaded area in the same figure. The work done in magnetising the iron is thus obtained by dividing the shaded area in Fig. 68 by 4π. If, now, the lines of force decrease, *dB* is negative and the work done is also negative, that is, energy is restored. The amount of energy thus restored is represented by the shaded area in Fig. 69. If the magnetising current be now reversed, the magnetising force *H* becomes negative, and the product $H.dB$ will again be positive, so that work must be done during this period corresponding to the shaded area in Fig. 70. With decreasing magnetisation a part of the energy is again restored as shown in Fig. 71.

The actual energy dissipated in a complete period is the resultant of the positive and negative amounts of energy. If we go continually through the cycle represented by the loop $ABCDEA$ in Fig. 67, the energy dissipated per cubic centimetre of iron in each cycle is, from equation (58), equal to the area of the loop divided by 4π. For the specimen of cast steel above mentioned the area was 170,000, whence the loss per cubic centimetre per cycle is

$$\frac{A}{V}=\frac{\Sigma H \,.\, dB}{4\pi}=\frac{170{,}000}{12{\cdot}5}=13{,}600 \text{ ergs per cubic centimetre.}$$

Although this method of finding the hysteresis loss by drawing and measuring the loop is of great importance in enabling us to understand the question of hysteresis, it is rarely adopted in practice on account of the time required to determine experimentally with a ballistic galvanometer the points on the hysteresis loop. It is now a rule to test the iron, as far as possible, under the conditions obtaining in actual practice. The magnetism is subjected to rapid reversals by means of an alternating current, and the energy consumed is measured on a wattmeter, the deflection of which is caused by the electrodynamic forces between a coil carrying the current and another coil connected across the mains and therefore carrying a current proportional to the potential difference between the mains.

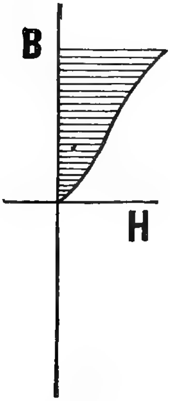

Fig. 68

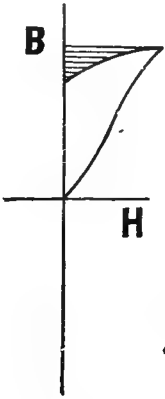

Fig. 69

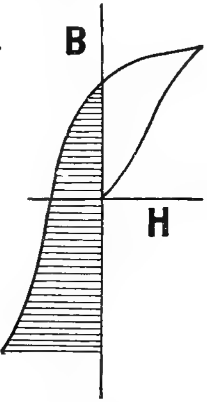

Fig. 70

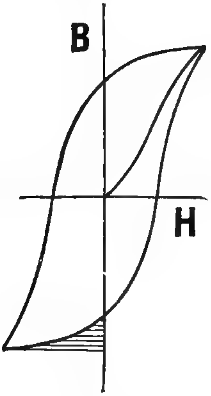

Fig. 71

It was found by Steinmetz that the hysteresis loss per cubic centimetre per cycle is approximately proportional to the 1·6th power of the maximum induction.

$$\frac{W}{V}=\eta \,.\, B_{\max}^{1{\cdot}6} \text{ ergs} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59).$$

The coefficient η is practically constant for a given specimen of iron, at least up to $B_{\max}=7000$. For the types of iron generally used it varies from 0·001 to 0·004. For the specimen of cast steel which we have considered, we found for $B_{\max}=18{,}250$ a value of $\frac{W}{V}=13{,}600$, whence

$$\eta=\frac{W}{V \,.\, B_{\max}^{1{\cdot}6}}=\frac{13{,}600}{18{,}250^{1{\cdot}6}}=0{\cdot}0028.$$

This value is, however, of little interest in the present case, as hysteresis need only be considered when dealing with alternating currents, and massive iron or steel is never used where it would be subjected to a rapidly alternating magnetisation.

Knowing η, Steinmetz's formula enables us to calculate the loss of power in watts due to hysteresis. If M is the mass of iron in kilogrammes, assuming a specific gravity of 7·7, we have for the volume in cubic centimetres,

$$V = \frac{M \cdot 1000}{7 \cdot 7}.$$

If the complete cycle is gone through f times per second the energy dissipated will be

$$\eta \cdot B_{\max}^{1 \cdot 6} \cdot \frac{M \cdot 1000}{7 \cdot 7} \cdot f \text{ ergs per second.}$$

Now, in Section 41 we shall see that

$$1 \text{ erg} = 1 \text{ centimetre-dyne} = \frac{1}{100} \cdot \frac{1}{981{,}000} \text{ metre-kilogramme,}$$

wherefore (see page 21)

$$1 \text{ erg per sec.} = \frac{1}{9 \cdot 81 \cdot 10^7} \text{ met-kg per sec.} = \frac{1}{10^7} \text{ watt.}$$

To obtain the loss in watts we have therefore to divide the loss in ergs per second by 10^7, hence

$$P = \frac{\eta \cdot B_{\max}^{1 \cdot 6} \cdot M \cdot f \cdot 10^{-4}}{7 \cdot 7} \text{ watts} \quad \dots\dots\dots\dots\dots\dots (60).$$

If, for example, $\eta = 0 \cdot 002$, $M = 100$ kg, $f = 50$ and $B_{\max} = 7000$, we have

$$P = \frac{0 \cdot 002 \cdot 7000^{1 \cdot 6} \cdot 100 \cdot 50 \cdot 10^{-4}}{7 \cdot 7} = 184 \text{ watts.}$$

Lately, however, Steinmetz's coefficient η has been used much less than formerly. When, as is now usual in practice, the hysteresis loss is determined by the wattmeter method, using alternating current, a new difficulty crops up, for the reading on the wattmeter includes not only the loss due to hysteresis but another loss due to the so-called Foucault currents or eddy currents. The total loss is now usually specified as being so many watts per kilogramme at a frequency of 50 cycles per second, the maximum induction being 10,000. This loss-coefficient, as it is called, lies between 3 and 4 for the iron generally used for stampings.

32. Dynamic Effect of Parallel Currents.

Two parallel wires are shown in Fig. 72, carrying currents in the same direction. Looking along the wires from A and B we shall see the feathers of the arrows, which are represented by crosses in the cross-sections of the wires. The lines of force due to the individual wires combine, as we have already seen in Section 27, to form longer lines encircling both wires. We have already likened the lines of force to stretched elastic bands, and have calculated, for one special case, the force with which they oppose any attempt to lengthen them. The lines will accordingly tend to contract in the case shown in Fig. 72 and thereby draw the conductors together. The attraction will be the greater because of the lines of force between the wires being in opposite directions, the resultant field vanishing at a point between the wires. We arrive therefore at the conclusion that: **Parallel wires attract each other when carrying currents in the same direction.**

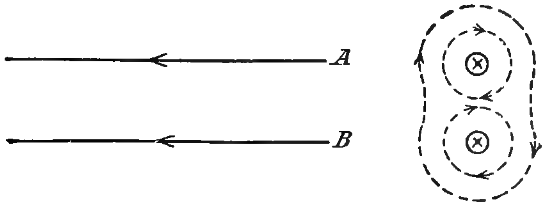

Fig. 72

If, however, the currents flow in opposite directions (Fig. 73), we shall see from *A* the point, and from *B* the feather of the arrows representing the direction of current. Drawing the lines of force due to each wire separately, we see that they are in the same direction between the wires and can therefore be added, whereas, in the space outside, they counteract each other. We have seen that lines of force in the same direction exert a lateral pressure, tending to mutually repel one another, and in the present case this will lead to the wires being mutually repelled. Hence, **parallel wires repel each other when carrying currents in opposite direction.**

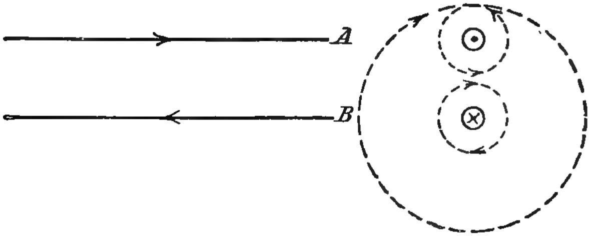

Fig. 73

In each case we observe that the tendency is to increase the total number of lines produced.

The application of this dynamic effect to the construction of wattmeters was mentioned in the previous section. It is also employed in the construction of ammeters and voltmeters of the so-called dynamometer type.

33. Induced Electromotive Force.

Imagine a conductor of length l cm perpendicular to the lines of force to be moved in a direction perpendicular both to the field of strength H, and to its own length. If the distance moved through in the time dt be ds, the velocity is

$$v = \frac{ds}{dt} \text{ cm per second.}$$

In Fig. 74 the lines of force pass normally through the paper from front to back, and are represented by points. The conductor l is moved along the metallic rails in the plane of the paper and it is found experimentally that an electromotive force is induced in it. This electromotive force only exists while l is in motion and vanishes directly l comes to rest. If the slide-rails are metallically connected, as shown in the figure, then the electromotive force produces a current, the direction of which is indicated by the dotted arrow, when l is moved downwards. During the time dt a certain amount of work is done by the electric current, which, from Section 10, is equal to the product of electromotive force, current, and time. We will now express the current in absolute units and the work also in absolute units, that is, in centimetre-dynes or ergs. We have therefore

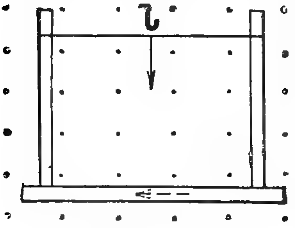

Fig. 74

$$dW = e \,.\, I' \,.\, dt \text{ ergs,}$$

where e is the electromotive force in some units not yet defined, but certainly not volts. This work dW is, naturally, only obtained by the expenditure of an equal amount of mechanical work. We know from page 54 that the current in l, and the magnetic field, exert a certain force f, which must be overcome when moving the conductor. It follows from Lenz's law that the induced current is always in such a direction that it opposes the motion inducing it. For the magnitude of this opposing force, we have from equation (32) on page 54

$$f = H \,.\, I' \,.\, l \text{ dynes.}$$

The mechanical work is the product of the force and the distance, so that

$$dW = H \,.\, I' \,.\, l \,.\, ds \text{ ergs.}$$

Equating the mechanical energy expended and the electrical energy developed, we have

$$e \,.\, I' \,.\, dt = H \,.\, I' \,.\, l \,.\, ds,$$

whence

$$e = H \,.\, l \,.\, \frac{ds}{dt} = H \,.\, l \,.\, v \text{ ....(61).}$$

Hence, the electromotive force induced in a conductor by its movement in a magnetic field is proportional to the field strength, to the length of the conductor, and to the velocity with which it moves. From equation (61) we can obtain the definition of the absolute unit of electromotive force. If we put $H = 1$, $l = 1$, and $v = 1$, then we get $e = 1$. Hence, **the absolute unit of electromotive force is induced when a conductor 1 cm long moves with a velocity of 1 cm per second perpendicular to its length and to the field of unit strength.**

We are now in a position to calculate in a simple manner the electromotive force of a machine. Suppose, for example, we require to find the maximum value of the E.M.F. of an alternator, that is, the E.M.F. at the moment when the armature wires are under the middle of the poles. Let the total number of armature wires be 400, each having an active induced length of 30 cm and let the field strength at the middle of the pole be 5000. Then, at a peripheral speed of 20 metres per second, i.e. 2000 cm per second, the electromotive force will be

$$e = 5000.30.400.2000 = 1200.10^8 \text{ absolute units.}$$

The absolute unit is far too small for practical purposes and a multiple of it is therefore used, viz. 10^8 absolute units, and is called **a volt.** This is the practical unit of pressure which was introduced in Section 2, but could not then be accurately defined. If e represents the electromotive force in volts, we have

$$e = H.l.v.10^{-8} \quad \dots\dots\dots\dots\dots\dots\dots\dots (62).$$

In the example calculated above the momentary electromotive force was therefore 1200 volts.

If, instead of being at right angles, the conductor is at an angle ϕ to the lines of force, it follows from equation (31) on page 54 that

$$e = H.l.v.\sin\phi.10^{-8} \quad \dots\dots\dots\dots\dots\dots\dots (63).$$

If, further, the motion is not perpendicular to the field, but at some angle, then v must be put equal to the component at right angles to the field.

The equations which we have thus established can be still further simplified. The product $l.ds$ in equation (61) represents the area swept out by the conductor, and the product $H.l.ds$ the total number of lines $d\Phi$ cut by the moving conductor. We write $d\Phi$ because we are dealing with an infinitesimal distance ds, the number of lines cut being therefore also infinitely small. For a single conductor, we get from equation (61)

$$e = \frac{d\Phi}{dt} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (64).$$

The electromotive force is thus obtained in absolute units by dividing the number of lines cut by the time taken to cut them, or, in other words, the electromotive force is equal to the rate at which the lines are cut.

If, in the time t, the flux Φ cuts T turns, it is equivalent to ΦT lines cutting a single wire, and the average value of the induced E.M.F. in volts is

$$E = \frac{\Phi . T}{t} . 10^{-8}.$$

The product of lines and turns linked by the lines is sometimes called the number of linkages. In the above example the linkages are $\Phi . T$. The electromotive force is therefore equal to the rate at which linkages are made or broken.

The absolute unit of electromotive force is induced when one line of force is cut per second. One volt is induced by the cutting of 10^8 lines per second.

Having determined the magnitude of the induced E.M.F. we must now consider its direction. This can be determined by means of a swimming rule, due to Faraday, which is as follows: **To a person swimming along the lines of force, that is, from north to south pole, and looking in**

the direction along which the conductor is moving, the induced electromotive force is from left to right. In Fig. 75 the lines of force are in the plane of the paper from right to left. They pass normally through a vertical plane in which the horizontal conductor moves downwards. If we swim from right to left with the face downwards our right hand will point in the direction of the dotted arrow, which is therefore the direction of the induced electromotive force. That this direction for the induced current is necessarily correct follows from a comparison of the two swimming rules of Ampère and Faraday respectively. If we swim with the current along the dotted arrow and look towards the north pole whence the lines of force proceed, the north pole would, according to Ampère's rule, tend to move towards our left, that is, downwards. If the pole is fixed the conductor tends to move upwards. Hence, the conductor tends to oppose the downward movement, as indeed it must, to conform to Lenz's law or to the principle of the conservation of energy. The direction found for the induced E.M.F. is therefore the correct one.

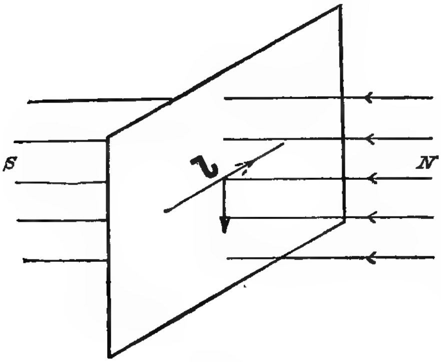

Fig. 75

We can apply this rule directly to the armatures of dynamos. In Fig. 76 is shown a hollow iron cylindrical or ring armature, with a closed winding of insulated copper wire, which is rotated in a clockwise direction between the poles *N* and *S* of an electromagnet. The lines of force leave the north pole and pass right and left through the armature to the south pole, leaving the air space inside the armature almost free from lines of force. It is therefore only the wires on the outside cylindrical surface that cut the lines of force. Applying Faraday's swimming rule we find that in the wires under the north pole the induced electromotive force is from front to back, while in those under the south pole it is from back to front. When current flows in the armature its direction will be as indicated by the arrows, viz. away from the south pole towards the north in the end connections facing us in the figure. We see that the current flows to the right both in the upper and lower half of the armature, meeting at a point halfway between the poles, where the currents from each half join and leave the armature together by means of the positive brush. After flowing through the external circuit the current enters the armature again by the negative brush, and divides between the upper and lower halves of the winding. For the sake of simplicity the brushes are shown rubbing on the external armature wires, from which the insulation would then have to be removed.

If the armature remains stationary while the poles rotate, the swimming rule must be applied to the relative motion. If, for example, in Fig. 77, which represents the usual arrangement of alternators, i.e. dynamos which generate alternating current, the north pole moves to the right, the effect is the same as if the pole were fixed and the armature wires moved to the left. If we imagine ourselves to be swimming with the field, that is, from *N* to *A* and looking in the direction of the relative motion of the conductors, that is, towards the left, the direction of the induced E.M.F. will be indicated by our right hand, that is, downwards through the paper. The current in the wires at *A* will therefore be represented by crosses, while in a similar manner we find that the current in the wires under the south pole is coming towards us and must be represented by dots.

Another simple rule for finding the direction of the induced E.M.F. is the right-hand rule. If the thumb and first two fingers of the right hand point mutually at right angles, like the three edges at a corner of a cube, and the first finger points along the lines of force, while the thumb points in the direction of motion, the second finger will give the direction of the induced electromotive force.

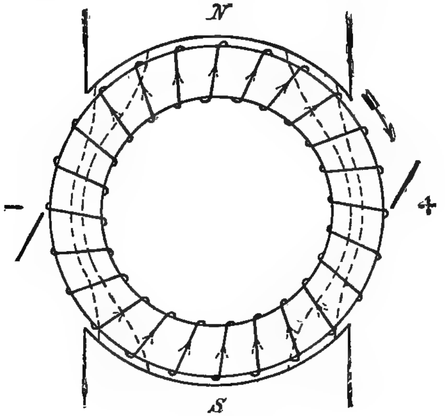

Fig. 76

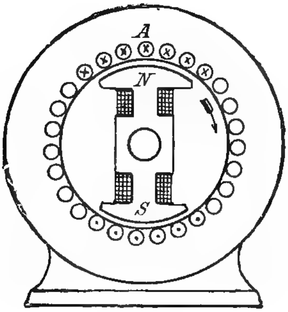

Fig. 77

We must be careful to notice, however, that, although we speak sometimes of the direction of the induced current, it is electromotive force and not current which is induced. In Fig. 74, for example, electromotive force was induced in *l* even when its ends were not metallically connected, and this electromotive force could be shown and measured on an electroscope. We must, for this reason, accustom ourselves to think of the electromotive force as the thing induced, its magnitude depending, as we have seen, on the field, on the length of conductor, and on its velocity, whereas the current is quite an arbitrary thing depending on the external resistance, that is, on the load taken by the customers of a station. The fact that the E.M.F. of a dynamo is not constant, but varies because of the effect of the current on the magnetic field, need not prevent us from seeing in the electromotive force the cause, and in the current, the effect.

In the foregoing, we have always looked upon the electromotive force as being caused by the cutting of lines of force. It is, however, often convenient to look upon the phenomenon from another point of view, according to which electromotive force is due to an increase or a decrease in the number of lines of force passing through a coil. In Fig. 74, for example, the number of lines passing through the coil, made up of the conductor l and the slide-rails, is decreased by the motion of l. A current is thereby induced which flows round the circuit in a clockwise direction. This current produces lines of force which pass down through the paper inside the circuit, and which are therefore in the same direction as those of the original field. **From this it follows that, if the number of lines through a circuit be decreased, a current is induced which tends to keep up the number of lines, whereas, if the lines be increased, a current is induced which tends to weaken the field. Lenz's law may thus be expressed in the following general form: The current induced by a change in the magnetic field is such as to oppose the change.**

For this reason, equation (64) must be re-written thus:

$$e = -\frac{d\Phi}{dt} \qquad \text{..............................(65).}$$

$d\Phi$ represents an increase in the number of lines. If this is positive, the electromotive force will be negative, that is, it will produce a current in such a direction as to weaken the field. If the field passes through more than one turn of the coil or circuit, Φ will represent the number of linkages and $d\Phi$ the change in the number of linkages.

34. The Laws of Mutual Induction.

So far, we have considered the electromotive force induced when a conductor cuts the lines of force produced by a magnet. We will now consider the effect of one conductor cutting the lines of force produced by another conductor. The induction in such a case is called mutual induction. In the primary wire I in Fig. 78 a current flows from right to left, producing lines of force as shown in the figure, viz. coming out from the paper at A, B and C. If the secondary conductor II is moved towards I, it will cut the lines of force at A, B, C, etc., and an electromotive force will be induced in it, the direction of which is found by the swimming rule or by the right-hand rule to be from left to right as shown by the dotted arrow.

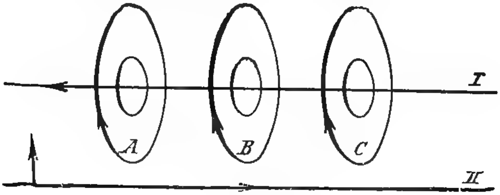

Fig. 78

Here again we can check our result by means of Lenz's law, which is only an application of the conservation of energy. If the ends of conductor II are externally joined by means of a conductor, a current will flow in the opposite direction to that in I. We have seen in Section 32 that the conductors will in this case repel one another, and therefore oppose the motion of *II* which is towards *I*. The work done in the latter case in forcibly bringing the wires nearer appears as heat in the secondary wire. We come therefore to the conclusion that, **on decreasing the distance between the wires, the induced secondary current is opposed to the primary, whereas on increasing the distance the secondary current is in the same direction as the primary current.**

It is, however, not necessary to move the wires mechanically nearer together, for an electromotive force is induced in the secondary whenever the primary current is increased or decreased, since its lines of force must thereby cut through the secondary conductor.

We are thus led to the most important conception, already used in Section 30, according to which the lines of force do not suddenly appear or disappear throughout a space, but, leaving the conductor, gradually spread out from it, to be followed by others, in much the same way as the ripples spread out from the point where a stone has been thrown into a lake. As the current in *I* increases the lines of force will be represented successively by Figs. 79 and 80, in which we see that they cut the secondary conductor in a downward direction. The result is the same as if the field were stationary and the conductor *II* were moved upwards towards *I*. From this it follows that the induced secondary current is opposed to the growing primary current, while it is in the same direction as the decreasing primary current.

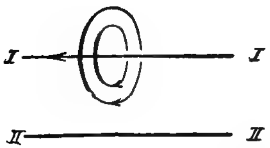

Fig. 79

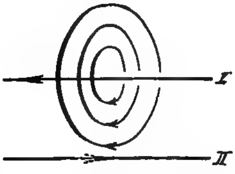

Fig. 80

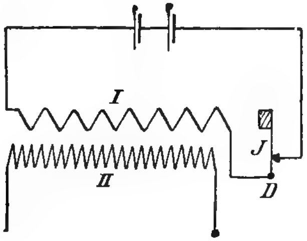

Fig. 81

It is possible to utilise this mutual induction for the transformation of direct into alternating current. Fig. 81 represents diagrammatically a so-called induction coil. It consists of a primary winding *I* supplied with direct current, one end of the winding being joined directly to the battery, while the other end is connected to the fulcrum *D* of the spring *J*. The circuit is completed by means of a contact point touching the spring and connected to the other pole of the battery. Directly the circuit is closed the coil acts as an electromagnet and attracts a piece of iron attached to the end of the spring, and thereby pulls the spring away from the contact point. The current being now broken, the coil loses its magnetism and allows the spring to go back and close the circuit once more. The current in the primary coil is thus intermittent. Wound over this primary coil is a secondary coil, which, for the sake of clearness, is shown beside the primary coil in Fig. 81. The primary lines of force grow and die away again in rapid succession and thus cut through the secondary windings first in one direction and then in the other, thereby inducing in them an electromotive force of rapidly alternating direction. If the secondary terminals are connected by means of a conductor, an alternating current will flow in the secondary circuit. This current increases from zero to a maximum, dies away again to zero and then reverses and goes through the same changes in the other direction. The current changes therefore both in magnitude and in direction. By increasing the number of secondary turns and making the primary current break as rapidly as possible, the secondary electromotive force can be made so large that the spark will jump across a considerable air-gap. In this way induction coils are now made to give a spark through the air over a yard long.

Of even greater importance from a technical point of view is the application of this principle of mutual induction to the construction of transformers which transform the high pressure alternating current transmitted over the line down to an alternating current of a low pressure suitable for ordinary use. We have seen that a very high pressure is essential for long-distance transmission in order to keep the losses small, or, if the losses are fixed, in order to keep the size of wire within practicable limits. To use such high pressures a great number of lamps would have to be put in series, in which case we would lose the independence of the separate lamps. In addition to this, the insulation for such high pressures would be extremely difficult, and pressures above 500 volts are very likely to give a fatal shock to a person touching the live wires. From the results of experiments it is known that the human body can, generally speaking, stand a current of about a hundredth of an ampere for a short time without any serious damage. If the resistance of the body, **when contact is lightly made with the finger-tips***, be estimated at 50,000 ohms, we see that the dangerous pressure begins at about 500 volts, thus,

$$E = I.R = \tfrac{1}{100}.50{,}000 = 500 \text{ volts.}$$

We see therefore that it is necessary to reduce the pressure to a value more suitable for the consumer, and this reduction of pressure is accomplished very simply by means of an alternating current transformer (Fig. 82). It consists of an iron core built up of insulated stampings, which carries a primary and a secondary winding. In our case the transformer is a step-down transformer, and the primary winding will therefore be connected to the high-pressure supply. The primary winding must, for this reason, consist of a great number of turns of fine wire. The lines of force

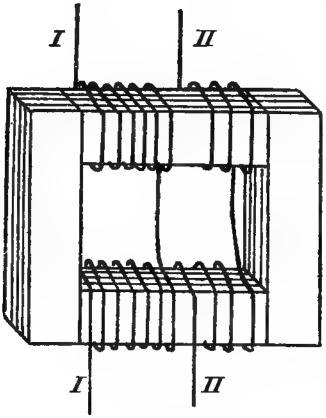

Fig. 82

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* If good contact be made with moistened hands a far lower pressure may prove fatal. Cases are on record where the voltage was below 100. It depends largely on the health of the individual.

produced by this winding will cut the secondary winding, which must consist of a few turns of thick wire, so that the electromotive force induced in it may be small. Now, the lines of force produced by the primary winding will, as they alternately grow and die away, cut the primary turns themselves, and induce in them an electromotive force proportional to the number of primary turns. We shall see later that this E.M.F. is almost exactly equal and opposite to that applied to its terminals. We see, therefore, that the primary and secondary pressures are proportional to the number of turns in the corresponding windings. The principles of the transformer are very easy to follow until current is taken from the secondary side, when the question becomes more complicated owing to the lines of force being now due to the combined action of both primary and secondary currents. We shall postpone a full investigation of the transformer until Chapter XI.

35. Self-Induction.

Towards the end of the previous section it was mentioned that the lines of force produced by a coil, in growing and dying away, cut, not only neighbouring wires, but also the turns of the coil itself. In Fig. 83 the dotted lines represent the growth of the lines of force due to a single turn of the coil. Each line of force emerges from the wire of the turn to which it is due, and gradually grows until it finally lies entirely in the iron ring, having cut, on its way, every turn in the coil. We will now calculate the magnitude of the electromotive force induced in the coil in this way.

Let T be the number of turns in the coil,
A, the cross-section of the iron,
and l, the length of the magnetic path in the ring.

We will assume that the permeability μ is constant, which is approximately true for low values of the induction, for which B is roughly proportional to H. If, now, the current of I' absolute units increases by an amount dI' in the time dt, the increase in the number of lines of force per sq. cm will be (see equation (42) on page 61)

$$dB = \frac{4\pi \, . \, T \, . \, dI' \, . \, \mu}{l},$$

and the increase in the total flux will be

$$d\Phi = \frac{4\pi \, . \, T \, . \, dI' \, . \, \mu}{l} \, . \, A.$$

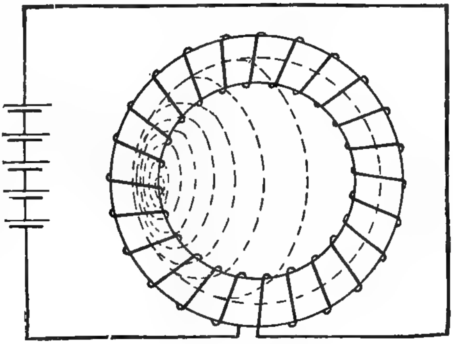

Fig. 83

These lines cut T turns in the time dt. **The electromotive force of self-induction thereby produced was seen in Section 33 to oppose whatever change of current produced it.** Since $d\Phi$ represents, as before, an increase in the number of lines, we must put a negative sign before the right-hand side of the E.M.F. equation. Substituting this value of $d\Phi$ in equation (65) on page 84,

$$e = -T \cdot \frac{d\Phi}{dt},$$

we have $$e = -\frac{4\pi \cdot T^2 \cdot \mu \cdot A}{l} \cdot \frac{dI'}{dt} \text{ absolute units} \quad \ldots\ldots\ldots\ldots\ldots (66).$$

If, now, the current is expressed in amperes, the right-hand side becomes 10 times too big, and must therefore be multiplied by 10^{-1}. To get the electromotive force in volts the right-hand side must be further multiplied by 10^{-8}. We get, then,

$$e = -\frac{4\pi \cdot T^2 \cdot \mu \cdot A}{l} \cdot 10^{-9} \cdot \frac{di}{dt} \text{ volts} \quad \ldots\ldots\ldots\ldots\ldots\ldots (67).$$

The expression $\frac{4\pi \cdot T^2 \cdot \mu \cdot A}{l} \cdot 10^{-9}$ gives us **the coefficient of self-induction** of the coil in practical units, that is, in henries. Denoting this by the letter L, we have

$$e = -L \cdot \frac{di}{dt} \text{ volts} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68).$$

Hence, if $e = 1$ when $\frac{di}{dt} = 1$, the self-induction must be 1 henry, i.e. if a coil has a self-induction of 1 henry, an E.M.F. of 1 volt will be induced by a steady change of current at the rate of 1 ampere per second.

The coefficient of self-induction can also be expressed thus:

$$L = \frac{0{\cdot}4\pi \cdot T \cdot \mu \cdot A}{l} \cdot T \cdot 10^{-8} \quad \ldots\ldots\ldots\ldots\ldots\ldots (68\,a).$$

Now $\frac{0{\cdot}4\pi \cdot T \cdot \mu \cdot A}{l}$ is the flux produced by a current of one ampere, and multiplying this by T gives the total number of linkages produced by one ampere. It follows from this that **the coefficient of self-induction in henries is the number of linkages produced by a current of 1 ampere, multiplied by 10^{-8}.**

The henry is thus the unit of self-induction and that coil has unit coefficient of self-induction for which the above expression works out to 1. The henry is a practical unit and can be introduced into calculations involving amperes and volts.

From equation (68) it is seen that the electromotive force of self-induction depends not only on the construction of the coil, but also on the rate at which the current is varying.

The expression $\frac{0{\cdot}4\pi \cdot \mu \cdot A}{l}$ represents the lines produced by one ampere-turn. This can be estimated in many cases, e.g. dynamo armatures, with sufficient accuracy. The flux produced by a current I, flowing in T turns, will be

$$\Phi = \frac{0{\cdot}4\pi \cdot \mu \cdot A}{l} \cdot T \cdot I.$$

If, now, the current in a coil is changed from $+i$ to $-i$ in a time t, due to the commutator segments to which it is connected passing under a brush, the number of linkages cut in the time t will be $2\Phi . T$. The average value of the electromotive force of self-induction will therefore be

$$E_{S\,\text{mean}} = \frac{2\Phi T}{t} . 10^{-8} = 2 . \frac{0{\cdot}4\pi . \mu . A . T^2 . I}{l . t} . 10^{-8},$$

or since

$$L = \frac{0{\cdot}4\pi . \mu . A}{l} . T^2 . 10^{-8},$$

$$E_{S\,\text{mean}} = \frac{2LI}{t} .$$

To find L we have simply to multiply the flux per ampere-turn, which, as we have already said, can often be estimated from past experience, by the square of the number of turns and by 10^{-8}.

As an example, we know that a wire embedded in an open slot in a dynamo armature produces a flux per ampere of about 6 lines per cm of length. Now, if the armature length be 30 cm, the embedded length of a coil will be 60 cm; and if the coil has 2 turns, the flux produced by a current of 1 ampere will be $2 \times 6 \times 60 = 720$. Hence, the linkages per ampere will be 1440, and $L = \frac{1440}{10^{-8}}$ henry, neglecting the ends of the coil. If the normal current in the armature conductors is 50 amperes, and the time of commutation $t = 0{\cdot}004$ second, we have

$$E_{S\,\text{mean}} = \frac{2LI}{t} = \frac{2 . 1440 \cdot 50}{0{\cdot}004 . 10^{-8}}$$
$$= 0{\cdot}36 \text{ volt.}$$

The effect of self-induction is seen in the gradual growth of current in a coil to which a certain fixed P.D. is applied. It also causes the current to die away slowly when the circuit is broken. It acts therefore as a sort of inertia, opposing any change in the current. Its effect is very considerable when the circuit of an electromagnet is suddenly broken, for the large number of lines in the magnet suddenly collapses, cutting all the turns of the magnet-winding at a high speed. Hence, in the equation

$$E = \frac{\Phi . T}{t} . 10^{-8} \text{ volts,}$$

which we established on page 81, both Φ and T are large, while t is very small. The electromotive force may be so great in such a case that the insulation of the coil is punctured. In any case, the spark produced on breaking such a circuit is very vicious owing to the electromotive force of self-induction trying to keep the current flowing, even across the air-gap of the switch.

For these reasons the field current of dynamos and motors is often greatly reduced by inserting resistance before breaking the circuit. In addition to this a resistance R (Fig. 84) is often put in parallel with the field winding R_m. When the circuit is broken, the current in the electromagnet, and the field, do not die away instantly, but gradually. As the field dies away it induces an E.M.F. of self-induction in the magnet-winding, in the same direction as the decreasing current. This E.M.F. drives the current round the circuit made up of R and R_m, and the current gradually falls to zero.

It is sometimes necessary to make a coil which shall have no self-induction. In one arrangement of the Wheatstone bridge the current is supplied from the secondary of an induction coil, while the galvanometer is replaced by a telephone receiver. The resistance to be measured must contain no self-induction, since the readings would then be dependent on the self-induction as well as on the resistance. This method can therefore only be applied to resistances having negligibly small coefficients of self-induction, e.g. glow-lamps and straight wires. The other resistances making up the bridge must naturally also be free from self-induction. This is arranged by doubling the wire on itself, and then winding the two parallel halves side by side as shown in Fig. 85. The magnetic effect of one wire is then neutralised by that of the neighbouring wire.

It was mentioned in Section 30 that, although no energy was expended in maintaining a magnetic field, yet a certain amount of work was done in building up the field, which remained stored in it until it collapsed, when the energy was restored. We are now in a better position to understand this point. We saw that the current in the ring-shaped coil was such as to oppose the movement of the gradually widening lines of force. We calculated the work done in overcoming this opposition, by multiplying the number of linkages by the current, and we asked ourselves the question as to how this energy was given to the coil. This question we can now answer. As the current increases, the electromotive force of self-induction e_s is opposed to it, and must be overcome by the applied terminal pressure V. If i is the current at any moment and R is the resistance of the coil, then, at that moment,

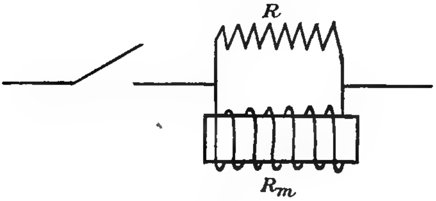

Fig. 84

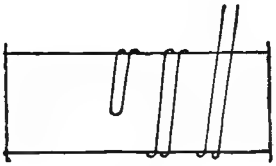

Fig. 85

$$V = e_s + i \, . \, R;$$

multiplying both sides by $i \, . \, dt$, we have

$$V \, . \, i \, . \, dt = e_s \, . \, i \, . \, dt + i^2 \, . \, R \, . \, dt.$$

Of these three terms, $V \, . \, i \, . \, dt$ is the total energy supplied to the coil in the time dt, while $i^2 \, . \, R \, . \, dt$ is the energy transformed into heat in overcoming ohmic resistance. The remainder $e_s \, . \, i \, . \, dt$ is used in forcing the lines of force through the inner wires of the ring winding. This energy is given back in the same form, viz. electrical energy, when the current decreases and the lines of force shrink up like a stretched elastic band. While the field is maintained, this energy is stored up in it, like the potential energy of a stretched spring. On breaking the circuit, the E.M.F. of self-induction keeps the current flowing just long enough to give back this store of energy. As a rule it is all transformed into heat, a large part of which is often evident in the spark. If the coil contains iron, only a part of the expended energy is given back, the remainder being converted into heat by the hysteresis of the iron.

We shall now consider more closely the manner in which the current changes from zero to its final steady value when a constant potential difference V is applied to a coil or circuit having a resistance R and inductance L. At any moment we have

$$V - L \, . \frac{di}{dt} = iR,$$

where i is the value of the current at the moment. Let $I = \frac{V}{R}$; then I is the steady value finally reached by the current, and the above formula may be written

$$\frac{R}{L} \, . \, dt = \frac{di}{\frac{V}{R} - i} = \frac{di}{I - i} \, .$$

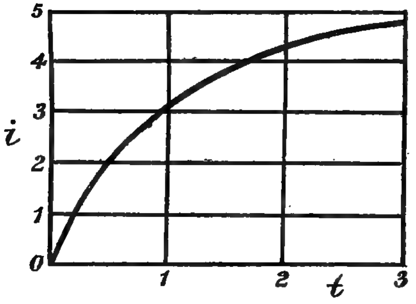

Fig. 86

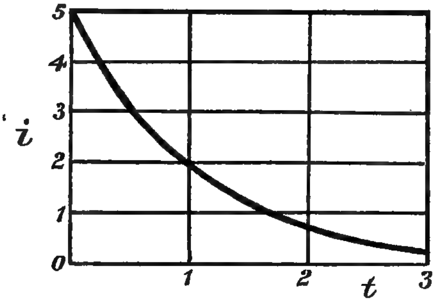

Fig. 87

Integrating, we have

$$-\frac{R}{L} \, . \, t = \log_\epsilon (I - i) + \log_\epsilon k = \log_\epsilon [k \, (I - i)],$$

where k is some constant, depending on the initial conditions. In our case we assume that when $t = 0$, $i = 0$ which makes $k = \frac{1}{I}$, and

$$\epsilon^{-R\frac{t}{L}} = k \, (I - i) = \frac{I - i}{I} = 1 - \frac{i}{I},$$

where $\epsilon = 2{\cdot}71828$, the base of the natural system of logarithms. Hence the value of the current at any moment t after closing the switch is given by the equation

$$i = I \, (1 - \epsilon^{-\frac{R}{L} . t}).$$

If the time be plotted horizontally and the current vertically we obtain the curve shown in Fig. 86. When the time which has elapsed is equal to $\frac{L}{R}$, we have $i = I(1 - \epsilon^{-1}) = 0{\cdot}63I$. Thus the ratio $\frac{L}{R}$ for any coil is really a time, viz. the time required for the current to reach 63 per cent. of its final value; this ratio is therefore called the time constant of the coil or circuit.

If a coil, in which a current is flowing, be suddenly short-circuited, the current will die away according to a very similar law. We have simply to put $V = 0$, for there is now no P.D. between the terminals of the coil. Our formula is therefore

$$\frac{R}{L} . dt = -\frac{di}{i}.$$

By integrating we have

$$-\frac{R}{L} . t = \log_\epsilon (k . i)$$

and

$$\epsilon^{-\frac{R}{L}.t} = k . i.$$

Now, when $t = 0$; $i = I$: which, as before, gives $k = \frac{1}{I}$,

so that

$$i = I . \epsilon^{-\frac{R}{L}.t}.$$

This result is shown graphically in Fig. 87, in which, as also in Fig. 86, $I = 5$ and $\frac{L}{R} = 1$. When $t = \frac{L}{R}$, $i = \frac{I}{\epsilon} = 0{\cdot}37I$. In this case, therefore, the time constant is the interval which elapses before the current falls to 37 per cent. of its initial value.

36. Eddy Currents.

The name of eddy or Foucault currents is given to those currents which do not flow along a particular prescribed path, but, following the line of least resistance, flow where they will in the material. Such currents are induced, for example, when a massive conductor is cut by a magnetic field, but their exact path and intensity cannot generally be accurately determined. We know, however, that they flow, in a general way, at right angles both to the magnetic field and to the direction of motion. If, for example, a turn of rectangular copper strip be wound on a massive iron cylinder, as shown in Fig. 88, and the cylinder be rotated so that the upper part in the figure comes out from the paper, the direction of the induced E.M.F. will be as shown by the arrow. An equal E.M.F. will, however, be induced in the iron below the copper strip. In consequence of the large cross-section and therefore the small resistance of the iron, the current produced by this latter E.M.F. will be very considerable. The heat thus generated will cause a prohibitive temperature rise in the iron, besides representing a great waste of power. The iron in Fig. 88 may be looked upon as a short-circuited dynamo and work must be done in turning it, because the induced currents oppose the motion to which they are due.

The energy consumed by eddy currents can be demonstrated in a very simple manner by means of a pendulum with a bob of sheet copper, so arranged that the latter can swing freely through the narrow gap between the poles of an electromagnet. On closing the exciting circuit of the magnet, the pendulum is brought rapidly to rest, as if the bob were immersed in a viscous fluid. The energy of the pendulum is used up in overcoming the opposing forces due to the induced eddy currents in the copper sheet, and the energy is converted into heat in accordance with Joule's law. It is evident that eddy currents represent, under all circumstances, a loss of energy, and to avoid this loss the armatures of machines are always built up of sheet iron stampings or discs, as shown in the lower half of Fig. 88. The plates are insulated from one another by means of varnish or thin paper, or sometimes by nothing more than the oxide on the surface. The iron must evidently be laminated in a plane perpendicular to the armature conductors, so as to break up the path along which the eddy currents would flow.

When the armature winding is placed in open slots it is often necessary to laminate the pole-shoes, as the lines of force go mainly through the teeth, and cross to the pole-face in tufts as shown in Fig. 152. The flux-density in the pole-shoe is greater opposite a tooth than opposite a slot, so that, as the armature rotates, each point of the pole-face is subjected to a rapidly varying field. In this way, electromotive forces are induced in the pole-shoe, necessitating its lamination.

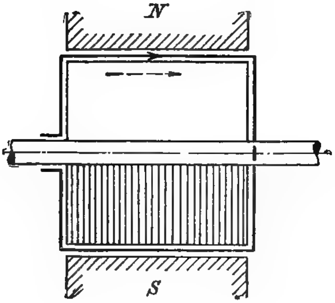

Fig. 88

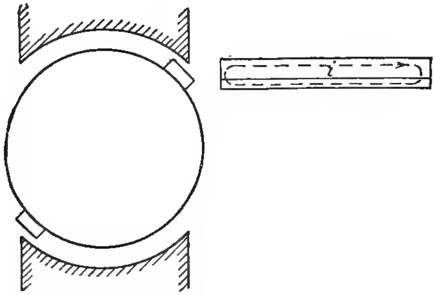

Fig. 89

Eddy currents can also occur in the copper bars or wires of the armature winding itself; especially if these be very massive, as shown, in a very exaggerated way, in Fig. 89, which may be regarded as a section through Fig. 88. In the position shown, one edge of the bar is moving in a strong field while the other edge is outside the pole and therefore in a weak field. The electromotive force induced in the part under the pole will drive current principally through the end connections, and thus through the external circuit, but, at the same time, it will drive current back along the other edge as shown by the dotted lines in the small side-view of the conductor. These eddy currents are diminished by rounding off the edges of the poles or by making the air-gap larger towards the edges than at the centre, thus causing a more gradual change in the field strength. They are, however, almost entirely avoided by using toothed armatures, in which the lines pass almost entirely through the teeth, as shown in Fig. 152. The field in the slot is therefore very weak and the E.M.F. induced by the motion of the conductor through this field is small. The main part of the E.M.F.

induced in the armature coils is due to the change in the flux passing through them by way of the teeth.

Although, in all the cases above mentioned, eddy currents were positively harmful, there are cases in which they are turned to account with great advantage. In one form of tramcar brake, an iron disc, keyed to the axle, revolves between the poles of an electromagnet. When the latter is excited the eddy currents induced in the disc oppose its motion, and the kinetic energy of the moving car is converted into heat in the disc.

Eddy currents are also used for damping galvanometers, the needles of which move in a hollow block of copper. The instrument is thus made aperiodic, that is, it takes up its new position without swinging about. If the instrument is of the moving coil type, as so many mirror galvanometers are, the coil is rapidly brought to rest by short-circuiting it. The currents induced in it, as it swings between the poles of the permanent magnet, oppose the motion producing them; these currents are not eddy currents, however, as they flow in the windings of the coil.

CHAPTER V

37. The units of length, mass, and time in the absolute system.—38. Dimensions and units of velocity, acceleration, and force.—39. Dimensions and units of pole-strength, field-strength, and magnetic flux.—40. Dimensions and units of pressure, current strength, quantity of electricity, and resistance.—41. Dimensions and units of energy, heat, and power.—42. Dimensions and units of the coefficient of self-induction, and of capacity.

37. The Units of Length, Mass, and Time in the Absolute System.

The unit of length in the absolute system is the centimetre. This is defined as the hundredth part of the standard metre preserved in Paris. The standard metre is approximately a ten-millionth part of the length of a quadrant of the earth, taken through Paris.

The absolute unit of mass is the gramme. This is defined as the mass of a cubic centimetre of water at 4° C., or better, as the mass which weighs as much as 1 cubic centimetre of water. Hence, the weight of a body in grammes gives its mass directly in absolute units. The student must be careful to distinguish between mass and weight. The mass of a cubic centimetre of water represents a certain quantity which is the same at any point on the earth's surface, or in space. Its weight, on the other hand, will vary from point to point, and will be appreciably different at the poles to what it is on the equator. The weight of a given mass is the force with which it is attracted by the earth. Since this force of gravity is found experimentally to be strictly proportional to the mass, for any given point on the earth's surface, the mass of a body is generally determined from its weight and expressed in the same units. We shall represent masses in grammes by the letter M.

The second is chosen as the absolute unit of time. It is defined as the 86,400th part of the mean solar day. A time expressed in seconds is represented by the letter t.

Most of the other quantities with which we shall be concerned, such as velocity, energy, etc., can be expressed as functions, or dimensions, of length, mass, and time. As the expression "dimension," used in this way, is apt to prove a difficulty to beginners, we will make its meaning plainer by means of a few simple examples. Area or surface is the second dimension of length, while volume is the third dimension of length. Area and volume are thus dimensions or functions of length, that is, they are quantities which can be calculated from measurements of length. Velocity is similarly a function or dimension of length and time, since its magnitude can be determined by dividing a length by a time. In this way, the majority of our physical conceptions are capable of expression in terms of length (L), mass

(M), and time (T). The dimensions of area, for example, are L^2, of volume L^3, and of velocity LT^{-1}.

It follows that the units for these various quantities in the absolute system of units can no longer be chosen at random, but are necessarily determined from the three fundamental units, viz. the centimetre, the gramme, and the second. Thus, the unit of area is necessarily the square centimetre, that of volume the cubic centimetre, while the unit of velocity must be one centimetre per second. Units derived in this way from these three fundamental units are known as C.G.S. units from the initial letters of the three units chosen. We proceed now to determine the dimensions of the various quantities, together with their absolute units, and to compare the latter with the practical units.

38. Dimensions and Units of Velocity, Acceleration, and Force.

(*a*) VELOCITY.

Velocity is defined as the ratio of the distance travelled to the time taken, or as the distance moved through in unit time. We have therefore

$$\text{Dimensions of velocity}: \frac{L}{T} = L \,.\, T^{-1}.$$

If v be the velocity in absolute units, we have

$$v = \frac{l\ \text{cm}}{t\ \text{sec.}} \quad \text{.................................(69).}$$

The absolute unit of velocity is one centimetre per second, or 1 cm/sec. The word "per" signifies as much as "divided by," so that the term "centimetre per second" indicates at once the dimensions of velocity and the method of calculating the velocity from the length and the time.

Example: The armature of an alternator has a diameter of 1·6 metres and runs at 300 revolutions per minute. What is its peripheral speed in absolute units?

We have circumference of armature $= 1{\cdot}6 \,.\, \pi = 5$ metres,
distance per minute $= 5 \,.\, 300 = 1500$ metres,

or, expressing length in centimetres and time in seconds,

$$l = 1500 \,.\, 100 = 150{,}000 \text{ cm.}$$
$$t = 60 \text{ seconds.}$$

The peripheral speed is therefore

$$v = \frac{l}{t} = \frac{150{,}000}{60} = 2500 \frac{\text{cm}}{\text{sec.}} = 2500 \text{ C.G.S units.}$$

(*b*) ACCELERATION.

Acceleration is the increase of velocity in unit time, or the ratio of increase of velocity to time:

$$\text{Acceleration} = \frac{\text{Increase of velocity}}{\text{time}}.$$

The dimensions of an increase in velocity are the same as those of velocity, viz. LT^{-1}. We get therefore

$$\text{Dimensions of acceleration}: \frac{L\,.\,T^{-1}}{T} = L\,.\,T^{-2}.$$

If v_1 be the initial velocity and v_2 the final velocity, we have for the acceleration, if uniform, in absolute units,

$$a = \frac{v_2 - v_1}{t} \text{ C.G.S. units } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70).$$

A body has unit acceleration when its speed increases in one second by the absolute unit, i.e. one centimetre per sec.

Example: A body starts from rest and attains in 3 seconds a speed of 29·43 metres per second. What is its acceleration in absolute units?

We have $\quad v_1 = 0, \qquad v_2 = 29{\cdot}43\,.\,100\,\frac{\text{cm}}{\text{sec.}}, \qquad t = 3.$

Wherefore $\quad a = \frac{v_2 - v_1}{t} = \frac{2943}{3} = 981\,\frac{\text{cm}}{\text{sec.}^2} = 981$ C.G.S. units.

We see, at once, that this is an example of a body falling freely under the action of gravity. The result is obtained in absolute units by putting each individual quantity in the calculation in absolute units. It is evident, moreover, that acceleration, whether due to gravity or not, can be expressed neither in cm nor in cm per second, but only in cm per sec. per sec. An acceleration can no more be expressed in units of velocity than can an area in centimetres. Similarly a horse-power is not equal to 76 metre-kilogrammes or 550 foot-pounds, but to 550 foot-pounds per second. Carelessness in the expression of units leads to great confusion. Only those quantities can be compared which have the same dimensions, and an error in calculation has often been discovered through a want of agreement between the dimensions of the two sides of an equation. It is therefore very important to master the principles of dimensions, and to gain that confidence in their application, which is the result of frequent use.

(*c*) FORCE.

Force is defined in mechanics as the product of mass and acceleration:

$$\text{Force} = \text{Mass}\,.\,\text{Acceleration}.$$

We have therefore

$$\text{Dimensions of force}: M\,.\,L\,.\,T^{-2} = L\,.\,M\,.\,T^{-2}\text{*}.$$

---

* The mechanical idea of force differs from the astronomical. In its simplest form the law of gravitation is expressed by the equation

$$f = \frac{m\,.\,m}{r^2},$$

where f is the force, m the mass, and r the distance. The dimensions of force in astronomical units are therefore $M^2.L^{-2}$. Dimensions of force in the two different systems cannot be equated, resembling in this respect the dimensions of quantity of electricity in absolute and in electrostatic units. The difference in the dimensions of force proves that the dimensions of a quantity are not based *a priori* on the nature of the quantity or, at any rate, only represent one point of view.

If f be the force, and M the mass in absolute units, then

$$f = M . a \quad \text{.................................(71).}$$

Hence, the absolute unit of force causes the absolute unit of mass, that is a cubic centimetre of water, to move with an acceleration of 1 cm/sec.^{2} This force is called a dyne.

Example: With what force does the earth attract a 1 kilogramme weight?

We have mass = 1 kilogramme = 1000 grammes,

acceleration = 9·81 met./sec.^{2} = 981 cm/second^2.

Wherefore $M = 1000, \qquad a = 981,$

and $f = 1000 . 981 = 981{,}000$ C.G.S. units or dynes.

Thus, the kilogramme weight, which is the practical metric unit of force, is equal to 981,000 dynes. If we represent a weight or force of 1 kilogramme by kg* we get

1 kg* = 981,000 dynes.

$$1 \text{ dyne} = \frac{1}{981{,}000} \text{ kg*} = 1{\cdot}02 \text{ milligrammes*} \quad \text{............(72).}$$

Similarly 1 pound* = 453·6 grammes* = 445,000 dynes.

39. Dimensions and Units of Pole-strength, Field-strength, and Magnetic Flux.

(*a*) POLE-STRENGTH.

The strength of a pole, or the quantity of "free magnetism" on a pole, is measured by the force exerted by the pole under certain conditions. According to Coulomb's law the force exerted by one pole on another is given by the equation

$$f = \frac{m_1 . m_2}{r^2},$$

where m_1 and m_2 are the strengths of the two poles and r is the distance between them. If we are not concerned with numerical values, we can omit the indices and write

$$f = \frac{m . m}{r^2},$$

or

$$m = r\sqrt{f}.$$

Hence, to find the dimensions of pole-strength we must take the square root of the dimensions of force and multiply the result by the dimensions of the distance r, i.e. by a length. We have therefore

Dimensions of pole strength: $L . \sqrt{L . M . T^{-2}} = L^{\frac{3}{2}} . M^{\frac{1}{2}} . T^{-1}$.

From Coulomb's law it follows that that pole has unit strength which exerts a force of 1 dyne on a similar pole placed 1 cm away from it. It has been proposed to call this unit a Weber, but the proposal has not been generally adopted.

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* See note, p. 44.

(*b*) FIELD-STRENGTH.

The force exerted on a pole by a magnetic field is, according to equation (26) on page 45, proportional to the strength of the pole and to the strength of the field, or

$$f = m . H.$$

Hence

$$H = \frac{f}{m}.$$

The student should accustom himself to reading equations, such as this, in words. The field-strength H is the force per unit pole, or the force exerted on a pole of strength 1. Its dimensions are therefore obtained by dividing the dimensions of force by those of pole-strength. We have therefore

$$\text{Dimensions of field-strength: } \frac{L . M . T^{-2}}{L^{\frac{3}{2}} . M^{\frac{1}{2}} . T^{-1}} = L^{-\frac{1}{2}} . M^{\frac{1}{2}} . T^{-1}.$$

That field has unit strength which exerts a force of 1 dyne on a pole of strength 1.

Example: The force on a north pole of 100 absolute units in a magnetic field is found to be 20 dynes. What is the strength of the field?

We have

$$H = \frac{f}{m} = \frac{20}{100} = 0{\cdot}2 \text{ C.G.S. unit.}$$

(*c*) MAGNETIC FLUX.

We have seen in Section 21 that the number of lines of force per square centimetre is equal to the strength of the field. The total number of lines, or the magnetic flux, is therefore obtained by multiplying field-strength by area:

$$\Phi = H . A.$$

The dimensions of magnetic flux can thus be found by multiplying the dimensions of field-strength by those of area. We have therefore

$$\text{Dimensions of magnetic flux: } L^2 . L^{-\frac{1}{2}} . M^{\frac{1}{2}} . T^{-1} = L^{\frac{3}{2}} . M^{\frac{1}{2}} . T^{-1}.$$

It is rather striking that the dimensions of magnetic flux should be the same as those of pole-strength. This is, however, in accordance with equation (28) on page 47, that is, with the fact that the total magnetic flux from a pole is found by multiplying the strength of the pole by 4π:

$$\Phi = 4\pi m.$$

If the values of pole-strength and magnetic flux only differ by a constant factor 4π, their dimensions must be the same. A line of force can be defined as the 4πth part of the flux emanating from a unit pole.

40. Dimensions and Units of Electromotive Force, Current, Quantity of Electricity, and Resistance.

(*a*) ELECTROMOTIVE FORCE.

Electromotive force can be defined as the number of lines, or the magnetic flux, cut per second. We have then

$$\text{Dimensions of electromotive force: } L^{\frac{3}{2}} . M^{\frac{1}{2}} . T^{-2}.$$

The absolute unit of electromotive force is induced when one line of force is cut per second. The volt is 10^8 absolute units.

$$1 \text{ volt} = 10^8 \text{ lines cut per second} = 10^8 \text{ C.G.S. units.}$$

If e be the electromotive force in volts induced in a coil of T turns by the introduction of $d\Phi$ lines in the time dt, we have

$$e = -\frac{d\Phi . T}{dt} . 10^{-8},$$

or, from equation (62) on page 81,

$$e = H.l.v.10^{-8}.$$

(b) CURRENT.

The force f experienced by a conductor of length l, which carries a current I' in a field H, is given by the equation

$$f = H.I'.l,$$

wherefore

$$I' = \frac{f}{H.l}.$$

The dimensions of current can therefore be found from those of force, field-strength, and length, thus

$$\text{Dimensions of current}: \frac{L.M.T^{-2}}{L^{-\frac{1}{2}}.M^{\frac{1}{2}}.T^{-1}.L} = L^{\frac{1}{2}}.M^{\frac{1}{2}}.T^{-1}.$$

That current has unit strength, which experiences a force of 1 dyne per cm of length in a field of unit strength. The tenth part of this unit has been chosen as the practical unit and called an ampere, wherefore

$$1 \text{ ampere} = \tfrac{1}{10} \text{ C.G.S. unit.}$$

The number of amperes is therefore always 10 times the number of absolute units in the same current.

The legal definition of the ampere is based on the fact that it deposits 1·118 mg of silver per second from a solution of silver nitrate.

(c) QUANTITY OF ELECTRICITY.

Since the current is defined as the quantity of electricity that flows through any cross-section of the conductor in one second, quantity of electricity must be the product of current and time. Hence, we have

$$\text{Dimensions of quantity of electricity}: L^{\frac{1}{2}}.M^{\frac{1}{2}}.T^{-1}.T = L^{\frac{1}{2}}.M^{\frac{1}{2}}.$$

An absolute unit of electricity flows in one second through any cross-section of a conductor carrying unit current. The tenth part of this quantity is the practical unit corresponding to the ampere, and is called a coulomb.

$$1 \text{ coulomb} = \tfrac{1}{10} \text{ C.G.S. unit of quantity.}$$

If Q represents the quantity of electricity in coulombs, then

$$Q = I.t.$$

Other units of quantity of electricity are as follows:

$$1 \text{ microcoulomb} = \frac{1}{10^6} \text{ coulomb} = 10^{-6} \text{ coulomb},$$

$$1 \text{ ampere-hour} = 3600 \text{ coulombs.}$$

The dimensions of quantity of electricity in the electrostatic system are not the same as in the above electromagnetic system. According to Coulomb's law for electricity, the force between two charges, or quantities of electricity, is given by the formula

$$f = \frac{m_1 . m_2}{r^2},$$

where m_1 and m_2 are the quantities expressed in electrostatic units. In this system, the unit is that quantity which exerts a force of 1 dyne on a similar quantity at a distance of 1 cm. This unit is very small, a coulomb being $3 . 10^9$ times as large, while an absolute electromagnetic unit of quantity is equal to $3 . 10^{10}$ electrostatic units. The dimensions of quantity of electricity in the electrostatic system are the same as those of pole-strength, since both are derived from the formula, $m = r\sqrt{f}$.

The units of quantity of electricity in the electrostatic and electromagnetic systems do not differ merely by the factor $3 . 10^{10}$, but also in their dimensions by the factor $L . T^{-1}$, i.e. by the dimensions of velocity.

The importance of this is shown by an experiment due to Rowland. If a ring with a static charge of 1 electrostatic unit per centimetre of periphery is rotated with a velocity of $3 . 10^{10} \frac{\text{cm}}{\text{sec.}} = 300{,}000 \frac{\text{km}}{\text{sec.}}$, the magnetic effect is exactly the same as if the ring were at rest, but carried a current of 1 absolute electromagnetic unit. It is worthy of note that 300,000 km per second is the velocity of light and of the electromagnetic waves used in wireless telegraphy.

(*d*) RESISTANCE.

Resistance is defined as the ratio of electromotive force to current,

$$R = \frac{E}{I}.$$

From this we have

$$\text{Dimensions of resistance}: \frac{L^{\frac{3}{2}} . M^{\frac{1}{2}} . T^{-2}}{L^{\frac{1}{2}} . M^{\frac{1}{2}} . T^{-1}} = L . T^{-1}.$$

Resistance has therefore the same dimensions as velocity, and, strange though it may seem, the absolute unit of resistance is equal to 1 centimetre per second. This is the resistance through which the very small absolute unit of electromotive force sends the relatively large absolute unit of current. The practical unit of resistance, the ohm, is 10^9 times as large as this absolute unit. This is the resistance through which 1 volt sends a current of 1 ampere.

$$1 \text{ ohm} = 10^9 \text{ C.G.S. units of resistance.}$$

Legally, the ohm is the resistance of a thread of mercury 1 sq. mm in cross-section and 106·3 cm long.

The legal volt is that pressure which produces a current of 1 ampere in a resistance of 1 ohm, or inversely, a volt is the potential difference between the ends of 1 ohm resistance when a current of 1 ampere is passed through it.

41. Dimensions and Units of Energy, Heat, and Power.

(*a*) ENERGY AND WORK.

Mechanical work is defined as the product of force and distance. Hence

$$\text{Dimensions of work: } L^2 . M . T^{-2}.$$

The absolute unit of work is done when a force of 1 dyne is overcome through a distance of 1 cm. This unit of work or energy is called a centimetre-dyne or an erg. If W represents the work done, in ergs, in overcoming a force of f dynes through a distance of s centimetres, we have

$$W = f . s \qquad (73).$$

Example: How many ergs of work are done in lifting a kilogramme a height of 1 metre?

We have

$$1 \text{ kg}^* = 981{,}000 \text{ dynes},$$
$$1 \text{ metre} = 100 \text{ cm}.$$
$$f = 981{,}000, \qquad s = 100.$$
$$W = f . s = 981{,}000 . 100 = 9{\cdot}81 . 10^7 \text{ ergs}.$$

Hence, the practical unit of work in the metric system, viz. a metre-kilogramme, is equal to $9{\cdot}81 . 10^7$ ergs. Similarly the British unit, a foot-pound, is equal to $1{\cdot}36 . 10^7$ ergs.

Since we saw in Section 10 that the product $E . I . t$ represents the electrical work, its dimensions must be the same as those found above for mechanical work. By multiplying together the dimensions of electromotive force, current and time, we find that this is so. This absolute unit of electrical energy is also, of course, the erg.

Now, a volt is 10^8 C.G.S. units, and an ampere is 10^{-1} C.G.S. units, from which it follows that the joule, which is the product of

$$1 \text{ volt} . 1 \text{ ampere} . 1 \text{ second}$$

must be equal to $10^8 . 10^{-1} = 10^7$ absolute units of energy, i.e. 10^7 ergs.

$$1 \text{ joule} = 10^7 \text{ ergs}.$$

We saw above that

$$1 \text{ metre-kg} = 9{\cdot}81 . 10^7 \text{ ergs},$$

and

$$1 \text{ foot-pound} = 1{\cdot}36 . 10^7 \text{ ergs}.$$

Therefore

$$1 \text{ metre-kilog.} = 9{\cdot}81 \text{ joules},$$

and

$$1 \text{ foot-pound} = 1{\cdot}36 \text{ joules}.$$

(*b*) HEAT.

Since heat is a form of energy its dimensions are necessarily the same as those of energy. Seeing, however, that the scale of the thermometer has been chosen quite arbitrarily, it is but natural that a numerical coefficient has to be introduced into the expression of Joule's law, whereas, by a suitable choice of units, such coefficients have been eliminated from the laws of Ohm, Coulomb, and Laplace. In choosing as the unit of heat the quantity of heat necessary to raise the temperature of 1 gramme of water through 1° C., we are taking a step quite outside the absolute system of units. This unit of heat is called the gramme-calorie. Its relation to the unit of mechanical work is what is known as the mechanical equivalent of heat, and was first determined by Joule.

$$1 \text{ gramme-calorie} = 0{\cdot}427 \text{ metre-kilog.}$$

Similarly, $$1 \text{ British thermal unit (F.}^\circ) = 778 \text{ foot-pounds.}$$

Since 1 metre-kilog. = 9·81 joules, we have

$$1 \text{ gramme-calorie} = 0{\cdot}427 \,.\, 9{\cdot}81 = \frac{1}{0{\cdot}24} \text{ joules}$$

or $$1 \text{ joule} = 0{\cdot}24 \text{ gramme-calorie.}$$

This is only another way of expressing Joule's law, according to which the quantity of heat in gramme-calories is given by the equation:

$$Q_h = 0{\cdot}24 E \,.\, I \,.\, t.$$

(*c*) POWER.

Power is the rate of doing work, i.e. the work done in a given time.

$$\text{POWER} = \frac{\text{work}}{\text{time}}.$$

We have therefore

$$\text{Dimensions of power: } L^2 . M . T^{-3}.$$

The absolute unit of power is 1 erg per second. This is an exceedingly small power, and a practical unit 10^7 times as large is used; it is called a watt.

$$1 \text{ watt} = 10^7 \text{ ergs per second} = 1 \text{ joule per second.}$$

Since the electrical work in joules is equal to the product $E . I . t$, the electrical power in joules per second, or in watts, must be equal to $E . I$. If P represents the power in watts, we have

$$P = E . I.$$

Example: How many watts are equivalent to one horse-power?

We know that

$$1 \text{ horse-power} = 550 \text{ foot-pounds per sec.,}$$

and $$1 \text{ foot-pound per second} = 1{\cdot}36 \text{ joules per second;}$$

therefore $$1 \text{ horse-power} = 550 \,.\, 1{\cdot}36 = 746 \text{ watts.}$$

42. Dimensions and Units of the Coefficient of Self-Induction and of Capacity.

(*a*) THE COEFFICIENT OF SELF-INDUCTION.

In Section 35 we saw that the coefficient of self-induction in absolute units was equal to

$$\frac{4\pi . T^2 . \mu . A}{l}.$$

As 4π, T, and μ are mere numbers, we have

$$\text{Dimensions of coefficient of self-induction: } \frac{L^2}{L} = L.$$

Hence, the dimension is simply a length, and the unit is a centimetre. If, however, in calculating the self-induction, the pressure be expressed in volts and the current in amperes, we get from equation (67), on page 88,

$$e = -\frac{4\pi . T^2 . \mu . A}{l} . 10^{-9} . \frac{di}{dt} = -L . \frac{di}{dt},$$

where $$L = \frac{4\pi . T^2 . \mu . A}{l} . 10^{-9} \quad \ldots\ldots\ldots\ldots\ldots\ldots (74).$$

This gives the coefficient of self-induction in practical units or henries. Since the number of henries is 10^9 times as small as the number expressing the same self-induction in absolute units, the henry must be 10^9 times as large as the absolute unit.

$$1 \text{ henry} = 10^9 \text{ C.G.S. units} = 10^9 \text{ cm}.$$

Now, 10^9 cm or 10,000 kilometres is the length of a quadrant of the earth. For this reason the practical unit of self-induction was formerly called a quadrant. According to equation (74) that coil has a self-induction of 1 henry for which the value of $\frac{4\pi . T^2 . \mu . A}{l} . 10^{-9}$ is equal to 1, or in which an electromotive force of 1 volt is induced when the current increases steadily at the rate of 1 ampere per second.

(*b*) CAPACITY.

A condenser consists of two metal plates very near together, but yet separated by a thin sheet of insulating material known as the dielectric. When the two plates are connected up to the terminals of any source of electricity, the condenser becomes charged by positive electricity flowing into one plate and negative into the other. This flow of electricity continues until the back pressure of the condenser exactly balances the terminal pressure of the supply. The quantity of electricity passing into, say, the positive plate of the condenser is greater, the greater the pressure applied, and also the greater the capacity of the condenser. This capacity is proportional to the area of the plates and inversely proportional to the distance between them. It is also dependent on the nature of the dielectric.

We have therefore

$$\text{Quantity of electricity} = \text{pressure} \times \text{capacity}.$$

The dimensions of capacity are therefore obtained by dividing the dimensions of quantity of electricity by those of electrical pressure or E.M.F.

$$\text{Dimensions of capacity: } \frac{L^{\frac{1}{2}} . M^{\frac{1}{2}}}{L^{\frac{3}{2}} . M^{\frac{1}{2}} . T^{-2}} = L^{-1} . T^2.$$

A condenser has unit capacity when a pressure of 1 absolute unit charges it with an absolute unit of quantity, or when unit quantity of electricity charges it to a pressure of 1 absolute unit. We have seen that the absolute unit of pressure is exceedingly small, viz. a hundred-millionth part of a volt, while the absolute unit of quantity of electricity is very large, viz. 10 coulombs. A condenser with a capacity of one absolute unit would therefore have enormous dimensions, in order that such a small pressure might produce such a large charge. For this reason the absolute unit of capacity is not used in practice, but is replaced by a practical unit called a farad. This is the capacity of a condenser which is charged with 1 coulomb by a pressure of 1 volt, or, in which a charge of 1 coulomb causes a back-pressure of 1 volt. If the capacity in farads be represented by C, and the quantity of electricity in coulombs by Q, then

$$Q = C\,.\,V \text{ coulombs,}$$

or

$$C = \frac{Q}{V} \text{ farads} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75).$$

The relation between the farad and the absolute unit of capacity can be found as follows:

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{10^{-1} \text{ C.G.S. unit}}{10^{8} \text{ C.G.S. units}} = 10^{-9} \text{ C.G.S. unit.}$$

The farad is thus the thousand-millionth part of the absolute unit.

Even the farad is too large a unit for ordinary practice, and a microfarad, i.e. a millionth of a farad, is used.

$$1 \text{ microfarad} = \frac{1}{10^{6}} \text{ farad} = 10^{-6} \text{ farad} = 10^{-15} \text{ C.G.S. unit.}$$

It was seen on p. 101 that the C.G.S. unit of quantity in the e.s. (electrostatic) system is $\frac{1}{3 \times 10^{10}}$ of the C.G.S. unit in the e.m. (electromagnetic) system. Since the product of quantity and potential difference gives energy, the unit of which is the erg in either system, it follows that the C.G.S. unit of potential difference in the e.s. system is equal to 3×10^{10} C.G.S. units in the e.m. system, that is, to 300 volts. Hence 1 e.m. unit of capacity is equal to 9×10^{20} e.s. units of capacity, and

$$\begin{aligned} 1 \text{ microfarad} &= 10^{-15} \text{ e.m. unit} \\ &= 9 \times 10^{5} \text{ e.s. units.} \end{aligned}$$

When the capacity of a condenser is calculated from its dimensions, the result is in e.s. units and requires division by 900,000 to convert it to microfarads.

CHAPTER VI

43. Bipolar ring-winding.—44. Bipolar drum-winding.—45. Multiple-circuit ring-winding.—46. Multiple-circuit drum-winding.—47. Two-circuit ring-winding.—48. Two-circuit drum-winding.—49. Series-parallel ring-winding.—50. Series-parallel drum-winding.

43. Bipolar Ring-winding*.

The principles of electromagnetic induction, and especially the induction of electromotive force by the movement of a conductor in a magnetic field, were first established by Faraday and published in 1831 and 1832 in his celebrated *Experimental Researches*. The first machines which were made consisted of a coil of insulated copper wire wound in two diametrically opposite slots in an iron cylinder. The ends of the wire were connected to two insulated slip-rings carried on the armature spindle. For the sake of clearness, these rings are shown one over the other in Fig. 90, instead of side by side. Two fixed brushes are held by springs against the slip-rings and maintain the connection between the coil and the external circuit.

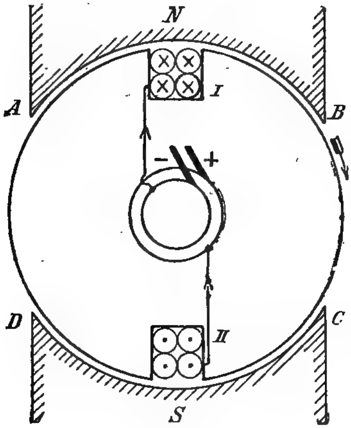

Fig. 90

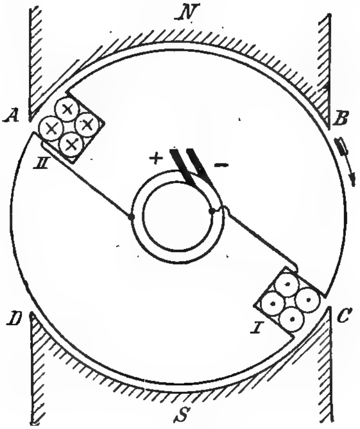

Fig. 91

On rotating the iron cylinder or armature between the poles of a magnet, the wires in the slots cut through the lines of force and induce electromotive force. In Fig. 90 the bunch of wires constituting a coil-side is shown under the middle of the pole. With a clockwise direction of rotation, an application of either the swimming or the right-hand rule shows the induced electromotive force in the wires under the north pole to be directed away from the observer. Since the electromotive force in the wires under the south pole is, at the same moment, directed towards the observer, the electromotive forces in the two sides of the coil can be added together. In a similar manner, the electromotive forces induced in each individual turn of the coil are added together. If the brushes are joined by means of an external resistance, a current flows in the direction indicated in the figure. This current leaves the machine and enters the external circuit by the outer slip-ring and brush, which is therefore marked positive. After flowing through the external circuit, the current re-enters the machine through the inner brush and slip-ring, which can therefore be called the negative terminal.

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* For a fuller discussion of armature windings see E. Arnold's *Armature Windings*.

The lines of force leaving the north pole cross the gap normally or radially into the iron of the armature and the strength of the field H is practically constant over the whole gap. So long as the side of the coil moves under the pole, the electromotive force induced in it can be found from equation (62) on page 81:

$$E = H.l.v.10^{-8} \text{ volts.}$$

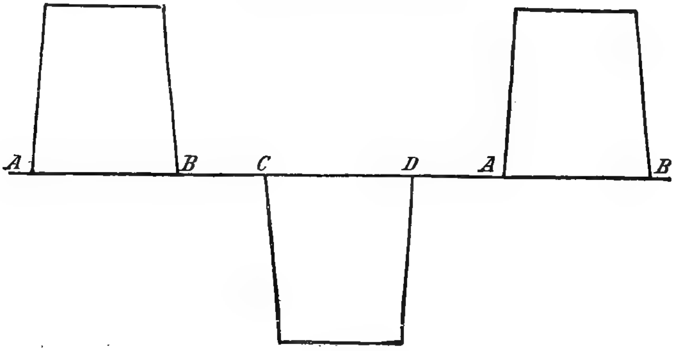

Fig. 92

l here represents the total length of wire under both poles, in centimetres, while v is the peripheral speed in cm per second. The wire not actually under the poles, such as the end connection of the coil, is inactive and must not be included in l.

As soon as the coil-side *I* leaves the pole-tip *B*, the electromotive force drops rapidly to zero and remains there as long as the coil-side moves between the pole-tips *B* and *C*. Directly it passes, however, under the south pole at *C* (Fig. 91) an electromotive force is induced in it towards the observer, and the direction of the current is reversed. Since the polarity of the brushes, and the direction of the current in the external circuit are also reversed at the same time, the machine gives an alternating current. The changes in the current can best be represented by plotting the circumference *ABCD* as abscissae and the electromotive forces induced when the coil-side is at these various points, as ordinates (Fig. 92). The current at any moment can be found by dividing the electromotive force at that moment by the total resistance of the circuit. It is evident that the machine gives current which not only reverses its direction, but which is also intermittent, that is, it ceases altogether for certain intervals.

The transition from the above machine to the modern dynamo is shown in Fig. 93, where the ends of the coil are connected to the two insulated halves of a single slip-ring. In Fig. 93 the direction of the current in the wires is such that the right-hand brush is positive, the current leaving the machine by this brush. It remains positive, however, when the coil-side *I* has passed from the north to the south pole, and the current in the coil has been reversed (Fig. 94). To obtain this result the brushes must be placed in the neutral zone, that is, they must touch the commutator at the extremities of a diameter perpendicular to the magnetic field. We have assumed that the insulation, which divides the slip-ring into two parts, is in the plane of the coil. On passing through the neutral zone, the direction of the current in the coil is reversed, as before, but its connections with the brushes and, therefore, with the terminals of the external circuit are reversed at the same time. The current in the external circuit flows now in one direction only, and instead of an alternating current we have an intermittent direct current (Fig. 95).

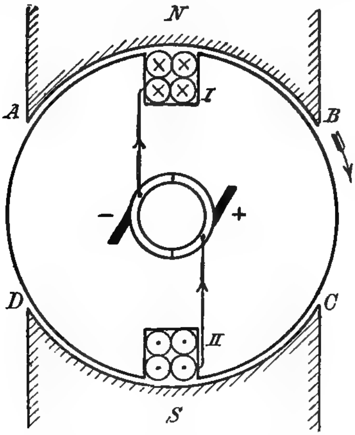

Fig. 93

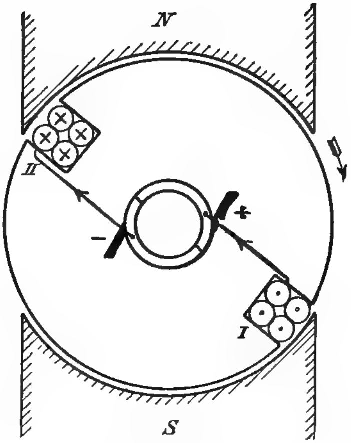

Fig. 94

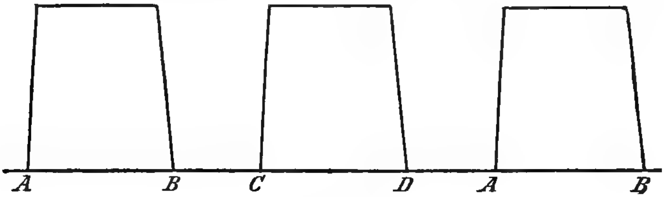

Fig. 95

In order that the external current may remain constant in magnitude as well as direction, the slip-ring must be divided into a greater number of parts, becoming thus transformed into the ordinary commutator with many segments. This arrangement was invented by Pacinotti in 1860, but was little used until taken up by Gramme. The Gramme ring armature consists of a hollow iron cylinder, through which a continuous spiral of insulated copper wire is wound. We must not imagine that the axial length of the armature is small, for it is quite considerable, and the armature is far more a hollow cylinder than a ring. In Fig. 96 the winding consists of 8 coils each of 2 turns. In an actual machine, however, the number of coils is much greater, the whole ring being closely covered with winding. Every spiral must be wound in the same direction.

The collector or commutator is mounted on the armature spindle. This is divided by planes passing through the axis into as many insulated copper segments as there are coils on the armature. As in Figs. 93 and 94 the insulation between the segments is shown opposite the coils.

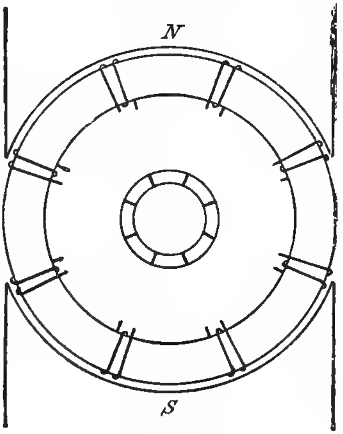

Fig. 96

The coils are now joined and a tap taken from each connecting wire down to a commutator segment (Fig. 97). The figure, thus obtained, is very simple and brings out clearly the continuous endless spiral. In actual practice, however, the number of soldered **T**-joints in the armature winding would be disadvantageous, and it is preferable to bring the end of each coil down to the commutator segment and to connect it there to the beginning of the next coil. The commutator segment itself constitutes, then, the connection between two adjacent coils.

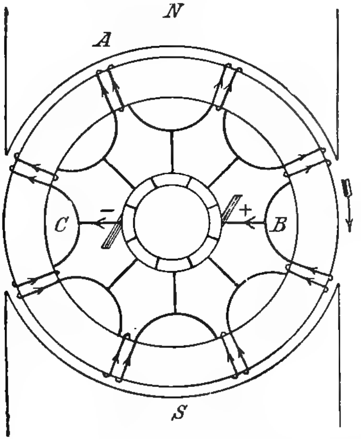

Fig. 97

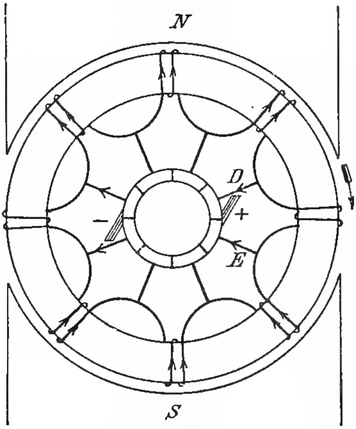

Fig. 98

If the ring is rotated between the poles of a powerful electromagnet, the wires on the periphery of the armature cut through the lines of force. Electromotive forces are induced in them, the direction of which can be found by either of the rules already given. Since the lines of force pass from the north to the south pole through the iron of the armature, and leave the internal air space almost free from magnetic field, only those wires cut through the lines of force which are situated on the external periphery of the ring. For a clockwise rotation the induced electromotive force at A in Fig. 97 is found to be away from the observer. Instead of indicating the direction of E.M.F. or current by means of dots and crosses we can put corresponding arrows on the end connections. **We see, therefore, that, for clockwise rotation, the current in the end connections facing the observer flows away from the south pole towards the north pole.**

This current can only flow, however, if suitable arrangements are made whereby it can leave the armature and flow through the external circuit. We see from the arrows in Fig. 97 that in both the upper and lower halves of the armature the electromotive forces act towards the point B. It is as if two equal forces opposed each other at B and mutually neutralised each other. If, however, brushes be placed on the commutator at B and C, and connected through an external resistance, current will leave the machine at B, flow through the external circuit, and enter the machine again at C, where it will divide between the two parallel halves of the armature winding and flow back to B. B is therefore the positive and C the negative brush. The brushes lie, as before, on a diameter at right angles to the magnetic field, that is, they are in the neutral zone. As before, we assume here that the insulation between two commutator segments is opposite the corresponding armature coil.

As in the previous example, the electromotive force of the machine can be found from the equation,

$$E = H.l.v.10^{-8} \text{ volts.}$$

Since the electromotive forces in the two halves of the armature cannot be added, but are in parallel, l must represent the length of active conductor under one pole. If the number of conductors on the periphery is very large, the number under one pole will be constant and the electromotive force will be practically constant. The advantages possessed by the Gramme machine with its commutator of many segments, compared with the earlier machines, is self-evident.

The electromotive force is not decreased when, in consequence of the rotation, the brush makes contact simultaneously with two adjacent segments (Fig. 98). At such a time the two coils lying in the neutral zone are cut out of the circuit since each one is short-circuited by a brush. The current flows, for example, directly from the points D and E in Fig. 98 to the positive brush. The electromotive force is not decreased, since the short-circuited coils are cutting absolutely no lines of force, assuming that the brushes are in their theoretically correct position. If the number of coils or commutator segments is very large, the change of armature resistance produced by short-circuiting a coil at the brush is quite negligible.

If D = diameter of armature in centimetres,
L = axial length ,, ,,

β = ratio of pole-arc to pole-pitch,
Φ = total flux leaving one north pole,
Z = total number of wires on external periphery,
n = revolutions per second,

then we have $v = D.\pi.n.$

The number of conductors under one pole is $\frac{Z.\beta}{2}$.

The length l of active wire under one pole is therefore

$$l = \frac{Z.\beta}{2}.L.$$

Substituting these values for v and l in the equation for E, we get the electromotive force of the machine

$$E = H.\frac{Z.\beta.L}{2}.D.\pi.n.10^{-8} \text{ volts.}$$

In this equation $\frac{D.\pi.\beta}{2}$ is the length of the polar arc, and $\frac{D.\pi.\beta}{2}.L$ is therefore the area of a pole, which, when multiplied by the flux-density H, must give the flux Φ leaving one north pole. Introducing the term Φ in the above equation, we have

$$E = \Phi.n.Z.10^{-8} \text{ volts} \ldots\ldots\ldots\ldots(76).$$

If, for example, $\Phi = 3.10^6$, $Z = 200$, and the speed be 1100 revs. per minute, then

$$E = 3.10^6.\frac{1100.200}{60}.10^{-8} = 110 \text{ volts.}$$

To find the armature resistance we must remember that the two halves of the winding are in parallel. If

l = total length of wire on armature in metres,

and A = cross-section of wire in square millimetres,

then the resistance of each half of the winding is $\rho.\frac{l/2}{A}$. The resistance of the two halves in parallel will be a half of this, viz.

$$R_a = \frac{\rho.l/2}{2A} = \frac{\rho.l}{4A} \ldots\ldots\ldots\ldots(77).$$

The specific resistance ρ of warm copper can be taken approximately as 0·02.

44. Bipolar Drum-winding.

The ring armature, already described, possesses the advantage of a simple winding, in which a single coil can be repaired without unwinding the whole armature. Since, moreover, the pressure between two adjacent wires is only a small fraction of the terminal pressure of the machine, the insulation can be easily and efficiently provided for. On the other hand, the ring armature has the disadvantage of requiring a large amount of inactive winding. The armature resistance and the weight of wire required are therefore both relatively large.

We have now to examine the drum armature of Hefner-Alteneck, which is an improvement on the ring armature so far as the proportion of active armature wire is concerned. Here, however, two wires having a large difference of potential may be very close together. In the drum-winding, after a wire has passed along the external cylindrical surface under a north pole, it passes right across the end surface to a point diametrically opposite, where it then passes along the armature under the south pole. The two diametrically opposite wires constitute the sides of an armature coil. The end of the first coil is connected to the beginning of the next coil, care being taken that the winding is distributed uniformly over the whole periphery.

To ensure this we divide the armature periphery into a suitable number of parts (in Fig. 99 we have taken 8), and number the points successively 1, 2, 3, etc. These are the starting points of the armature coils. To wind the first coil we start at 1, pass down the armature, that is, away from the observer, and should then pass across the back to a point diametrically opposite. This point is, however, already taken up by the beginning of coil 5, and we therefore take the back end-connection of coil 1, as shown by the dotted line, to a point 1′ lying near point 5. The wire is then brought up to the front along the surface of the armature at 1′. The wires 1—1′ together with their end-connections constitute one turn of the first coil. From 1′ we pass across the front, back to 1, and wind another turn 1—1′, repeating this until the first coil is completely wound. For the sake of simplicity the coils in Fig. 99 consist of a single turn. There is, however, no reason why each turn in the figure should not represent a coil of many turns.

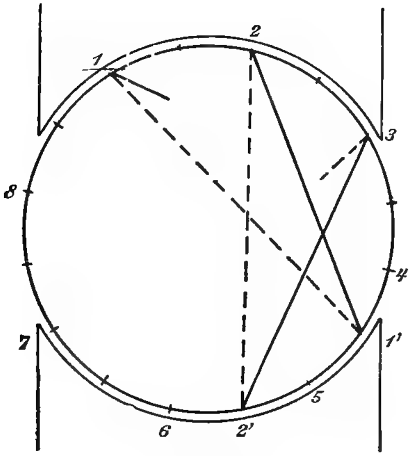

Fig. 99

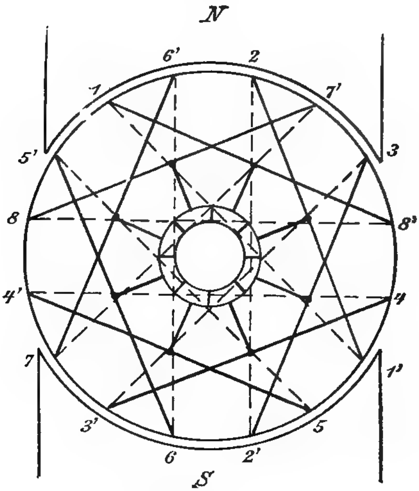

Fig. 100

After completely winding the first coil, we pass across the front to the beginning of the second coil at the point 2. After passing down the armature at point 2 we should cross the back end to point 6, but as this is already occupied we go to the neighbouring point 2′, as shown by the dotted line.

The coil 2—2′ is then wound in exactly the same way as coil 1—1′, and when it is finished we pass to coil 3, and so on round the whole armature.

The winding can be represented in tabular form in the following manner:

8′	4′
/	/
1—1′	5—5′
/	/
2—2′	6—6′
/	/
3—3′	7—7′
/	/
4—4′	8—8′

The horizontal lines represent connections at the back of the armature, such as 1—1′, while the slanting lines represent connections at the front or commutator end, such as 1′—2.

When completely wound in this way there will be sixteen wires or coil-sides on the periphery, dividing it into sixteen equal parts, of which each coil spans 7. In winding coil 1, for example, we started at the point 1 and stepped forward over 7 divisions to the point 1′. The forward step, which we shall call y_1, is therefore equal to 7, or

$$y_1 = 7.$$

From 1′ we step back over 5 divisions to the point 2, whence the backward step y_2 is equal to 5, or

$$y_2 = 5.$$

If the sixteen wires had been consecutively numbered from 1 to 16 as in Fig. 101, the steps y_1 and y_2 would still have been 7 and 5, and starting from 1, the first coil would have gone to $1 + 7 = 8$ and have stepped back to $8 - 5 = 3$. The winding table for Fig. 101 would be as follows:

6	14
/	/
1—8	9—16
/	/
3—10	11—2
/	/
5—12	13—4
/	/
7—14	15—6

If the coils have many turns the end-connections of the coil last wound will lie over the end-connections of the coils already wound, and the resistances of the various coils will be unequal. This can be avoided by winding a half of the first coil, a half of the second and a half of the third, then all the fourth coil, and finally the remaining halves of the third, second, and first coils. In modern armatures, however, the coils are usually all alike, being wound on formers in such a way that they can be slipped one over the other.

After completing the last coil, its end must be connected to the beginning of the first coil, thus completely closing the winding, as shown in Fig. 100

and the following figures. We have then to connect the wires joining successive coils to the commutator segments. It is simplest to draw the insulation between the segments opposite to the corners of the regular octagon formed by the front end-connections. Each front end-connection is then joined to the nearest segment.

It is interesting to notice that the drum-winding is identically the same in principle as the ring-winding. The end of each coil is connected to the beginning of the next or adjacent coil. In the drum-winding, however, space is left between the beginnings of successive coils for the ends of other coils. The agreement becomes plainer when we notice that, except for the negligibly small lack of symmetry, the coil-sides 1′, 2′, etc. in Fig. 100 are exactly equivalent to the coil-sides 1, 2, 3, etc. They have the same position relatively to the poles and have the same electromotive force induced in them at any moment. In considering the drum-winding we could therefore neglect the coil-sides 1′, 2′, 3′, etc., and in their place imagine the number of wires in the coil-sides 1, 2, 3, etc., to be doubled. In this way we obtain a number of coil-sides around the periphery of the armature, connected successively in series, as in the ring armature.

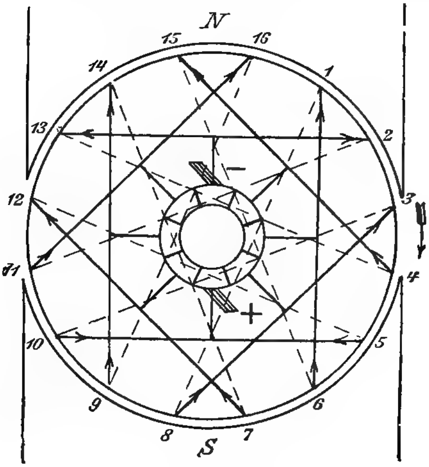

Fig. 101

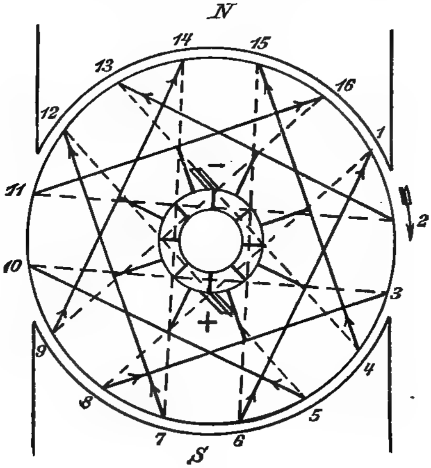

Fig. 102

In order to trace out the path of the current in the drum-winding we will consider in the first place the position of the armature shown in Fig. 101. We saw in the previous section that for clockwise rotation of the armature the current in the front end-connections flows from the south pole towards the north. If arrows are drawn in accordance with this rule, as in the figure, it is evident that the electromotive forces in the turns of any coil, and also in the various coils, act, in general, in the same direction and can be added. In the end-connection 5—10, however, the two forces oppose each other, and a brush must therefore be placed on the corresponding segment. When the external circuit is closed, current will leave the machine by this brush, which must therefore be marked +. Similarly, the negative brush must be placed on the segment connected to the end-connection 2—13.

The current enters the machine at the negative brush, divides and flows through two parallel paths to the positive brush. At the moment represented in Fig. 101 the paths of the current through the armature can be shown in the following manner:

$$-\left|\begin{array}{cccccccc} 2 & 11 & 16 & 9 & 14 & 7 & 12 & 5 \\ 13 & 4 & 15 & 6 & 1 & 8 & 3 & 10 \end{array}\right|+$$

We will now consider the case when each brush touches two segments simultaneously (Fig. 102). At this moment the coils 2—11 and 3—10 are short-circuited by the negative and positive brushes respectively, and are thereby cut out of the circuit. No arrows are drawn, therefore, to indicate the current in the end-connections of the wires 2, 11, 3 and 10. The paths of the current through the armature are as follows:

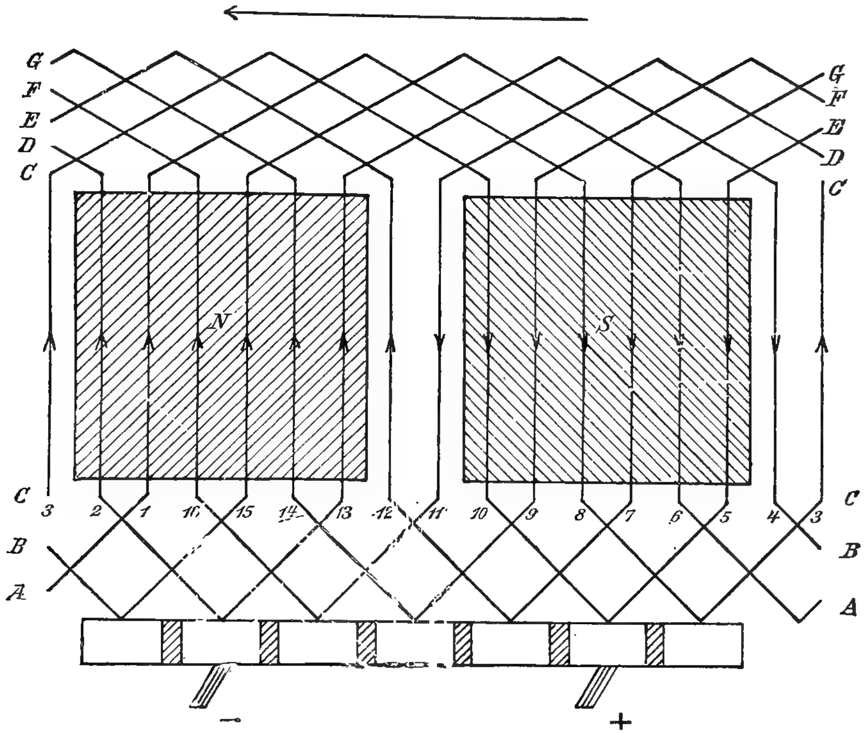

Fig. 103

$$-\left|\begin{array}{cccccc} 16 & 9 & 14 & 7 & 12 & 5 \\ 13 & 4 & 15 & 6 & 1 & 8 \end{array}\right|+$$

As before, we see that the short-circuited coils lie in the neutral zone, if the brushes are in their correct position.

The path of the current is clearly shown by developing or unwinding the periphery of the armature and spreading it out flat, as is done in Fig. 103

for the moment represented in Fig. 101 when each brush touches one segment only. We must imagine the winding to move in the direction of the upper arrow in front of the stationary poles N and S, while the commutator slides along the fixed brushes. A sixteenth of a second later each brush will touch two segments and short-circuit two coils lying in the neutral zone.

In modern armatures the ends of the coils are not taken across the ends of the armature as shown in Figs. 101 and 102, but are arranged on a cylindrical prolongation of the armature surface. A good idea of the arrangement is obtained by imagining Fig. 103 to be wrapped round the cylindrical armature surface.

The electromotive force of a drum armature is naturally the same as that of a ring armature with the same number of external wires. We see the advantage of introducing the number of external wires, instead of the number of turns or coils, into the formula for the electromotive force. We have then for the bipolar drum armature

$$E = \Phi . n . Z . 10^{-8} \text{ volts.}$$

The formula for the armature resistance is also exactly the same as for the ring armature.

45. Multiple-circuit Ring-winding.

We have previously mentioned that the electromotive force is a fundamental characteristic of the ordinary machine, depending on its construction and the speed of rotation. The current, on the other hand, is dependent on the external resistance, which is entirely in the hands of the consumers. A limit is set to the current, however, by the amount of heat allowable in the armature without endangering the insulation. Machines for large currents must therefore have a low armature resistance and a sufficiently large cooling surface. This leads to conductors of large cross-section, and copper strips or bars are used instead of round wire.

The losses due to eddy currents in massive copper bars would be very considerable. Moreover, in machines of large size the bipolar magnets would be very clumsy and unfavourable to ventilation. Finally, if the current in each armature coil is too large, the short-circuiting of the coils at the brushes leads to sparking. Large machines are therefore made multipolar and the armature winding has as many parallel paths as there are poles. The simplest form of such a winding is the continuous spiral ring-winding. It is exactly the same as for the bipolar machine. The magnet system is arranged so that the poles are alternately north and south. If the armature is turned in the clockwise direction, we know that the current in the front end-connections flows towards the north poles and away from the south poles. We see that the current flows from both sides towards the points A and B, and from here, by means of the commutator, to the positive brushes. Both positive brushes are connected together and to the positive terminal of the external circuit. In the same way the two negative brushes are connected together and to the negative terminal of the external circuit.

The armature winding is thus divided into four parallel paths, so that if

l = total length of wire on armature in metres,
p = number of pairs of poles, and
A = cross-section of wire in sq. mm,

the resistance of one path between two brushes of opposite sign will be $\frac{\rho \, . \, l}{2p \, . \, A}$. Since there are $2p$ such paths in parallel the resistance of the whole armature will be $2p$ times as small, that is,

$$R_a = \frac{\rho \, . \, l}{4p^2 \, . \, A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78).$$

This multiple-circuit or parallel winding has, thus, the advantage of giving a very small armature resistance. Since the external current divides in $2p$ parts, the current in each coil is relatively small, viz. $I_a/2p$. Let us consider, as an example, a 4-pole machine for 110 volts and 100 amperes armature current. If the total length of wire on the armature is 200 metres and its cross-section 10 sq. mm we have, since $p = 2$,

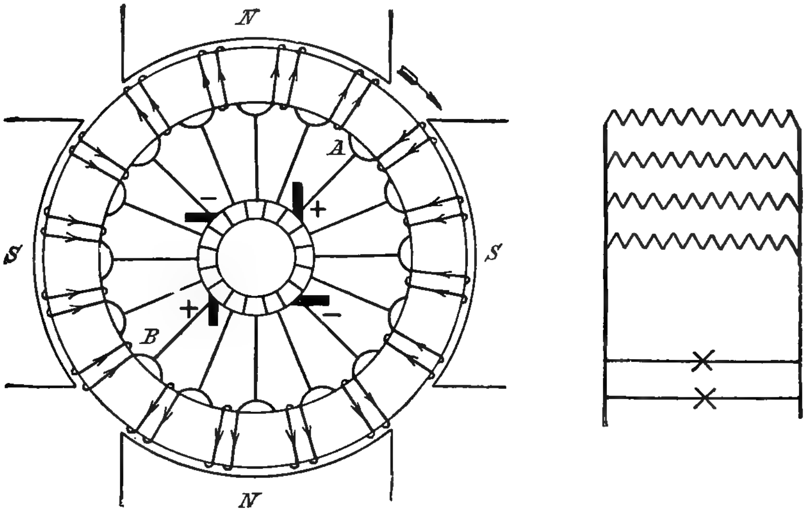

Fig. 104

$$R_a = \frac{\rho \, . \, l}{4p^2 \, . \, A} = \frac{0\cdot 02 \, . \, 200}{4 \, . \, 4 \, . \, 10} = 0\cdot 025 \text{ ohm.}$$

The heating or so-called copper loss is therefore

$$I_a^2 R_a = 100^2 . 0\cdot 025 = 250 \text{ watts.}$$

This is about 2·5 per cent. of the output of the dynamo. The pressure drop in the armature is

$$I_a R_a = 100 . 0\cdot 025 = 2\cdot 5 \text{ volts,}$$

and the current density in the armature winding

$$\frac{I_a}{2p \, . \, A} = \frac{100}{4 \, . \, 10} = 2\cdot 5 \text{ amps. per sq. mm.}$$

To calculate the electromotive force of armatures with parallel winding we observe that the number of lines of force cut by a conductor in one revolution is p times as great as in a bipolar machine having the same number of lines per pole. On the other hand, the number of wires connected in series is p times as small as in a bipolar machine with the same total number of wires. The electromotive force of a multipolar dynamo with parallel winding and Φ lines leaving each north pole is therefore

$$E = \Phi . n . Z . 10^{-8} \text{ volts}$$

as in the bipolar machine.

With regard to the number of brushes we see that there is a brush in each neutral zone, which gives a total of $2p$ brushes. This number can be reduced to 2 by interconnecting all segments separated by an angle of $\frac{360}{p}$ degrees. These connections can easily be placed on the end of the commutator towards the armature. They are shown in Fig. 105 for the commutator of a six-pole parallel wound machine ($p = 3$). In this case every three segments separated by $\frac{360}{3} = 120°$ are connected together, the connection thus made replacing that which would otherwise be made between the three brushes of like sign. If the connecting strips are cut from sheet copper and bent, so that the parts shown by heavy lines lie in a different plane to those shown by light ones, contact between the various strips is easily avoided.

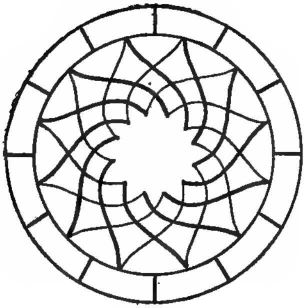

Fig. 105

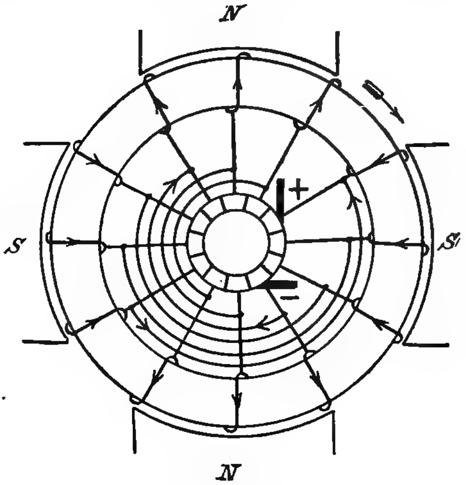

Fig. 106

In this way we have not only connected the three segments on which the brushes would lie at any moment, but also all the other segments in groups of three. Since, however, all interconnected points have always the same potential, their connection will have no effect on the distribution of current in the armature, and no current will flow through any connection except those of the segments under the brushes at the given moment. This is, perhaps, more evident from Fig. 106, in which the equipotential connections are made on the wires leading from the armature to the commutator segments. These connections were first introduced by Mordey.

A reduction in the number of brushes is, of course, only permissible when the current density under the brushes is small. Equipotential connections are used to a large extent in modern dynamos, not for the purpose of reducing the number of brushes, but for equalising the distribution of current in the armature, and between the various brushes, so as to prevent sparking at the commutator due to any one brush being heavily overloaded. It is for this reason that they are generally known as equalising connections.

46. Multiple-circuit Drum-winding. (Lap-winding.)

In a multipolar lap-winding, the wire, after passing under a north pole, returns under the adjacent south pole, passes across the end to the point from which it started and repeats the process until the first coil is completely wound. The wire then passes to a point very near the starting point of the first coil, and the second coil is wound in exactly the same way, its position being but slightly displaced from that of the first coil. If the return wires under the south pole be neglected the similarity, in principle, of the lap- and ring-windings can be seen, and it is evident that, like the ring-winding, the lap-winding must lead to multiple-circuit or parallel connection. We shall first consider:

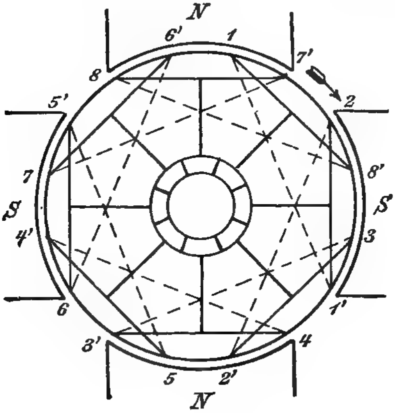

Fig. 107

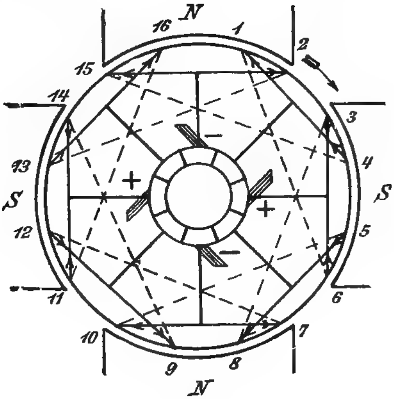

Fig. 108

(*a*) Lap-winding with Long Coils.

In this winding each coil spans, at least, the pole pitch. Assume, for example, that there are 4 poles and 8 armature coils. As shown in Fig. 107 the periphery is divided into 8 parts and the points at which the 8 coil-sides will commence are numbered 1, 2, 3, etc. A wire is passed from front to back at 1, bent round, and brought up from back to front at a corresponding point under the south pole. As this point 3 is already taken for the beginning of the third coil, the return side of the first coil must come up at a point near it such as 1′. When the coil 1—1′ is completely wound we pass across the front from the end 1′ to the beginning of the second coil at 2. Proceeding in this way the successive coils are all in series, and finally the end of the eighth coil is joined up to the beginning of the first. The wires joining successive coils, such as 1′—2, are connected to the commutator segments. Since there are 8 coils, there are 8 segments.

The step from 1 to 1′ is equal to 5, while that from 1′ back to 2 is equal to 3. If, as before, y_1 represents the forward step and y_2 the backward step, then

$$y_1 = 5 \text{ and } y_2 = 3.$$

In general, if there are $2p$ poles, the forward step should theoretically be equal to the $2p$th part of the periphery, but practically 1 division greater or less than this. Hence, if there are U coil-sides on the periphery, for a lap-winding with long coils we have

$$y_1 = \frac{U}{2p} + 1, \qquad y_2 = \frac{U}{2p} - 1,$$

where y_1 and y_2 refer to coil-sides and not to separate conductors; each coil may consist of several turns.

U must evidently be divisible by $2p$ in order that y_1 and y_2 may be whole numbers, and, in addition to this, y_1 and y_2 must be odd numbers. The necessity for this last condition is evident if we number the coil-sides successively from 1 to 16 as shown in Fig. 108 and write down the corresponding winding table, thus:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
 & & 4 & & & 12 \\
 & \diagup & & & \diagup & \\
1 & — & 6 & 9 & — & 14 \\
 & \diagup & & & \diagup & \\
3 & — & 8 & 11 & — & 16 \\
 & \diagup & & & \diagup & \\
5 & — & 10 & 13 & — & 2 \\
 & \diagup & & & \diagup & \\
7 & — & 12 & 15 & — & 4
\end{array}$$

Had the step been even, commencing at 1, we should have arrived at uneven numbers only, and the winding would have closed without using the evenly numbered coil-sides.

As before, it is easily seen that for clockwise rotation the currents in the front end-connections flow towards the north poles and away from the south poles. The path of the current in Fig. 108 can be represented as follows:

$$- \left| \begin{array}{cccc} 7 & 12 & 9 & 14 \\ 10 & 5 & 8 & 3 \\ 2 & 13 & 16 & 11 \\ 15 & 4 & 1 & 6 \end{array} \right| +$$

Figures 107 and 108 represent smooth armatures or toothed armatures with a single coil-side per slot. Smooth armatures are now little used, since the number of wires which can be placed on the periphery is very limited. In addition to this, it is much cheaper to wind the coils on formers and lay them in the slots than to wind the smooth-cored armatures. Another advantage is the reduction of loss due to eddy currents as pointed out in Section 36. Modern machines have therefore, almost exclusively, toothed armatures and several coil-sides in each slot. There is no reason, however, why Fig. 108 and the winding-step which we have given for it should not represent such an armature. We have merely to imagine the coil-side 2 placed below coil-side 1 instead of beside it, and in finding the step to count both upper and lower coil-sides in each slot.

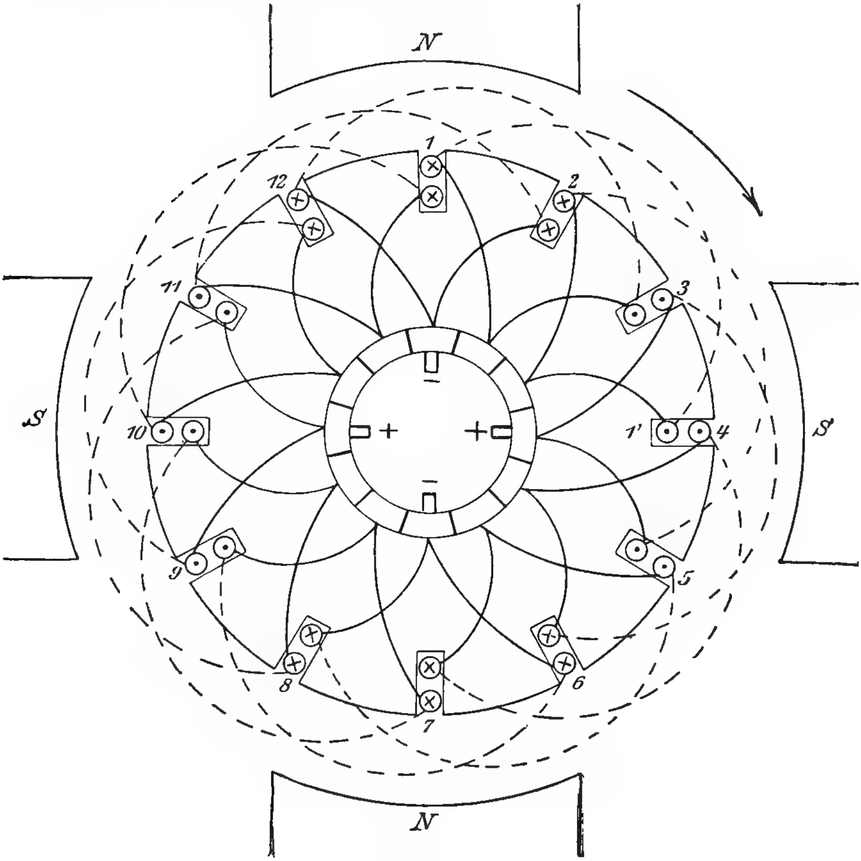

Fig. 109

It is simpler, however, to count the step by the number of slots and not by the number of coil-sides. We assume, as is practically always the case, that, of the two sides 1 and 1′ of a coil, one occupies the top of a slot and the other the bottom of another slot (Fig. 109). With the winding arranged in two layers, the step with respect to coil-sides is therefore necessarily odd. We see then that for toothed armatures we need not calculate the step at all but need only make the coils equal to the pole pitch in width. After finishing coil 1—1′ we go back to slot 2, to the coil occupying the same position in this slot as the previous coil occupied in slot 1. This is shown in

Fig. 109 for a 4-pole machine with 12 slots. Comparison with Fig. 107 will show that both are the same in principle and in the method of numbering. For the sake of clearness the back end-connections are drawn outside the armature.

(*b*) SHORT-CHORD LAP-WINDING.

The width of the armature coils need not be so great as the pole pitch. By making them considerably narrower we obtain the short-chord winding

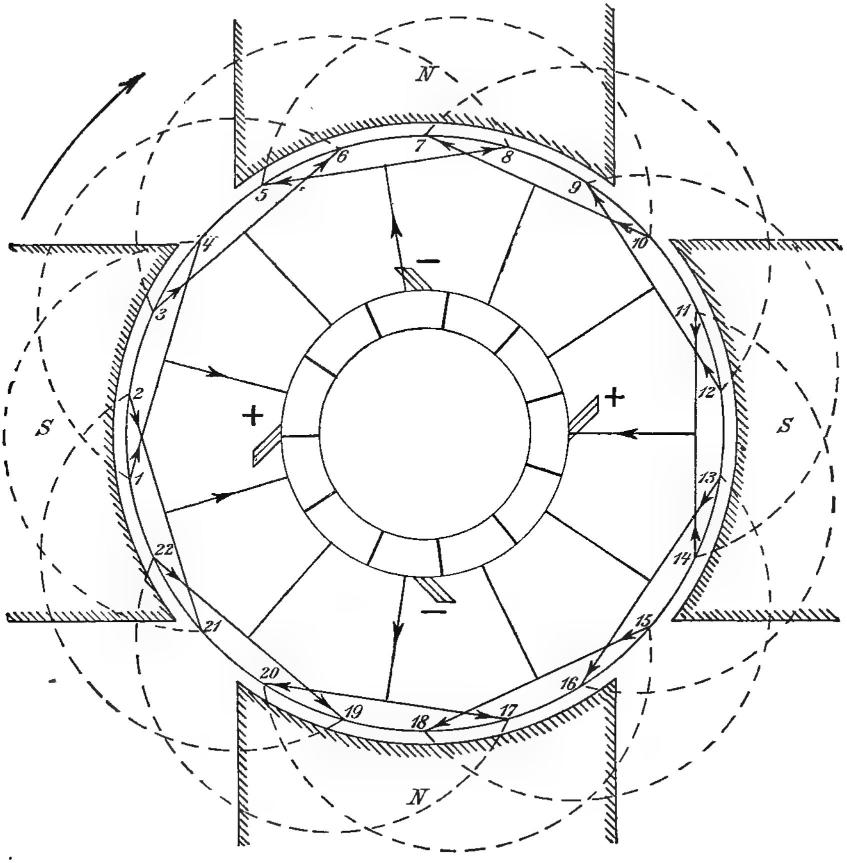

Fig. 110

of Swinburne. For a smooth armature the winding-step with respect to the consecutively numbered coil-sides can be found at once from the following formulae:

$$y_1 = \frac{U-b}{2p} + 1, \qquad y_2 = \frac{U-b}{2p} - 1,$$

where b is any even number giving odd whole numbers for y_1 and y_2. The larger we choose b the narrower are the coils. If, for example, $U = 22$, $p = 2$, and $b = 6$, we get the winding shown in Fig. 110, for which we have

$$y_1 = 5, \qquad y_2 = 3.$$

The position of the brushes can be determined by putting arrows on the front end-connections to indicate the direction of the current. We see that current flows from the wires 11 and 14 towards the right-hand commutator segment, and we therefore place the positive brush on this segment and arrange the other brushes every $\frac{360}{2p}$ degrees. The left-hand brush short-circuits the coil 21—4, which lies in the neutral zone. The path of the current through the armature is as follows:

$$-\left|\begin{array}{ccccc}5 & 10 & 7 & 12 & 9\\ 8 & 3 & 6 & 1 & \\ 20 & 15 & 18 & 13 & 16\\ 17 & 22 & 19 & 2 & \end{array}\right|+$$

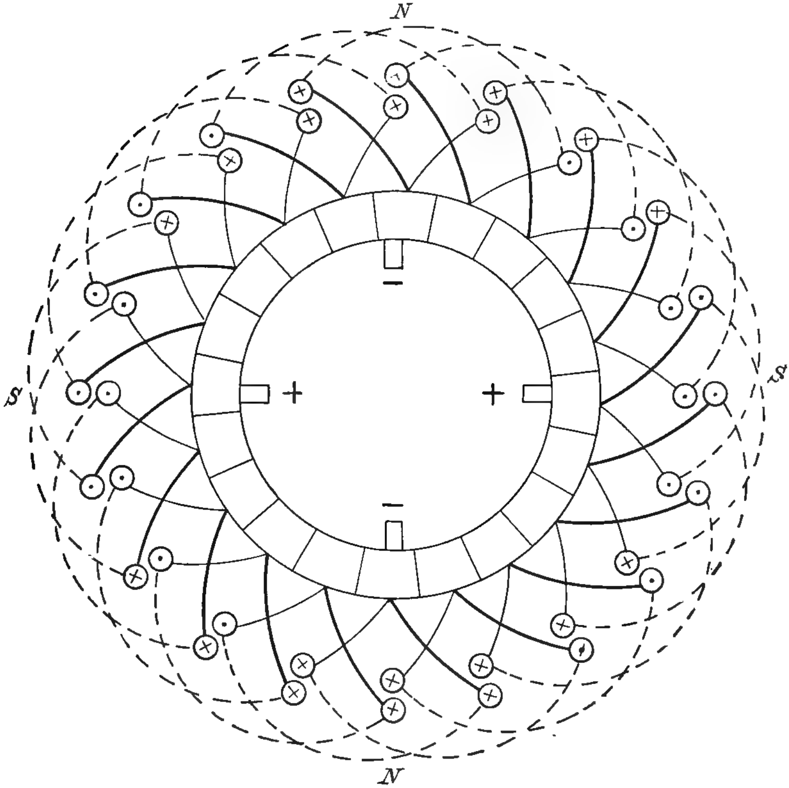

Fig. 111

In this case, only one coil is short-circuited at the same time. This is always so if the number of segments is not divisible by the pairs of poles. Generally speaking, this would be an advantage from the point of view of sparkless commutation, but a disadvantage if the number of coils in the different armature branches is thereby made to differ considerably. The number of segments should preferably be divisible by the number of pairs of poles; if still indivisible by the number of poles, all the positive brushes will short-circuit coils at the same time, and all the negative brushes at another time.

The advantage of the short-chord winding lies in the fact that adjacent wires in the neutral zone carry currents in opposite directions and thus have no demagnetising effect (see Section 54). This is seen very clearly in the slot-winding shown in Fig. 111, which also shows the simplicity of this type of winding. No calculation of the winding-step is required, and it is only necessary to stipulate that, of the two sides of a coil, one shall occupy the upper and the other the lower position in a slot. For a total number of slots $S = 20$ and 4 poles, we have a slot-step of 5 forwards and 4 backwards. If, however, we choose a short-chord winding with slot-steps of 3 and 2, we obtain the winding shown in Fig. 111. The wires in the neutral zone are seen to carry currents in opposite directions. The shortening of the coils is here, however, carried too far and would cause a decrease in the electromotive force.

47. Two-circuit Ring-winding.

In multipolar machines with two-circuit or series winding the current passes through the armature by two parallel paths as in bipolar machines. If Φ represents, as before, the magnetic flux leaving one north pole, the induced electromotive force for the same number of wires Z and the same speed n will be p times as great as that of a two-pole machine. We have thus for two-circuit winding,

$$E = p.\Phi.n.Z.10^{-8}.$$

It is convenient to have, if possible, the same formula for the electromotive force for both series and parallel windings. If now we let a represent the number of pairs of parallel paths through the armature, we have in both cases,

$$E = \frac{p}{a}.\Phi.n.Z.10^{-8} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79).$$

The resistance of a series-wound multipolar armature is, of course, the same as if it were bipolar, viz.

$$R_a = \frac{\rho.l}{4A},$$

where l is the total length of wire in metres, and A is its cross-section in sq. mm. It is easy to see from the formulae for electromotive force and resistance that the series winding is specially suitable for machines with a high pressure and small current.

The principle of the series ring-winding is as follows: A coil under a north pole is connected in series with the coil similarly situated under the next north pole and this again with the similar coil under the next north pole, and so on. After making p steps, we have been once round the armature and must come to a coil next to that from which we started. Thus, if there are U coils on the armature and each step is equal to y, we have,

$$py = U \pm 1,$$

or

$$y = \frac{U \pm 1}{p}.$$

In order that the winding may close on itself after the whole armature is wound, y and U must have no common factor. The number of coils U must always be odd.

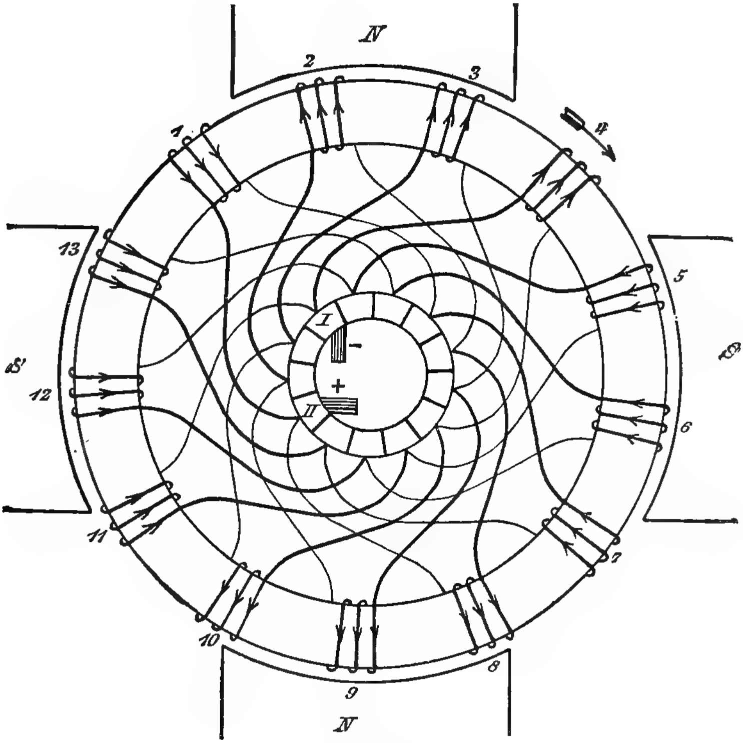

Fig. 112

In the bipolar winding each coil was connected to its neighbour. This is also true for the multipolar winding, except that the corresponding coils under the other similar poles are connected between them. The consideration of the multipolar winding can thus be referred to that of the bipolar winding. In every complete circuit round the armature the winding creeps forward or backward by one coil. Hence, there are p coils connected between two adjacent commutator segments. If, for example, $p = 2$ and $U = 13$, then y must be 7 or 6. In Fig. 112, we have chosen $y = 6$. Commencing at any segment, the beginning of a coil passes across the front of the armature, while the end of the coil is brought up to the front through the interior of the ring and connected, at the commutator segment, to the beginning of the corresponding coil under the next similar pole. The winding can be tabulated as follows:

1—7	11—4
/	/
13—6	10—3
/	/
12—5	9—2
/	/
11	8—1

We now draw the arrows on the front end-connections in the usual manner, omitting those on coils 1 and 4 in which no electromotive force is induced. There can be no doubt as to the direction of the current in the other coils, even coil 7 being safely considered as under a south pole.

The direction of the current in coil 11 shows that the negative brush must be placed on segment I, while from coil 8 we see that the positive brush must be on segment II. The brushes are thus not diametrically opposite, but subtend an angle of 90° or $\frac{360}{2p}$ degrees. The direction of the current in coils 1 and 4 can now be seen, and the path of the current through the armature may be represented as follows:

$$-\left|\begin{array}{ccccccc} 11 & 5 & 12 & 6 & 13 & 7 & 1 \\ 4 & 10 & 3 & 9 & 2 & 8 & \end{array}\right|+$$

The small want of symmetry in the two halves of the armature is of no practical importance.

It is easily seen that the brushes might have been placed on segments diametrically opposite to those on which we have placed them. The current in the coils 1 and 4, which lie in the neutral zone, would thereby have been reversed. In general, there are p possible positions for each brush, separated by an angle of $\frac{360}{p}$ degrees. We can, however, go a step further and place brushes on the commutator at each of these points and thus have p positive and p negative brushes. The coils lying in the neutral zone will then be short-circuited through pairs of similar brushes. With series windings, however, multipolar machines can be, and often are, run with two brushes only.

A moment later than that represented in Fig. 112, the positive brush will make contact with two segments and thereby short-circuit the coils 1 and 7 connected in series. Since, from the nature of the series ring-winding, we start at any segment and pass through p coils to get round the armature to the adjacent segment, it follows that p coils in series must be short-circuited whenever a brush touches two segments simultaneously.

If we wish the brush to short-circuit one coil at a time, instead of p coils in series, we must make the number of segments p times as large as the number of coils. The number of coils can then be kept relatively small. The beginning and the end of every coil are connected to separate, but adjacent, segments. The connection of the coils in series is effected by the interconnection of the commutator segments, every set of p equally spaced segments being connected together. The result is the same as if the coils had been connected together in accordance with the formula

$$y = \frac{U \pm 1}{p}.$$

In Fig. 113, for example, $U = 7$ and $p = 3$. The commutator has, therefore, 21 segments, which are connected together in sets of three, separated from each other by 120 degrees. Only a few of these connections are shown in the figure, the remainder being indicated by letters.

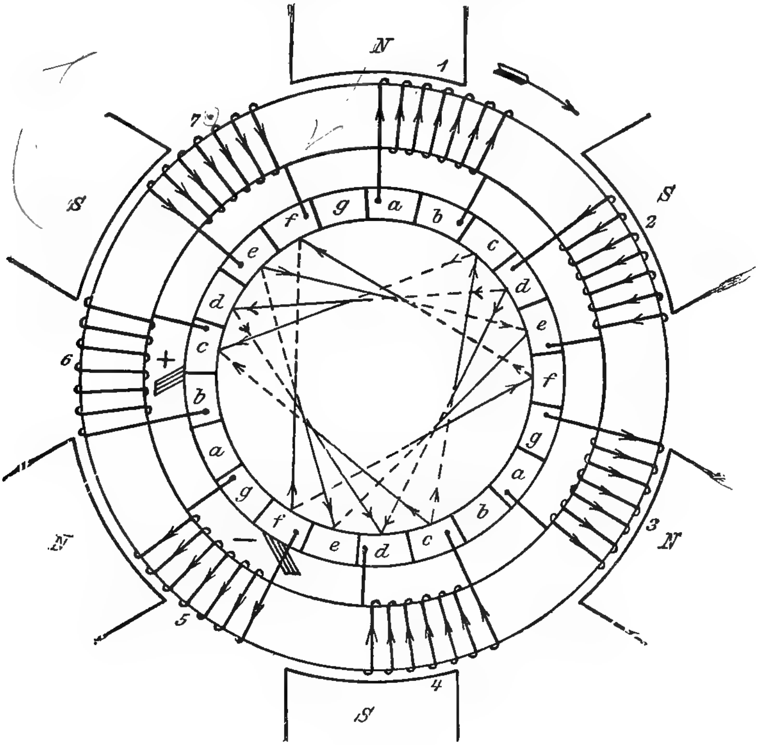

Fig. 113

At the moment shown in the figure the coil 6 is exactly in the neutral zone, and is short-circuited by the positive brush. The negative brush lies 60° away from it. The path of the current is, then, as follows:

$$-\left|\begin{array}{ccc} 7 & 2 & 4 \\ 5 & 3 & 1 \end{array}\right|+$$

This shows very clearly that p positive and p negative brushes can be used, since the p segments are interconnected and equivalent to one large segment.

It is important to notice the positions, with regard to the poles, of the coils constituting one of the two paths through the armature. Coil 4, for example, lies partly under the beginning of a south pole, coil 2 lies wholly under a south pole, while coil 7 lies partly under the end of a south pole. The three coils are therefore equivalent to a single coil with three times as many turns, spread out over a whole pole-pitch.

48. Two-circuit Drum-winding.

In the series or two-circuit drum-winding, a coil consists of a coil-side under a north pole and a coil-side in a somewhat similar position under the adjacent south pole. The end of the coil is not brought back to a point near the beginning, but is connected to the beginning of a coil occupying a corresponding position under the next pair of poles. After proceeding round the armature in this way and commencing to continue round a second time, the coil-sides are found to lie next but one to the corresponding coil-sides of the first round. If each coil-side consists of a single wire or bar, the complete coils disappear and the winding becomes a simple wave-winding (Fig. 114).

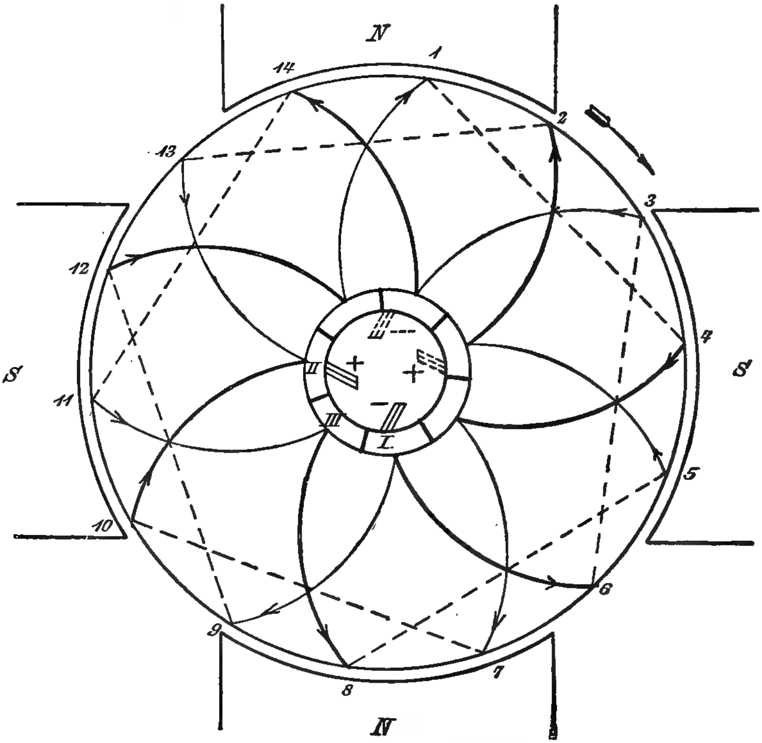

Fig. 114

From these considerations the formula for a series drum-winding should be

$$p\,(y_1 + y_2) = U \pm 2,$$

or

$$y_1 + y_2 = \frac{U \pm 2}{p}.$$

Both winding-steps are continuously in one direction and there is no backward step. Both y_1 and y_2 must be odd since, otherwise, beginning with coil-side 1 we should either come to odd coil-sides only, or find the winding impossible. As in all drum-windings U must be an even number. If the winding-steps y_1 and y_2 be made equal, we have

$$y = \frac{U \pm 2}{2p} \text{ or } 2py = U \pm 2,$$

where U is even and y is odd.

The choice of unequal values for y_1 and y_2 may be due to two causes. If the number of wires or coil-sides is fixed, it may be necessary to make y_1 and y_2 unequal in order to make them odd. If, for example, $U = 214$ and $p = 6$, we have

$$y_1 + y_2 = \frac{214 + 2}{6} = 36.$$

Now, we cannot take $y_1 = y_2 = 18$, as the winding would then close on itself before the armature was completely wound. We should therefore make $y_1 = 19$ and $y_2 = 17$. By making y_1 and y_2 widely different we obtain a similar result to that obtained by Swinburne's short-chord winding, viz. a reduced armature reaction.

A series drum-winding is shown in Fig. 114 in which $U = 14$ and $p = 2$. We have

$$y_1 + y_2 = \frac{U \pm 2}{p} = \frac{14 \pm 2}{2} = 8 \text{ or } 6.$$

We choose $y_1 + y_2 = 6$ and make $y_1 = y_2 = 3$.

The winding scheme is then simply 1—4—7—10—13—2, etc. Starting from the front at 1 we pass down the armature under the north pole, across the back-end to 4, up to the front and then across the front to 7, making connection with the commutator on the way. The back end-connections are drawn dotted in the figure.

We put in the arrows to indicate the direction of current in the front end-connections, omitting, for the present, those for wires 13 and 6, which lie in the neutral zone. Because of the direction of the current in wire 9, we place the negative brush on segment I, and the positive brush, because of the current in wire 10, we place on segment II. This settles the current in wires 13 and 6, and the current path through the armature is as follows:

$$-\left|\begin{array}{cccccccc} & & 9 & 12 & 1 & 4 & 7 & 10 \\ 6 & 3 & 14 & 11 & 8 & 5 & 2 & 13 \end{array}\right| +$$

The odd number of coils causes, at times, a slight dissymmetry between the two armature paths.

Here again 4 brushes could be used, or, in general, $2p$ brushes separated by $\frac{360}{2p}$ degrees. In Fig. 114, for example, two more brushes are shown dotted, lying diametrically opposite to those on segments I and II. Since the brushes of like sign will be connected together, the coils 3—6 and 2—13, lying more or less in the neutral zone, will be short-circuited at the moment represented in the figure. The path of the current through the armature will then be as follows:

$$-\left|\begin{array}{cccccc} 9 & 12 & 1 & 4 & 7 & 10 \\ 14 & 11 & 8 & 5 & & \end{array}\right|+$$

As in the series ring-winding p coils are connected in series between two neighbouring segments. This can be obviated, with a view to improving the commutation, by making the number of segments p times the number

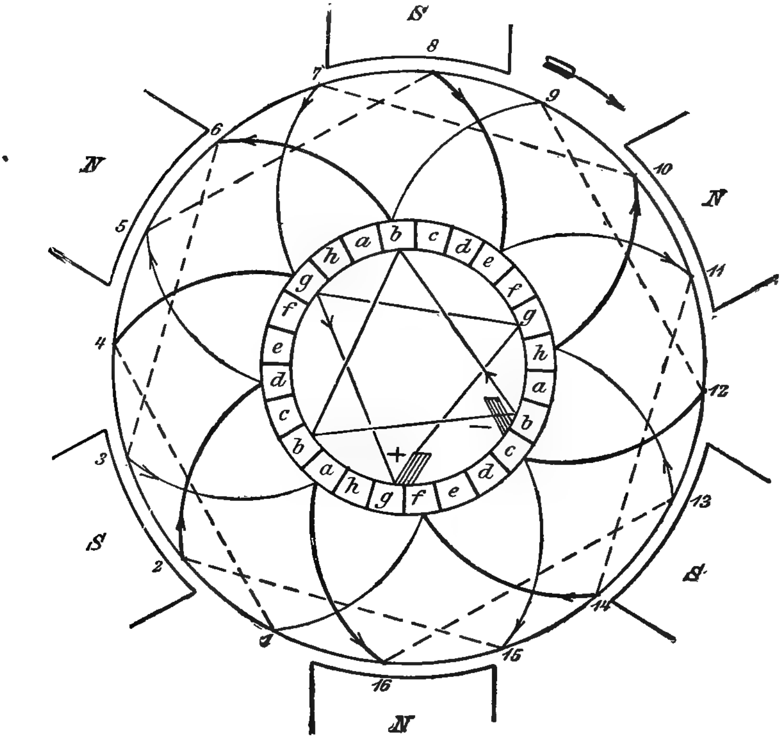

Fig. 115

of coils. Those segments which are separated by $\frac{360}{p}$ degrees are then interconnected. In Fig. 115, for example, $U = 16$ and $p = 3$, whence

$$y = \frac{16 + 2}{6} = 3.$$

The winding proceeds as follows: 1—4—7—10, etc. Of the connections within the commutator, only those are shown which are of importance at the moment. The others are clearly indicated by means of the letters on the segments.

The arrows are now drawn on the front end-connections. Assuming that the coils 1—4 and 9—12, lying in the neutral zone, are short-circuited at the moment chosen, we get the position for the brushes shown in the figure. The path of the current is then as follows:

$$- \left| \begin{array}{cccccc} 6 & 3 & 16 & 13 & 10 & 7 \\ 15 & 2 & 5 & 8 & 11 & 14 \end{array} \right| +$$

Naturally, as the segments are connected in groups of p, the number of brushes can be increased to $2p$.

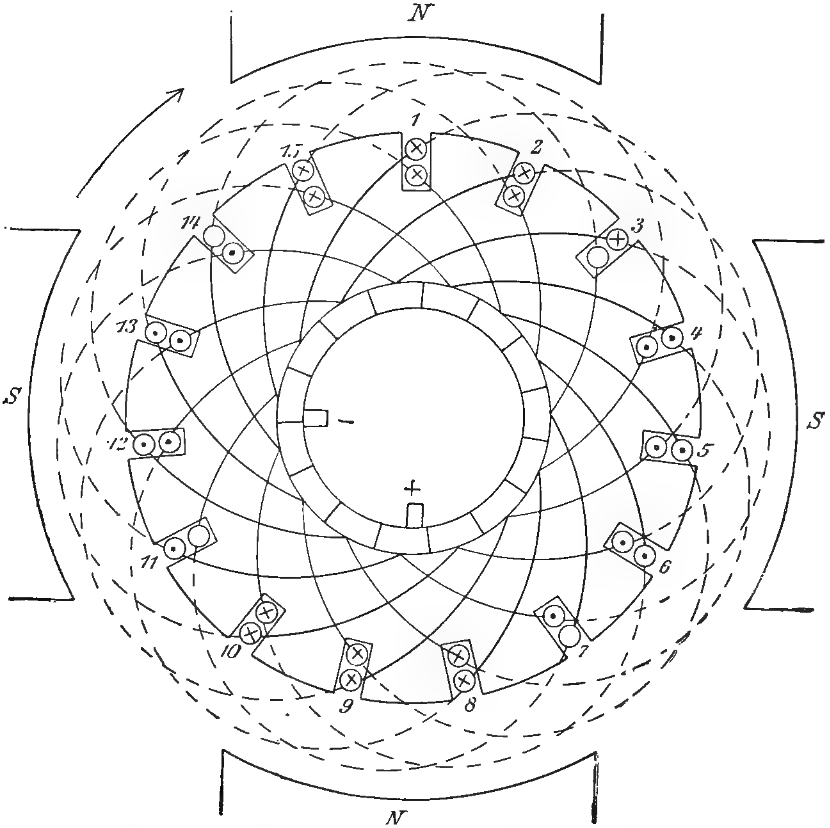

Fig. 116

As before, the slot-winding with two coil-sides per slot is extremely simple. After going once round the armature, that is, after $2p$ steps, we arrive at the slot next to that from which we started. The result is the same as when, in a smooth-cored armature, we arrive at the coil-side next but one to that from which we started. If y_{s1} and y_{s2} represent the winding-steps with regard to the slots, and the number of slots be S, then the above consideration leads to the formula

$$p \cdot (y_{s1} + y_{s2}) = S \pm 1,$$

or

$$y_{s1} + y_{s2} = \frac{S \pm 1}{p}.$$

If, for example, $S = 15$ and $p = 2$, we have

$$y_{s1} + y_{s2} = \frac{15 \pm 1}{2} = 8 \text{ or } 7.$$

We choose $y_{s1} = y_{s2} = 4$ and obtain the winding shown in Fig. 116.

49. Series-parallel Ring-winding.

Up to the present we have divided armature windings into parallel or multiple-circuit windings and series or two-circuit windings. Of these, the former are suitable for machines for large currents, and the latter for machines for high pressures. As a general rule, that type of winding is chosen which gives a suitable current in the armature wires. If, for example, the armature current be 300 amperes, a series winding would give a current of 150 amperes in each armature path. Experience shows this to be a suitable value, requiring a copper strip of large cross-section and wasting little space in insulation. A series winding would therefore be chosen in such a case. If, on the other hand, the armature current be 600 amperes, the current per armature wire would be too heavy with a series winding, and it would be necessary to adopt a parallel winding.

A difficulty which occurs with all the parallel windings already described is due to the fact that each branch of the armature winding lies entirely under one pair of poles. If now the various pairs of poles have different numbers of lines of force, the electromotive forces induced in the different branches will be different. This may cause great differences between the currents in the various branches. As an example, let us assume that an electromotive force E_1 of 115 volts is induced in the upper half, consisting of two parallel branches, in Fig. 117. As the result of inequality in the magnetic flux, let the electromotive force E_2 in the lower half be 114 volts. Let the resistance R' of each double branch be 0·05 ohm. What will be the current in each branch when the total current is 100 amperes?

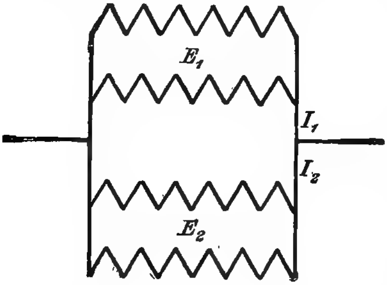

Fig. 117

Since the terminal pressure is the same for each branch, we have

$$E_1 - I_1 R' = E_2 - I_2 R' = V,$$

or

$$115 - I_1 . 0{\cdot}05 = 114 - I_2 . 0{\cdot}05.$$

Wherefore

$$I_1 - I_2 = 20.$$

As the total current is 200 amperes, we have

$$I_1 = 60 \text{ amperes},$$

$$I_2 = 40 \text{ amperes}.$$

The greater load and therefore the greater currents under some pole-pairs may lead to destructive sparking at the commutator. It is therefore best to provide such armatures with equalising connections by means of which these differences between the currents in the different branches are equalised within the armature itself, and the same current is carried by each brush.

The difficulty can, however, be overcome to a large extent by using Arnold's series parallel windings. Each branch of the armature winding is distributed around the whole armature, so that any inequality between the poles acts in a similar manner on every branch. The winding progresses in one direction only, and, after finishing a coil, passes to the corresponding coil under the next pair of poles.

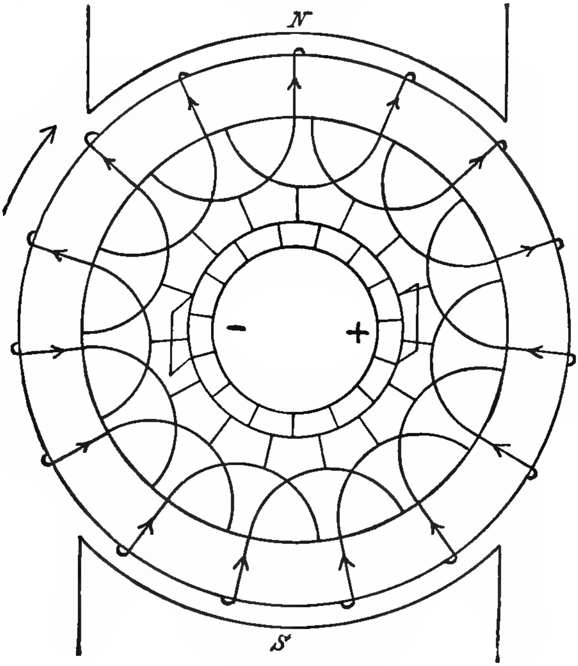

Fig. 118

Up to this point the winding does not differ from the series windings already described. These latter were, however, two-circuit windings, whereas Arnold's winding has more than two parallel paths. This result is obtained by so arranging the winding that, after going once round the armature, we arrive at the coil next but one, or next but two, as the case may be, to the coil from which we started; in the two-circuit winding we arrived at the next coil.

To make this plain, we shall consider a bipolar machine and examine the effect of winding successively every other coil, so that it is necessary to go twice round the armature to complete the winding. This is shown in Fig. 118, where the step is 2, and the number of coils 15. U and y have no common factor, and the winding is therefore simply closed on itself, although electrically it is divided into four parallel paths. The brushes must be wide enough to cover more than one segment. In general, a winding-step of a gives a winding with $2a$ parallel paths.

In this way we are led to the following rule for multipolar machines with ring-windings: If, after passing once round the armature, we come to the ath coil from that at which we started, the winding will be divided into $2a$ parallel paths. It is easy to see that the series ring-winding previously described was merely a special case of the series-parallel winding, in which a was made equal to 1.

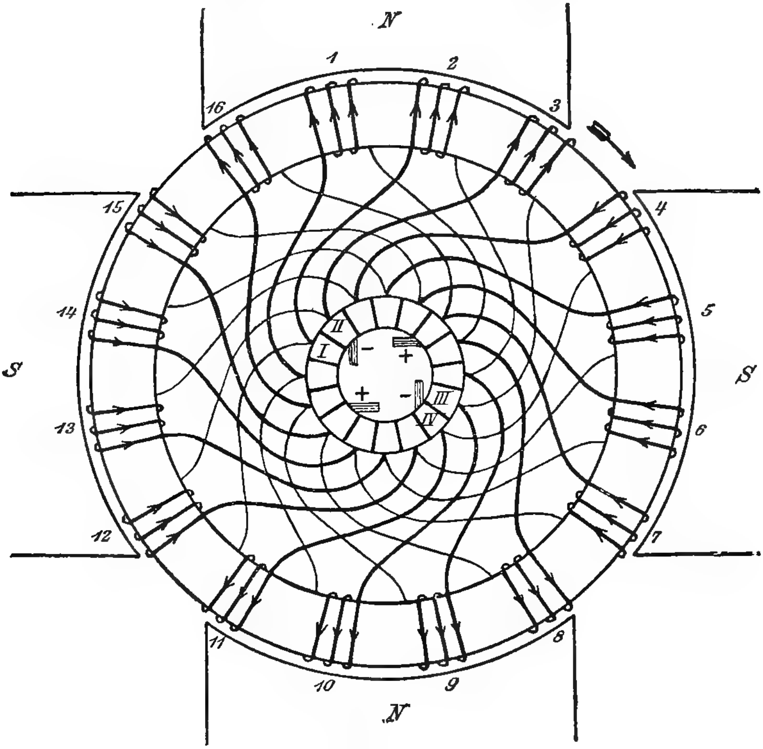

Fig. 119

We shall confine ourselves to the more important case in which the number of parallel paths is equal to the number of poles, or $p = a$. In a ring armature with series-parallel winding, after going once round the armature, that is, after making p steps each equal to y coils, we must arrive at the pth coil from that at which we started. We have, therefore,

$$py = U \pm p,$$

or

$$y = \frac{U \pm p}{p} = \frac{U}{p} \pm 1.$$

In this formula U may be odd or even according to circumstances. If the winding is to be singly closed U and y must have no common factor. In Fig. 119, for example, $U = 16$ and $p = 2$. That gives $y = 9$ or 7. We have chosen 7, and the winding scheme is simply 1—8—15—6—etc.

The connections between the coils themselves and with the commutator are made at the front of the armature by means of connectors in two planes.

Putting in arrows to show the direction of the current in the front end-connections of the coils, we see that it flows towards the point joining coils 8 and 15, and also towards the point joining coils 16 and 7. This determines the position of the positive brushes, and gives us the following path for the armature current:

$$- \left| \begin{array}{cccc} 11 & 2 & 9 & 16 \\ 4 & 13 & 6 & 15 \\ 12 & 5 & 14 & 7 \\ 3 & 10 & 1 & 8 \end{array} \right| +$$

A moment later each of the four brushes will make simultaneous contact with two segments. The negative brushes, for example, will short-circuit the coils 3 and 11, the short-circuited path being as follows:

I, II, 3, III, IV, 11, I.

Thus, the coils 3 and 11, lying in the neutral zone, are short-circuited at the given moment by the negative brushes. In general, both the positive and negative brushes short-circuit p coils in series. Since a closed circuit can be broken at one point without destroying the electrical connection between all the paths of the circuit, it is possible to remove one of the positive and one of the negative brushes. The remaining brushes must be wider than one segment. Each brush is connected to the segment, from which the other similar brush has been removed, through the coil lying in the neutral zone.

50. Series-parallel Drum-winding.

We have already drawn attention to the similarity in principle between ring- and drum-windings, and we will now use this fact to find a formula for the winding-step of series-parallel drum-windings. We saw in the last section that, for a ring-winding to have $2a$ parallel paths, it is necessary that a complete circuit round the armature should bring us to the ath coil from that at which we started. Now, the space occupied by a coils on the ring is represented by $2a$ coil-sides on the drum. In a complete circuit of the drum armature we make p double steps of $y_1 + y_2$ coil-sides. For a pure parallel winding ($p = a$) we have, therefore,

$$p\,(y_1 + y_2) = U \pm 2p.$$

Since the two sides of a coil always embrace an even number of other coil-sides, the steps y_1 and y_2 must both be odd. The sum $y_1 + y_2$ is therefore even, and $y_1 + y_2$ and U have always the common factor 2. Except for this they must have no common factor if the winding is to be singly closed.

If the winding-step is the same both back and front, we have

$$y = \frac{U}{2p} \pm 1.$$

Here U and y must have no common factor. If, for example, $U = 24$ and $p = 3$, we get $y = 5$ or 3. The latter value, i.e. $y = 3$, would not give a singly closed winding, since U and y would have a common factor 3. Starting at 1, we would only make eight steps before arriving again at 1, and the winding would thus be closed after only a third of the armature had been wound. The complete armature winding would thus be triply closed, that is, it would consist of 3 distinct windings. If, on the other hand, we choose the value $y = 5$, we get the winding 1—6—11—16—etc., which is singly closed.

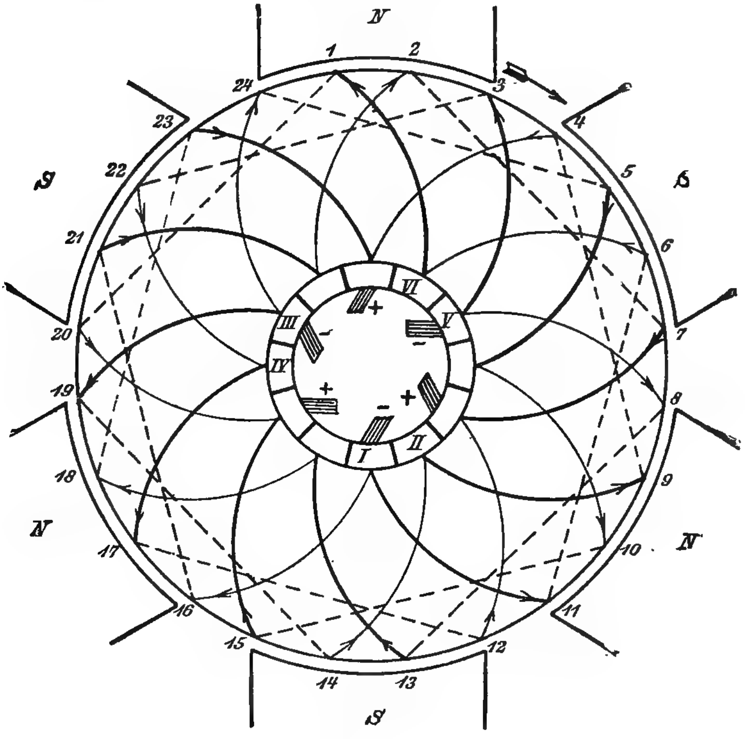

Fig. 120

This winding is shown in Fig. 120, in which the back end-connections are shown dotted, while those at the front are arranged in two planes in the usual way. The number of segments is half the number of coil-sides. Putting in the arrows to show the direction of the current in the front connections, we find the position of the brushes. Positive brushes are placed where the currents flow together from both sides, while negative brushes are placed where the currents flow away in both directions. The paths followed by the current are as follows:

$$-\left|\begin{array}{rrrr} 11 & 6 & 1 & 20 \\ 16 & 21 & 2 & 7 \\ 8 & 13 & 18 & 23 \\ 3 & 22 & 17 & 12 \\ 24 & 5 & 10 & 15 \\ 19 & 14 & 9 & 4 \end{array}\right|+$$

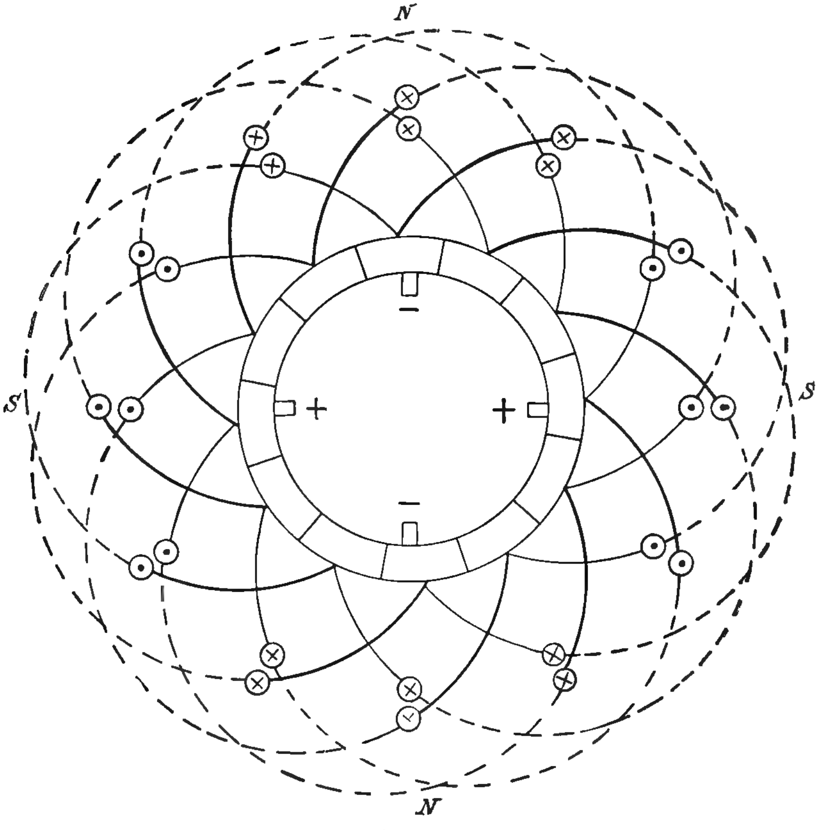

Fig. 121

A moment later the brushes will make simultaneous contact with two segments, and the negative brushes, for example, will short the following circuit and cut it out of the main armature circuit:

I, II, 14, 19, III, IV, 22, 3, V, VI, 6, 11, I.

Three coils will be short-circuited at the same time by the positive brushes. The coils short-circuited in this way lie in the neutral zone. In general, p coils or $2p$ coil-sides in series are short-circuited by both the positive and negative brushes. What was said concerning the leaving off of some of the brushes in the ring armature holds equally well in the case of drum armatures.

As in other cases, this winding is very simple when applied to toothed armatures with the slots numbered consecutively and each containing two coil-sides. The rule for a series-parallel winding is then: If, after going once round the armature, we come to the *a*th slot from that at which we started, the armature will have $2a$ parallel paths. If y_{s1} and y_{s2} are, as before, the winding-steps with regard to the slots, we have, for a pure parallel winding ($p = a$), the formula

$$p\,(y_{s1} + y_{s2}) = S \pm p,$$

or

$$y_{s1} + y_{s2} = \frac{S}{p} \pm 1.$$

If, for example, as in Fig. 121, the number of coil-sides is 24, so that the number of slots S is 12 and the pairs of poles $p = 2$, then we have

$$y_{s1} + y_{s2} = \tfrac{12}{2} \pm 1 = 7 \text{ or } 5.$$

We have chosen $y_{s1} = 3$, $y_{s2} = 2$ and obtain the winding shown in Fig. 121.

It is found in practice, however, that sparkless commutation is difficult to obtain with series-parallel windings, especially when the winding is such that, commencing on the neutral zone, and passing once round the armature, we arrive at a coil-side under a pole-tip. To ensure successful operation, the use of equalising connections is to be recommended. If the number of segments is divisible by the number of pairs of poles, which is desirable for the sake of symmetry and sparkless commutation, there can be no difficulty in determining which segments should be interconnected. It is by no means necessary that all the segments should be so connected, since a small fraction of the total number of possible connections is generally found to suffice.

CHAPTER VII

51. The excitation of dynamos.—52. The field magnets.—53. Position of the brushes.—54. Armature reaction.—55. Sparkless commutation.—56. Three-wire dynamos.

51. The Excitation of Dynamos.

The earliest machines in which electromotive force was induced by the motion of a wire in a magnetic field were provided with permanent magnets of hardened steel. They were built up of laminations which gave much stronger magnets than could be obtained from massive steel. Even these compound magnets, however, gave a relatively small magnetic flux, so that the electromotive force and current of a machine of reasonable dimensions were small.

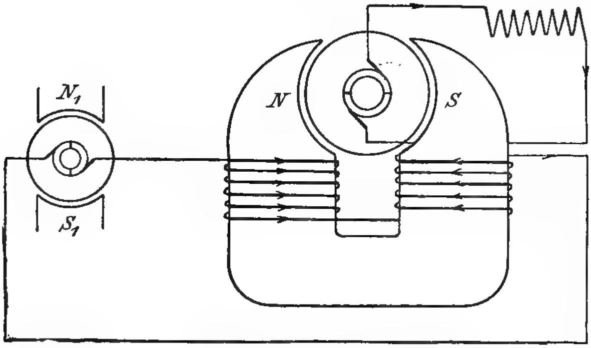

Fig. 122

It was a great advance, therefore, when Dr Wilde of Manchester passed the current obtained from the two-part commutator of a small permanent-magnet dynamo around the wrought-iron poles of a second dynamo. Since wrought-iron has a high permeability, it follows that the magnetic flux in this latter machine, and therefore also the electromotive force induced, were very large. An arrangement similar to that of Wilde is shown in Fig. 122, in which N_1 and S_1 represent the steel magnet of the exciting machine. The current taken from this machine is passed through the field coils of a larger dynamo with magnets of wrought-iron. The coils must be wound in such a way that one pole is north and the other south. When, for example, the bottom end of the left-hand coil lies in front, the bottom end of the right-hand coil, to which it is connected, must be behind. The substitution of electromagnets for permanent magnets had been proposed and carried out by Wheatstone and Cooke in 1845, the exciting current being obtained from batteries.

One of the greatest factors in the enormous progress made by electrical engineering during the latter years of the nineteenth century was the discovery of the self-exciting power of a dynamo with wrought-iron magnets. This discovery was made almost simultaneously by several workers in different countries, and the name of *dynamo* was coined by Werner von Siemens in 1866 in his paper describing such a machine. This discovery, together with that of the commutator with a large number of segments, laid the foundation of modern electrical engineering.

The word *dynamo* is now used in a broader sense to signify all machines for generating electromotive force by electromagnetic induction, whether these machines be self-exciting or not. It is usually confined, however, to direct-current generators, and is rarely applied to alternators.

The process of self-excitation in a direct-current dynamo is as follows. On starting, the armature conductors cut the lines of force due to the remanent magnetism, and thereby induce a small electromotive force. If the circuit is closed as in Fig. 123 this produces a small current, which, passing round the field coils, strengthens the remanent magnetism. The electromotive force is thereby increased, and with it the current, and so on. In this way the machine gradually builds up its field. The strength of field and electromotive force finally reached depend on the construction of the machine and on the conditions under which it is working. It might appear at first sight that the magnetic field would increase until the iron was completely saturated. This is, however, not the case, quite apart from the fact that the saturation of the iron is a very indefinite idea. As a matter of fact, the current increases until the product of current and resistance of circuit is equal to the electromotive force induced by the corresponding flux.

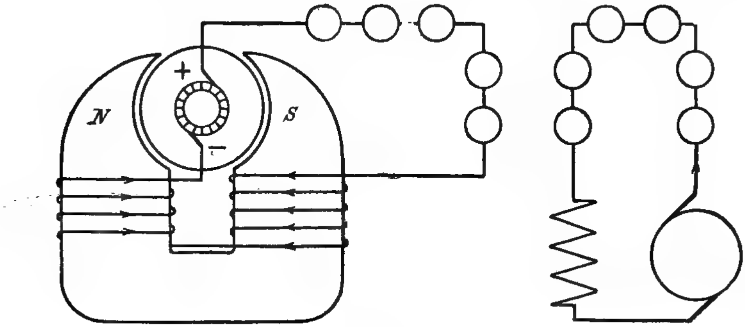

Fig. 123

This increases the difficulty of understanding the action of the dynamo, since the flux, which produces the electromotive force, is itself dependent on the current. We are thus led to the conclusion that although the current, in accordance with Ohm's law, is dependent on the electromotive force and the resistance, yet, on the other hand, the electromotive force itself is dependent on the current and therefore on the resistance. We must therefore give up our previous view, according to which the electromotive force and the resistance of the circuit are the given quantities, from which we can calculate the current. The electromotive force of a dynamo is not merely a function depending on constants of the machine such as flux, speed, or number of wires on the armature. Over and above all these, it is dependent on the conditions under which it is run, that is, on the resistance between its terminals, or the current taken from it.

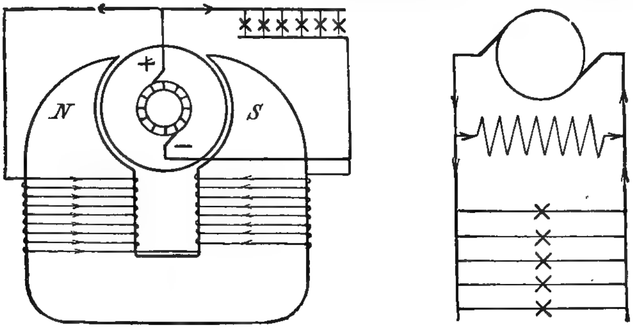

Fig. 124

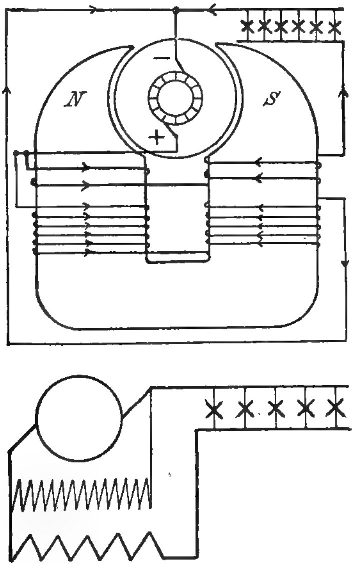

Fig. 125

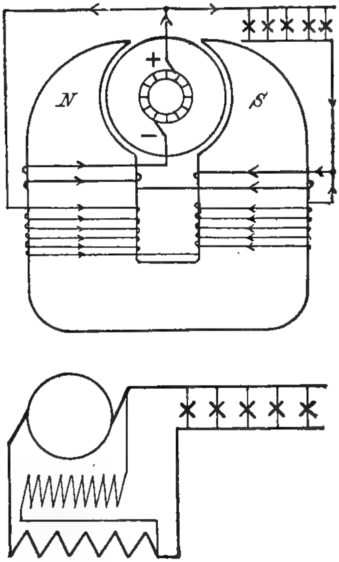

Fig. 126

There are several ways in which the magnet winding of a self-excited dynamo may be arranged. In the series dynamo (Fig. 123) it consists of a few turns of thick wire, through which the main current passes. Hence the armature, field and external circuit are all connected in series.

In the shunt dynamo (Fig. 124) the field winding consists of a large number of turns of fine wire. This winding is connected directly to the brushes, and is thus in parallel with the external circuit. Since the resistance of the field winding is large, the current through it is very small compared with the normal current in the external circuit.

In the compound dynamo (Figs. 125 and 126) the field winding consists of a combination of shunt and series windings. It is really a shunt dynamo, the magnetic field of which is strengthened by means of a few series turns. The coils must be wound in such a way that the currents in the two windings are in the same direction. In Fig. 125 the shunt winding is connected directly to the brushes, and is thus in parallel with the series winding and the external circuit. This is known as the short-shunt arrangement. In Fig. 126 the shunt winding is connected to the terminals of the machine, and is simply in parallel with the external circuit. This is known as the long-shunt arrangement. The object of the compound winding is to maintain a constant or even rising pressure at the terminals of the machine in spite of the increasing drop in the armature with increasing current.

It is of special interest to determine the conditions under which a machine may refuse to excite. They are as follows:

I. If the initially induced electromotive force is too small, i.e.
  (1) with too little remanent magnetism,
  (2) at too low a speed.

II. If the magnetising current produced is too small, i.e.
  (1) with large brush-contact resistance,
  (2) with large external resistance on series dynamos,
  (3) with large field-circuit resistance on shunt dynamos,
  (4) with very small external resistance on shunt dynamos, such as a short-circuit, which would cause the terminal pressure to sink to zero and the magnetising current to vanish.

III. If the flux produced either does not strengthen the remanent magnetism at all or not sufficiently, i.e.
  (1) with large air-gaps between poles and armature,
  (2) with wrong field connections for a given direction of rotation,
  (3) with a wrong direction of rotation for given field connections.

A reversal of the polarity of a machine does not prevent its excitation, but simply reverses the polarity of the brushes.

52. The Field Magnets.

As a rule the field magnets consist of a number of parts of different material bolted together. The materials used are stampings, wrought-iron, cast-iron, cast-steel, and ingot-iron.

Wrought-iron could be used throughout for the magnets of separately excited dynamos or for motors, and would be preferable in some cases on account of its high permeability. It could not be used throughout in a self-exciting dynamo on account of its small remanent magnetism, which would probably be insufficient to enable the field to build up. Its use might therefore be confined to the pole-cores, which must be kept as small as possible, not only to save iron, but to make the length of each turn of the field coil as short as possible and thus save copper.

Stampings of sheet-iron are used for the pole-shoes in order to diminish the eddy losses caused in them by the armature teeth.

Cast-iron was formerly largely used for the field magnets of dynamos and motors. On account of its low permeability, it is now little used except for small machines, in which the magnet frame and bed-plate are cast in one. In this case the cheapness of construction and the low price of the cast-iron more than compensate for the increased weight of copper and iron due to the low permeability.

In large modern machines, however, cast-iron has been replaced by cast-steel. It possesses the twofold advantage of a high remanent magnetism and a high permeability, hardly differing from that of wrought-iron. The flux-density can be carried much higher than in cast-iron, while requiring relatively few ampere-turns on the poles. Whereas it was formerly common to find a half of the total number of ampere-turns used up in driving the flux through the cast-iron field magnet, it is now common in large machines for the whole path in iron, including armature core and teeth, to require no more than a quarter of the total ampere-turns*. The weights of iron and of copper on the field magnets are thus both reduced by the use of cast-steel. This makes the machine lighter and less clumsy in appearance and improves the ventilation. Moreover, the higher flux-densities which can be used have an appreciable influence on the commutation, as we shall see later on. Against these advantages we have to place the higher price of cast-steel and the probable increase in cost of manufacture due to the magnet frame being no longer in one piece with the bed-plate.

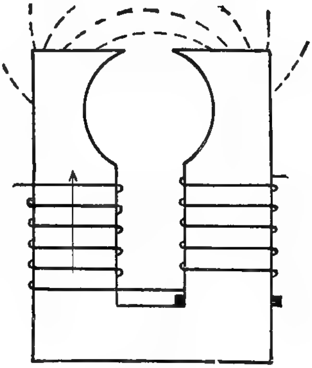

Fig. 127

Turning now to the shape of the field magnets, we find that the early machines were of the horse-shoe type. It was soon seen that the long pole-cores, first introduced, were quite unnecessary and only lengthened the path of the magnetic lines. It was further found that the best form from a magnetic point of view was a well-rounded type following the path of the lines of force. So long as a line of force remains in the same medium it never forms a sharp corner, and the sharp edges in Fig. 127 only add to the weight of the machine. The edges should be rounded off as shown in Fig. 122.

The Manchester type shown in Fig. 128 is of special historical interest. The two field coils are connected so that their magnetomotive forces act in parallel. The lines of force flow upward in both cores and produce at *N* a so-called consequent pole. In calculating the necessary ampere-turns, it is best to divide the machine into two parts by a vertical plane through the spindle and calculate the ampere-turns for each half separately. Each core carries a half of the armature flux. The ampere-turns so calculated must be placed on each core, since the magnetic effects of the two cores are in parallel and cannot be added.

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* This depends, however, on the purpose for which the machine is designed. In machines which have to light glow lamps without a battery connected in parallel, a half of the total ampere-turns may be required for the iron and for armature reaction. (See Section 59.)

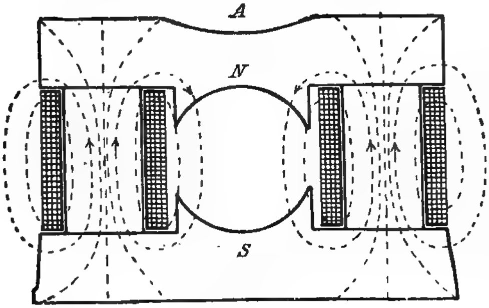

Fig. 128

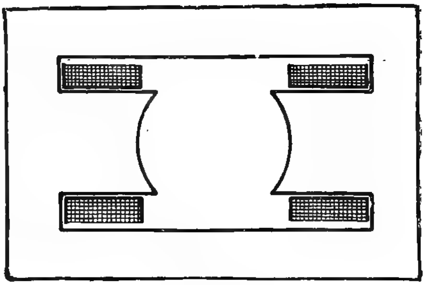

Fig. 129

The arrangement shown in Fig. 129 is commonly known as the Lahmeyer type, and is now very largely used. The winding is completely enclosed by the iron. Practically all multipolar machines are built on this principle, each pole carrying a field coil (see Fig. 151). In Fig. 130 is shown an arrangement whereby a four-pole machine can have two field coils. This arrangement has been used in traction motors where space in one direction is very limited. The unsymmetrical arrangement may, however, lead to sparking at the commutator.

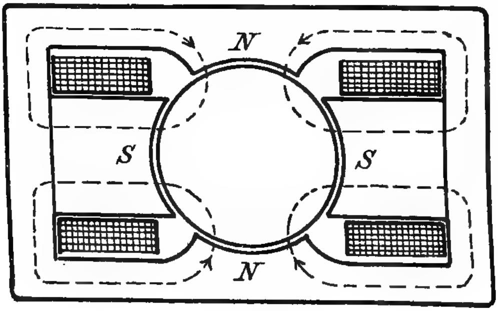

Fig. 130

A great number of machines were made in Germany with the poles inside the armature. Mechanical difficulties with the rotating external ring armature caused this type to be abandoned. The usual type of large alternating current generator consists of an external stationary armature and an internal rotating field magnet system.

We must now compare these various types of machines with respect to magnetic leakage.

By the term *leakage flux* we mean those lines of force which pass from pole to pole through the air instead of going through the armature. As a result of this leakage, the flux through the magnet cores is much greater than that through the armature, and it is only this latter which is usefully employed. The uselessly increased flux through the poles and yoke necessitates a greater number of ampere-turns in the field coils.

If Φ = the flux entering the armature from a north pole,

Φ_l = the flux leaking from the pole on all sides,

$\Phi_m = \Phi + \Phi_l$ = the total flux in the pole core,

then $\nu = \frac{\Phi_m}{\Phi}$ = the leakage coefficient ....(80).

To determine this leakage coefficient experimentally, a small secondary coil is wound round the pole-core as shown at the bottom of the right-hand core in Fig. 127. The ends of this coil are connected to a ballistic galvanometer. When the exciting circuit is broken the field collapses, and the lines of force cut through this secondary coil. The swing of the ballistic galvanometer is a measure of the flux which has collapsed in this way. The experiment is repeated with the secondary coil wound round the armature so as to enclose the armature flux. The ratio of the swings obtained in the two cases gives the leakage coefficient ν.

As average values we find that

$\nu = 1{\cdot}5$ for the Manchester type,

$\nu = 1{\cdot}36$ for the horse-shoe type,

$\nu = 1{\cdot}1$ to $1{\cdot}2$ for the Lahmeyer and multipolar types.

The leakage is greatest in the Manchester type, because the field coils are on the part of the magnetic circuit most remote from the armature, and the top and bottom masses of iron are of large extent and have the full difference of magnetic potential between them. It has little effect to cut away the metal at *A* (Fig. 128), and so follow a little more closely the natural path of the lines.

The leakage of the horse-shoe type is considerable, since the magnet cores run close together and the polar surfaces are large and cause a lot of external leakage. Rounding the poles, as shown in Fig. 122, reduces the leakage by causing the leakage paths to be longer and less convenient.

The Lahmeyer type (Fig. 129) is more favourable, since the field coils are close to the armature and the leakage paths have consequently a small cross-section.

We must remember, however, that the values given above for ν are only rough approximations, and that the leakage coefficient for machines of the same type may vary considerably. The leakage will be greater, for example, with rectangular poles than with round poles under the same conditions, and greater when unlike poles are near together than when they are far apart. The leakage coefficient is greatly influenced by the magnetic reluctance of the path of the useful flux. If, for example, the air-gap between poles and armature is specially large, a great number of ampere-turns will be required to drive the main flux through this reluctance. This causes a correspondingly large leakage flux, and therefore a high leakage coefficient.

This becomes clearer if we look upon the leakage and useful fluxes as two parallel currents of strengths inversely proportional to the magnetic resistances of the paths. The ratio of the leakage to the useful flux is thus equal to the ratio of the reluctance of the main path to that of the leakage path. It is evident that machines of the same type may have very different leakage coefficients.

On the other hand, the value of ν may vary for the same machine with the conditions of load, etc., since the flux through the armature is not always the same. For the same reason the saturation of the teeth, and therefore their reluctance, varies with the load on the machine. In addition to this, the ampere-turns on the armature itself represent a magnetic back pressure equal to X_b ampere-turns, as we shall see in Section 54. These back ampere-turns increase with the load and necessitate an increased number of ampere-turns on the field, which leads, as we have already seen, to an increased leakage.

This can be taken into account by calculating the magnetic reluctance of the leakage path, if the shape of the machine lends itself to simple geometrical measurement. If A_l and l_l are the cross-section and length of the leakage path, we have seen in Section 29 that its reluctance is

$$R_l = \frac{l_l}{A_l} \quad \text{.................................(81).}$$

If $X_a + X_t + X_g$ ampere-turns are necessary to overcome the reluctance of core, teeth and gap, and X_b ampere-turns are required to overcome the back ampere-turns of the armature, then the magnetomotive force between the poles will be

$$\text{M.M.F.} = 0{\cdot}4\pi\,(X_a + X_t + X_g + X_b).$$

From the formula for the magnetic circuit, we have for the leakage flux

$$\Phi_l = \frac{\text{M.M.F.}}{R_l} = 0{\cdot}4\pi\,\frac{X_a + X_t + X_g + X_b}{R_l} \quad \text{..............(82).}$$

The total flux in the pole-cores is then $\Phi_m = \Phi + \Phi_l$. An example of such a calculation of the leakage is given in Section 29. Too much reliance, however, must not be placed on such calculations, since they are only roughly approximate.

When the machine does not lend itself to geometrical measurement the matter is more complicated and the results even less trustworthy. In multipolar machines it is safe to assume that the polar arc is 70 per cent. of the pole-pitch, so that the interpolar space is 30 per cent. of the pitch. The length l_l of the leakage path is accordingly proportional to the pitch $\frac{\pi D}{2p}$. The cross-section A_l of the leakage path is certainly proportional to the axial length L and roughly proportional to the radial length of the pole. This latter may be taken as approximately proportional to the pole-pitch, so that the cross-section of the leakage path is roughly proportional to

$L \cdot \frac{D\pi}{2p}$. In the same way as we find the electrical resistance from the formula $\rho \frac{l}{A}$, we get the magnetic reluctance to be

$$R_l = c_1 \cdot \frac{D\pi/2p}{L \cdot D\pi/2p} = \frac{c_1}{L}.$$

c_1 is here a factor having different values depending on the type of machine. It might be called the specific reluctance of the leakage path. Kapp uses the geometrical mean of the armature length and the pole-pitch instead of the armature length L. This alters the above formula to the following:

$$R_l = \frac{c}{\sqrt{\frac{D}{p} \cdot L}} \quad \text{.............................(83),}$$

where c is a constant for which Kapp gives the following values:

Manchester type ... ... $c = 0{\cdot}15$,
Horse-shoe type ... ... $c = 0{\cdot}36$,
Multipolar or Lahmeyer type ... $c = 0{\cdot}35$—$0{\cdot}55$.

If, for example, the diameter of the armature is 60 cm, its length 40 cm, the pairs of poles 2, and the value of c be estimated at 0·35, we get

$$R_l = \frac{c}{\sqrt{\frac{D}{p} \cdot L}} = \frac{0{\cdot}35}{\sqrt{1200}} = 0{\cdot}01.$$

If the number of ampere-turns required to drive the main flux from pole to pole be 10,000, we get for the leakage flux

$$\Phi_l = 0{\cdot}4\pi \frac{10{,}000}{0{\cdot}01} = 1{\cdot}26 \cdot 10^6 \text{ lines.}$$

We must remember here again that such calculations of the leakage are only rough approximations, but are generally sufficiently accurate for ordinary design.

53. Position of the Brushes.

We saw in Section 43 that the current in each division of the armature flowed towards the neutral zone. The brushes must therefore be placed in the neutral zone, or rather, since the commutator connections are often bent, in such a position that a coil passes from one division of the armature to another while it is in the neutral zone. If the brushes are placed in the wrong position the individual coils of a branch of the armature winding lie under opposite poles and are therefore the seat of opposing electromotive forces, which neutralise each other and lower the electromotive force of the machine. The correct position of the brushes is on the diameter at right angles to the lines of force through the armature. This is called the neutral axis.

If the brushes are displaced, not only is there a loss of electromotive force, but the coil, or coils, short-circuited by the brush are moving in a magnetic field and are therefore the seat of an induced E.M.F. The resistance of the short-circuited coil is very small, and the current produced may be very large. When the short-circuit is broken by one of the segments leaving the brush there is a vicious spark which rapidly destroys the commutator. Hence, to obtain sparkless commutation, it is necessary that the brushes be placed more or less in the neutral axis.

We find, however, that when the armature carries a current, the magnetic flux through it is distorted and twisted round in the direction of rotation. The neutral axis found at no-load, which we may call the geometrical neutral axis, since it is geometrically symmetrical, is no longer the magnetic neutral axis or the point where no E.M.F. is induced in the moving armature coil. This true magnetic neutral axis is twisted round with the field in the direction of rotation This is shown in Fig. 131; it can be shown experimentally by means of iron filings.

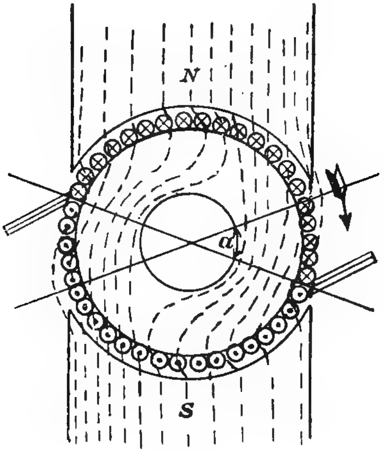

Fig. 131

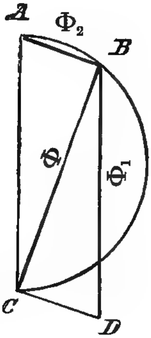

Fig. 132

The direction of the magnetic neutral axis can be found by considering that the armature itself is an electromagnet tending to drive lines of force through its interior in the direction of the diameter through the brushes. The resultant flux Φ is compounded of the flux Φ_2 produced by the armature and the flux Φ_1 produced by the field magnets.

By applying either the swimming or the screw rule, the direction of Φ_2 is found to be *BA*, while that of Φ_1 is evidently *BD* (Fig. 132). Since, for the correct position of the brushes, the brush-axis is perpendicular to the resultant flux Φ, and Φ_2 acts necessarily along the brush-axis, it follows that the angle *ABC* is a right angle.

If α represents the double displacement angle (Fig. 131), the angle through which the brush is moved is $\frac{\alpha}{2}$, and we have

$$\sin\frac{\alpha}{2}=\frac{\Phi_2}{\Phi_1}.$$

With the brushes in this position the machine gives its highest terminal pressure for the given armature current. It is important to notice that the reluctance of the path of the lines generated by the armature is very different to that of the lines generated by the field-magnets. A calculation of the theoretically correct position of the brushes by means of the above formula is practically impossible. In practice, however, the brushes are pushed forward until they are almost under the pole-tip. The current in the coil under the brush has to reverse its direction during the time of short-circuit. This reversal is retarded by the self-induction of the coil (see Section 35). The brushes of a dynamo are therefore moved beyond the magnetic neutral axis (Fig. 133), so that an E.M.F. is induced in the short-circuited coil in such a direction as to neutralise the effect of self-induction and make the commutation sparkless. The small loss of E.M.F. thus caused is quite negligible.

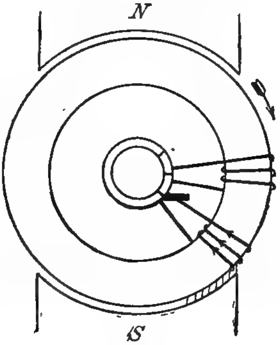

Fig. 133

54. Armature Reaction.

If the brushes are placed in the neighbourhood of the pole-tip, the whole armature winding may be looked upon as consisting of two parts, of which the first lies within the angle α (Fig. 131) and the second within the remaining angle $180° - \alpha$. The windings constituting the first part lie within the neutral zone while those of the second part lie mainly under the poles. As can be seen from Fig. 134 the windings in the neutral zone are directly opposed to the field magnets and tend to produce a flux in the direction of the dotted arrows. This can also be seen from Fig. 132 where the armature flux Φ_2 has a component acting in the opposite direction to the magnet flux Φ_1. The actual resultant flux, which passes from pole to pole through the armature, is due to the difference between the field ampere-turns and these back ampere-turns X_b of the armature. If Z is the total number of wires on the armature, the number of wires included in the angle α is $\frac{Z \cdot \alpha}{360}$ (Fig. 134). This is therefore the number of back armature turns. If there are $2a$ parallel paths through the armature, the current in each armature wire is $\frac{I_a}{2a}$, and for the back ampere-turns we have

$$X_b = \frac{I_a}{2a} \cdot \frac{Z \cdot \alpha}{360} \quad \text{..............................(84).}$$

This equation is equally applicable to series or parallel windings, drum or ring armatures. In the ring armature shown in Fig. 135 the coils in each neutral zone embrace the single cross-section of the ring, and a pair of opposite coils could be replaced by a single coil embracing the whole armature as in a drum-winding. In either case two external wires represent one turn opposing the field magnets. The effect of the armature back ampere-turns is to decrease the flux as the load comes on the dynamo, and thus cause a decrease in the electromotive force. In calculating the necessary ampere-turns for the field magnets when designing a machine, it is necessary to add the back ampere-turns X_b to those required to overcome the reluctance of the iron and air-gap. As a rule X_b is from 10 to 15 per cent. of the total excitation, but as modern dynamos work with highly saturated iron, that is, on the flat part of the magnetisation curve (Fig. 150), a large number of back ampere-turns causes but a small drop in the electromotive force. The demagnetising effect is largely reduced by the use of short-chord windings (p. 122).

We turn now to the remaining windings lying within the angle 180° − α and practically all under the poles. These produce what are known as cross ampere-turns, since they tend to produce a flux through the armature at right angles to the main flux. These lines of force may be looked upon as leakage; they follow paths through the air-gap and pole as shown in Fig. 136. They have their greatest density under the pole-tips, since the lines which cross the gap at this point link all the coils lying under the pole. At the point where the armature passes under the pole they oppose the main flux, while at the point where the armature leaves the pole they act in the same direction as the main flux. The cross ampere-turns do not therefore directly either strengthen or weaken the armature flux, but merely distort it, as shown in Fig. 131. Now, the pole-tip at which the armature passes under the pole is of special importance, since it is in this neighbourhood that the short-circuited coil has to find a suitable external field to reverse its current and give sparkless commutation (see Section 55). If, then, the cross-magnetisation is too strong, the field under this pole-tip will be too weak, or may even become negative, making sparkless commutation impossible. As a rough approximation, it may be assumed that a flux density of 1500 lines per sq. cm is necessary in the case of drums and 2500 in the case of ring-armatures to ensure sparkless commutation.

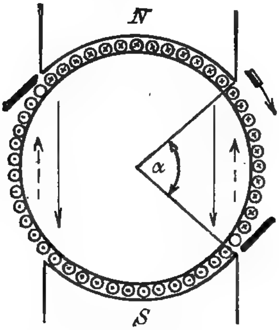

Fig. 134

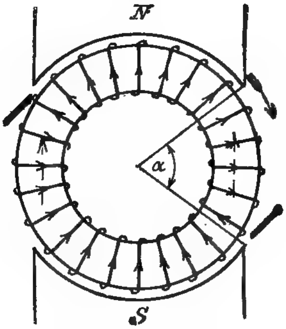

Fig. 135

The fact that a stronger field is required for the commutation of a ring-armature does not indicate that the cross-magnetisation is greater in this case. The reason is to be found in the greater self-induction of the ring-winding. To overcome this greater self-induction during the interval of short-circuit the ring-armature requires a stronger commutating field than the drum-armature.

If B_g represents the mean flux-density in the air-gap and B_c the flux-density under the pole-tips due to the cross-magnetisation, then the resultant flux-density under the commutating tip is

$$B = B_g - B_c.$$

Now, the ampere-turns of cross-magnetisation are

$$X_c = \frac{Z \,.\, \beta}{2} \,.\, \frac{I_a}{2a} \quad \text{..............................(85).}$$

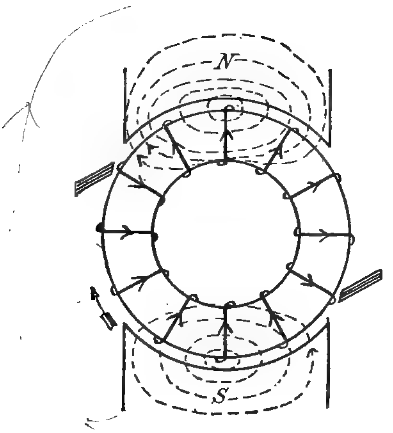

Fig. 136

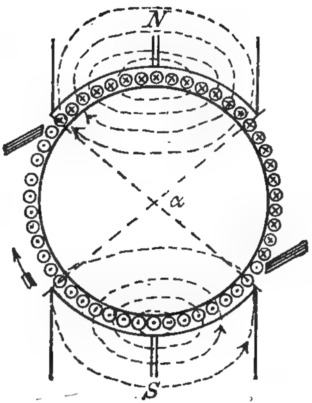

Fig. 137

The principal reluctance in the path of the cross ampere-turns is that due to the air-gap, and, neglecting the reluctance of the iron, we have from equation (43) on page 61

$$B_c = \frac{0{\cdot}4\pi \,.\, X_c}{l_g},$$

where l_g is double the air-gap in cm.

Combining this with the equation

$$B_g = \frac{0{\cdot}4\pi \,.\, X_g}{l_g}.$$

we have for the actual strength of the commutating field

$$B = B_g - B_c = 0{\cdot}4\pi \,.\, \frac{X_g - X_c}{l_g}.$$

The empirical conditions for sparkless commutation, which were mentioned above, may therefore be expressed as follows:

$$0{\cdot}4\pi \,.\, \frac{X_g - X_c}{l_g} \geqq 1500 \text{ for drum-armatures} \quad \text{...........(86),}$$

$$0{\cdot}4\pi \,.\, \frac{X_g - X_c}{l_g} \geqq 2500 \text{ for ring-armatures} \quad \text{..............(87).}$$

We see from these equations that the influence of the cross ampere-turns increases with the armature current, and that we have here a limit to the overload capacity of the dynamo. This limit is not merely due to the fact that very large loads would cause a prohibitive temperature rise in the armature, but is fixed by the number of cross ampere-turns beyond which sparkless commutation is impossible. In designing a dynamo it is necessary above all things to keep the cross-magnetisation small, and to this end the following means are adopted:

1. From equation (85) it can be seen that a small value of the ratio β is an advantage. As a rule the polar-arc is about $\frac{2}{3}$ of the pole-pitch.

2. It is an advantage to have a small number Z of wires on the armature. To obtain the necessary electromotive force, this will necessitate a large magnetic flux.

3. The high flux-density in the air-gap thereby rendered necessary (8000—10,000) will lead to a large value of X_g, which, again, will make it easier to fulfil the equations (86) and (87).

4. The value of X_g can be increased by increasing the air-gap, the two having a constant ratio. Since X_c is thereby unaffected, the result is to increase the left-hand side of equations (86) and (87). It must be borne in mind, however, that large fluxes and long air-gaps necessitate a great number of field ampere-turns.

5. By making the teeth narrow, so that their flux-density is very high (18,000—24,000), their permeability is largely reduced. This is practically equivalent to an increase in the air-gap, and its effect is the same as in number 4*. The common idea that the highly saturated teeth reduce the self-induction of the short-circuited coil is entirely wrong, since, when the coil is short-circuited, the teeth on either side of it are not saturated.

6. Attempts have been made to reduce the cross-magnetisation by making a slit across the pole as shown in Fig. 137. These attempts have not proved successful even when the slit was extended so as to entirely divide the magnet system. The failure is possibly due to the large area and narrow width of the slit, whereby its reluctance to the cross-flux is very small.

7. The cross-magnetisation is neutralised in a very effective manner by means of auxiliary poles placed in the centre of the neutral zone. These poles are excited by the main current, and produce a commutating field the strength of which is proportional to the armature current, that is, to the current which has to be reversed in the short-circuited coil. The brushes can therefore be fixed in the geometrical neutral axis. This idea, originally suggested by Menges, has been developed by Swinburne, Fischer-Hinnen, and, latterly, by many others†. In Fig. 138 the cross-magnetisation acts from right to left, while the magnetomotive force of the auxiliary poles acts from left to right. Machines of this type can be very heavily overloaded without causing sparking at the brushes, and the maximum output is limited solely by the heating of the armature. The use of auxiliary poles is extending very rapidly, since the designer can then economise in many directions without fear as to the commutation, and may thus more than compensate for the cost of the auxiliary poles.

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* See Fischer-Hinnen, *Gleichstrommaschinen*, 5th edit., 1904.

† See several articles in the *Elektrotechnische Zeitschrift*, 1905–1906.

Another method, due to Déri, is theoretically more perfect than the foregoing, but is more difficult to carry out. The field magnet has no salient

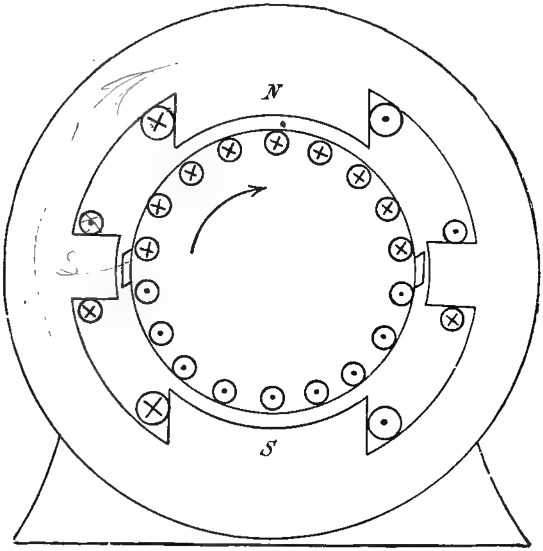

Fig. 138

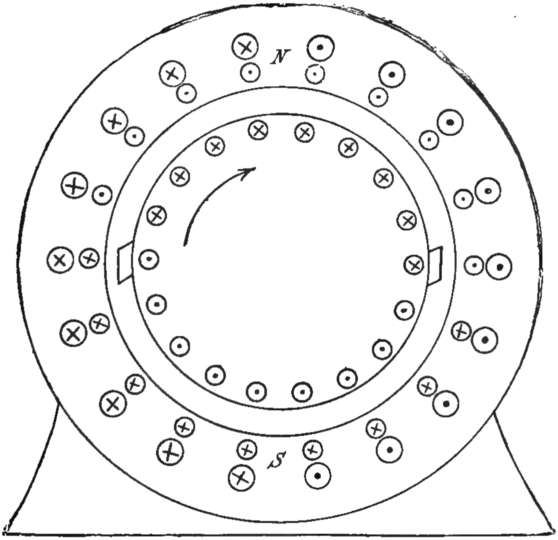

Fig. 139

poles, but, like the stator of an induction motor, is a ring carrying a drum-winding (Fig. 139). This drum-winding is represented by the outer ring in the figure, and produces a field running vertically through the armature from top to bottom. In the same slots is the compensating winding, which carries the main current, and which would produce a field running horizontally from left to right. This field is, however, exactly equal and opposite to that which would be produced by the cross-magnetisation. The latter is therefore completely neutralised. By making the axis of the compensating winding slightly inclined to the horizontal in the direction of rotation, it can be made to strengthen the main field, and thus to counteract the drop of pressure with increasing load. The dynamo is therefore compounded*.

55. Sparkless Commutation.

In order that we may clearly understand the process of commutation and find the conditions for sparkless running, we shall confine ourselves to the simplest case, in which a brush is never in contact simultaneously with more than two segments. We shall also neglect the influence of neighbouring wires which may be undergoing commutation at the same time under another brush. We shall consider

1. the reversal of the current in the short-circuited coil under the influence of the brush contact resistance, that is, neglecting self-induction;
2. the effect of self-induction;
3. the conditions for sparkless commutation in the magnetic neutral axis;
4. the neutralisation of the effect of self-induction by shifting the brushes.

1. The coil is assumed to have no self-induction, and commutation is assumed to occur in the magnetic neutral axis. The width of the brush is taken to be that of one segment, and the resistance of the coil and commutator connections is neglected because of its smallness. We shall make use of the following symbols:

I	the current in one branch of the armature,
t	the time, reckoned from the moment of short-circuit,
T	the duration of short-circuit,
i	the short-circuit current at the moment t,
i_1	the current in the segment coming under the brush,
i_2	the current in the segment leaving the brush,
R	the contact resistance of the whole brush,
r_1	the resistance between brush and segment coming under it,
r_2	the resistance between brush and segment leaving it.

If we neglect the variation of contact resistance with current density, the resistances r_1 and r_2 are inversely proportional to the contact surfaces. We have then

$$r_1 = R \cdot \frac{T}{t}, \qquad r_2 = \frac{R \cdot T}{T - t}.$$

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* See Arnold, *Die Gleichstrommaschine* (1902), vol. 1. p. 405.

The current $2I$ flows from the external circuit to the brush and there divides into two unequal portions. The current flowing through the resistance r_2 is seen from Fig. 140 to be made up of the current I of the right-hand half of the armature and the current i of the short-circuited coil. We have then

$$i_1 = I - i \quad \text{and} \quad i_2 = I + i.$$

These equations will still remain true when the current i has been reversed and become negative. Applying Kirchhoff's second rule to the short circuit, and neglecting the small resistance of the coil itself, we have

$$- i_1 r_1 + i_2 r_2 = 0.$$

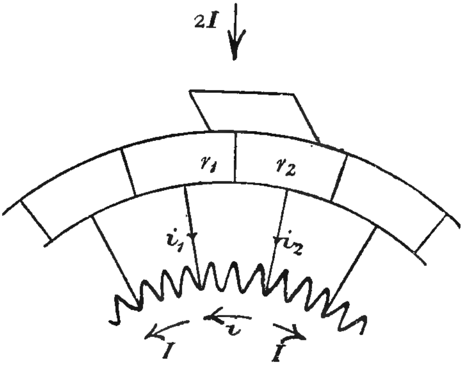

Fig. 140

Substituting the values found above for i_1, i_2, r_1 and r_2, we have

$$- (I - i)\frac{R \, . \, T}{t} + (I + i)\frac{R \, . \, T}{T - t} = 0.$$

By solving this equation for i, we find that

$$i = I - 2I \, . \frac{t}{T} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

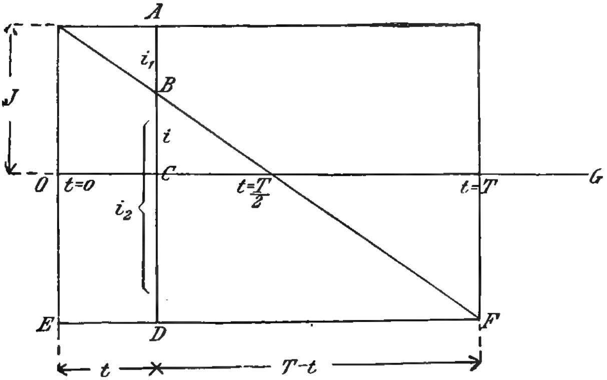

Fig. 141

In Fig. 141 the time t is plotted along the horizontal axis OG, while the corresponding values of the short-circuit current i are set up as ordinates. From equation (a) we see that, when

$$t = 0, \qquad i = I,$$
$$t = \frac{T}{2}, \qquad i = 0,$$
$$t = T, \qquad i = - I.$$

We thus get the straight diagonal line shown in the figure. At any time t from the commencement of short-circuit, represented by OC, we have

$$i = BC, \quad i_1 = I - i = AB, \quad i_2 = I + i = BD.$$

The ordinates measured between the horizontal line EF and the diagonal represent the current i_2 in the trailing tip of the brush. Now, the area of contact of the trailing tip is proportional to $T - t$. The current density under this brush tip is therefore proportional to $\frac{i_2}{T - t}$, which we see from the figure is a constant ratio equal to $\frac{2I}{T}$. We see, then, that the current density under the brush is constant at every stage of the commutation. There is therefore no sparking since the current in the trailing tip decreases proportionately with the decreasing area, until as the brush leaves the segment the current has fallen to zero. At the same moment the current in the coil is $-I$, so that the coil passes into the new branch of the winding without undergoing any change.

2. As a matter of fact the conditions are not nearly so favourable, as the short-circuited coil produces a flux, which, on the collapse of the old current, and the establishment of the new reverse current, cuts the coil itself and induces in it an E.M.F. opposing the change of current. This retardation of the process of commutation, due to the self-induction of the coil, causes the current to depart from the straight line or uniform rate of change, and follow a curve as shown in Fig. 142. The short-circuit current no longer passes through its zero value at the time $\frac{T}{2}$, but much later.

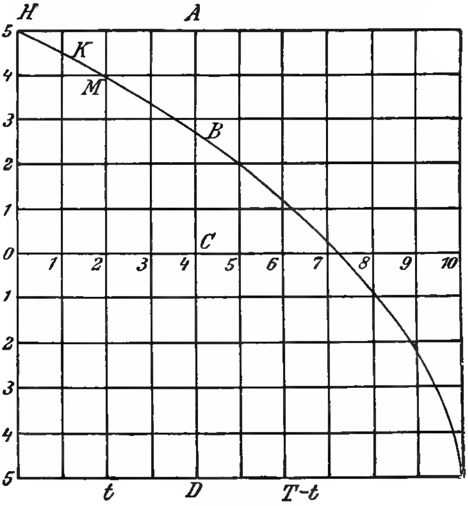

Fig. 142

To find the shape of the above curve we apply Kirchhoff's second law to the short-circuit, taking into account its self-induction. We know that the E.M.F. of self-induction E_s is in the direction of the decreasing current, that is, at the moment shown in Fig. 140, in the direction of i_2. We have then

$$E_s = i_2 r_2 - i_1 r_1.$$

Substituting for E_s the equivalent $-L \cdot \frac{di}{dt}$ from equation (68) on page 88, and putting in the values found for r_1 and r_2 on page 154, we have

$$-L \cdot \frac{di}{dt} = i_2 \cdot \frac{R \cdot T}{T - t} - i_1 \cdot \frac{R \cdot T}{t}.$$

It follows from this that

$$\frac{di}{dt} = -\frac{R \, . \, T}{L}\left(\frac{i_2}{T-t} - \frac{i_1}{t}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

where $i_2 = I + i$ and $i_1 = I - i$. The solution of this differential equation has been given by Arnold. We shall obtain the short-circuit curve, however, by a graphical method*. For the sake of simplicity, we shall assume that

$$T = 10, \quad I = 5, \quad L = 1, \quad R = \tfrac{1}{10},$$

so that

$$\frac{R \, . \, T}{L} = 1.$$

In the first place we shall determine the slope of the curve at its beginning, i.e. for $t = 0$. Assuming that this part of the curve is straight, we see from Fig. 142 that

$$-\frac{di}{dt} = \frac{i_1}{t}.$$

Now, when $t = 0$, $i_2 = 2I$ and $T - t = T$, so that equation (*b*) becomes

$$\frac{i_1}{t} = \frac{R \, . \, T}{L}\left(\frac{2I}{T} - \frac{i_1}{t}\right),$$

or

$$\frac{i_1}{t} = \frac{2I \, . \, R}{L\left(1 + \dfrac{R \, . \, T}{L}\right)}.$$

This equation gives us the slope of the curve for the moment $t = 0$. In our case we have

$$\frac{i_1}{t} = \frac{10 \, . \, \frac{1}{10}}{1 + \frac{1}{10} \, . \, 10} = \tfrac{1}{2}.$$

We draw then from the point H in Fig. 142, a straight line having this inclination to the horizontal and cutting the ordinate $t = 1$ in the point K. HK is the first part of the curve, and we must now determine the next part. When $t = 1$, we have

$$i_1 = 0{\cdot}5, \quad i_2 = 9{\cdot}5, \quad t = 1, \quad T - t = 9.$$

From equation (*b*) we have

$$\frac{di}{dt} = -1\left(\frac{9{\cdot}5}{9} - \frac{0{\cdot}5}{1}\right) = -0{\cdot}555.$$

This, then, is the trigonometrical tangent of the angle made with the horizontal by the geometrical tangent to the curve at the point K. We draw the line KM having this slope, and have now, with sufficient approximation, got the curve from H to M. At the point M we have

$$i_1 = 1{\cdot}055, \quad i_2 = 8{\cdot}945, \quad t = 2, \quad T - t = 8.$$

From these values we can calculate $\frac{di}{dt}$ and draw the next portion of the curve, and so on, until the whole curve is obtained. At any given moment we have $AB = i_1$, $BC = i$, $BD = i_2$. The value of i finally becomes negative. It is evident that the current changes very rapidly towards the end of the short-circuit.

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* See Rothert, *E. T. Z.*, 1902, p. 865.

3. To determine the conditions for sparkless commutation in the magnetic neutral axis we shall find the current density under the trailing brush-tip. We have already mentioned that the current density is inversely proportional to the contact area, that is, to $T-t$.

We have then, neglecting a constant coefficient,

$$c=\frac{i_2}{T-t},$$

where c is the current density.

During the last moments of commutation, when t is nearly equal to T, the value of i_1 will approach $2I$, since practically the whole current passes through the new segment. If we assume that the end of the curve is a straight line making the angle α with the horizontal, we have

$$\tan\alpha_{(t=T)}=-\frac{di}{dt}=\frac{i_2}{T-t}.$$

At this final moment $$c=\frac{i_2}{T-t}=-\frac{di}{dt},$$

or, using equation (*b*) and putting $\frac{i_1}{t}=\frac{2I}{T}$,

$$c=\frac{R.T}{L}\;c-\frac{2I}{T}\Big).$$

Solving for c, we have $$c=\frac{2I.R}{R.T-L} \quad\text{..............................(88).}$$

It follows from this equation that with commutation in the neutral zone the current density under the trailing brush-tip can only then have a positive finite value, when $\frac{R.T}{L}$ is greater than 1. This result is found to agree very well with the conditions for sparkless commutation derived from actual practice when carbon brushes are employed. With copper brushes, however, the resistance R is very small and the above condition cannot be fulfilled, but yet the machines run sparklessly. This may be due to a large increase in the contact resistance with a slight and inappreciable amount of sparking.

From the condition $$\frac{R.T}{L}>1,$$

we see, in the first place, that, to obtain sparkless running, it is advantageous to have large brush contact resistance, and perhaps even to increase the resistance of the short-circuited coil by the use of high-resistance connectors between armature and commutator. Carbon brushes are consequently used in almost every case in preference to those of copper. In electrolytic dynamos, however, where the currents are large and the pressures very small, carbon brushes cannot be used on account of the pressure drop they introduce.

We see further the unfavourable effect of high speed in reducing the time available for commutation. This can be overcome to some extent by making the brush wider and thus lengthening the period of commutation. The fact that several neighbouring coils will then be short-circuited at the same time considerably complicates the problem.

The effect of high speed is specially felt in designing turbo-dynamos, where the great frictional loss with carbon brushes makes the use of copper brushes desirable. The difficulties have, however, been largely overcome by the use of commutating poles and compensating windings.

It is of prime importance to reduce the self-induction of the coils as much as possible. To this end the armature slots are open and coils of a single turn used wherever possible.

To calculate the self-induction of the coil, it is necessary to multiply the flux produced per ampere by the number of turns in the coil and by the number of such coils short-circuited in series. Very often another conductor in the same slot and carrying current in the same direction is short-circuited at the same time under a brush of opposite polarity. This has the effect of doubling the inductance of the part of the coil within the slot. The calculation is greatly simplified by making an assumption due to Hobart, who showed that the flux produced per cm per ampere was about 4 lines within the slot and about 0·8 line in the end-connections.

The effect on the above relations of making the brush wider than a single segment can hardly be followed mathematically. Hobart assumes that the self-induction increases proportionately with the number of coils simultaneously short-circuited. This, however, hardly appears compatible with the good results obtained by using wide brushes. These good results are due to the lengthening of the time T and to the effect of a neighbouring short-circuited coil in reducing the self-induction of the active coil. This latter effect is due to the currents induced in the neighbouring short-circuited coil, which by Lenz's law oppose any rapid change of flux.

If the inequality $\frac{R.T}{L} > 1$ be multiplied by I, we get the condition

$$2I.R > \frac{2I.L}{T}.$$

The left-hand side represents the ohmic pressure-drop in the contact resistance of the brush when the current is uniformly distributed, while the right-hand side is equal to the average electromotive force of self-induction of the short-circuited coil (see p. 89). The condition for sparkless commutation is therefore that the ohmic drop should be greater than the mean E.M.F. of self-induction.

It is customary, however, to discuss the problem from the point of view of another E.M.F., the so-called reactance voltage. This is determined by means of a formula which we shall use in connection with alternating currents, viz.

$$E_r = \omega.L.I.$$

I is the current in one branch of the armature winding, and $\omega = 2\pi f$ is the angular velocity of the alternating current vector. f is the number of complete periods per second, and since in our case a half-period occupies the time T, the number of periods per second is given by the equation

$$f = \frac{1}{2T}.$$

The reactance voltage, calculated in this way, should not exceed 1 volt for sparkless commutation in the neutral zone. There is, however, much uncertainty in such predictions as to whether a machine will run sparklessly or not, and we shall not enter into an actual numerical example.

4. The conditions just established for sparkless running refer to machines with their brushes in the neutral zone, such, for example, as reversible motors. The brushes of generators are displaced in the direction of rotation, so that commutation takes place in a field of suitable strength (Fig. 133). We have already seen that this commutating field induces an electromotive force E_k, which opposes and counteracts the electromotive force E_s of self-induction.

To show the effect of the commutating field we shall consider two specially simple cases. We saw above under (1) that a constant current density under the brush, that is, a straight line short-circuit curve, is advantageous. We shall assume that, under the joint action of the electromotive forces E_s and E_k, the current curve becomes linear. We see then from Fig. 141 that the constant value of $\frac{di}{dt}$ is $\frac{2I}{T}$, and we have, at every moment,

$$E_s = -L \, . \frac{di}{dt} = -L \, . \frac{2I}{T} \, .$$

Hence, to obtain a straight line short-circuit curve and constant current density under the brush, the externally induced electromotive force E_k must be constant, that is, the short-circuited coil must move in a field of constant strength during the whole time of commutation. If this is so it is evident that the self-induction may be as large as we please. As a matter of fact, however, such a field cannot be obtained, and the strength of the field, in which commutation occurs, varies with the load, owing to the distortion produced by armature reaction.

It would evidently be advantageous if the current density under the trailing brush-tip were already reduced to zero at the moment of leaving the segment. To find the condition for this we apply Kirchhoff's second law:

$$E_s - E_k = i_2 . r_2 - i_1 . r_1 .$$

We make the following substitutions from our previous considerations:

$$E_s = -L \, . \frac{di}{dt} \text{ and } i_2 r_2 - i_1 r_1 = R \, . T \, . \left(\frac{i_2}{T-t} - \frac{i_1}{t} \right) .$$

At the last moment of short-circuit we have

$$\frac{i_2}{T-t} = -\frac{di}{dt} \text{ and } \frac{i_1}{t} = \frac{2I}{T} .$$

Our expression of Kirchhoff's law may therefore be written thus:

$$L \, . \frac{i_2}{T-t} - E_k = R \, . T \, . \left(\frac{i_2}{T-t} - \frac{2I}{T} \right) .$$

Solving for $\frac{i_2}{T-t}$, which is the current density s, we get

$$s = \frac{2IR - E_k}{R . T - L} \quad \text{.............................(89).}$$

This equation shows that the current density is infinitely large when $\frac{RT}{L} = 1$, even when commutating in an external field. There is, however, an exception, and that is when $E_k = 2IR$.

For $\frac{RT}{L} > 1$ the current density under the trailing edge of the brush is zero, if the externally induced electromotive force E_k becomes equal to $2IR$, that is, exactly equal to the pressure drop in the brushes. Whether this condition can also be obtained when $\frac{RT}{L}$ is less than one is not quite certain, although equation (89) certainly points to this conclusion*.

The whole question depends, therefore, on the production of a commutating field of suitable strength. The most obvious way of obtaining this is to suitably shape the pole-shoes. The passing of a coil-side from the neutral zone into the active field must be gradual. Bad sparking has often been cured simply by rounding the edges of the poles. Skewing the sides of the pole shoes so that they no longer lie parallel to the armature slots appears to facilitate the correct placing of the brushes. Dolivo-Dobrowolsky's idea of connecting up all the pole-shoes by means of an iron ring encircling the armature appears to be obsolete. By this means the change in the strength of the field at the pole-tips was made very gradual, but the commutating field was very sensitive to armature reaction. It was therefore very satisfactory when the brushes could be shifted with changing load.

The requirements of modern specifications, whereby machines must run sparklessly from no-load up to 25 per cent. overload without shifting the brushes, have greatly increased the difficulties. As a fact, most machines will work sparklessly at half-load, but at no-load and full load, with the brushes in the same position, the commutation depends largely on the effect of the cross-magnetisation (see page 152).

56. Three-wire Dynamos.

The advantages of the three-wire system of distribution were pointed out on page 25. At the same time we mentioned the earliest and simplest method of obtaining the intermediate pressure, viz. by connecting two dynamos in series. Since, however, two small machines are dearer than a single large one, a number of other methods have been tried and are in use to-day.

It is simple to obtain an intermediate pressure when a battery works in parallel with the dynamo. The middle or neutral wire is then joined to the mid-point of the battery. This arrangement has, however, the disadvantage of unequally working the two halves of the battery.

The method most commonly adopted is to connect two similar motors, which are rigidly coupled together, in series and to connect the neutral wire to their common terminal. Such a set is called a balancer (Fig. 143.). When both sides of the system are equally loaded, the two machines run light, and with similar construction and equal excitation the pressure across the mains will be equally shared between them. If now the two sides of the system are unequally loaded, as shown in the figure, the machine I will act as a generator and deliver the extra current to the positive, overloaded side of the system. The motor II will take enough current to drive the generator I and to cover its own losses. To make the action of the set as clear as possible we shall assume that the ohmic resistance of the machines is negligible. If the no-load current of each motor is I_0 and the pressure across each half of the mains e volts, then the no-load power taken by the balancer set is $2e.I_0$. If now the machine I acts as a generator and supplies a current x, the load on it is $e.x$. The power required by the motor II is therefore $2e.I_0 + e.x$. The current through it will be $2I_0 + x$. From Kirchhoff's first law we know then that the current in the middle wire is $2I_0 + 2x$. This must be the difference between the currents I_1 and I_2 in the outers. We have then

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* The validity and meaning of the above equations have been warmly discussed in the *E. T. Z.*, 1906.

$$I_1 - I_2 = 2I_0 + 2x,$$

or

$$x = \frac{I_1 - I_2}{2} - I_0.$$

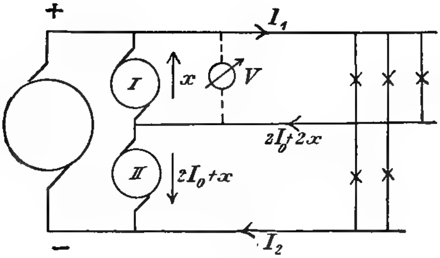

Fig. 143

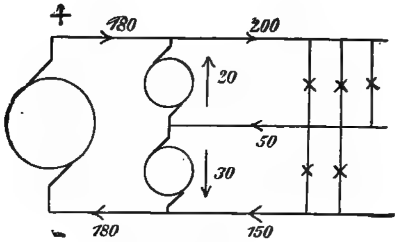

Fig. 144

If, for example, $I_1 = 200$, $I_2 = 150$, $I_0 = 5$, then the current in the generator I will be

$$x = \frac{200 - 150}{2} - 5 = 20.$$

The machine II will take the current

$$x + 2I_0 = 20 + 10 = 30,$$

and the current in the main dynamo (Fig. 144) will be 180 amperes.

As we shall see in Section 64, a back E.M.F. is induced in the motor, which, if we neglect the armature resistance, is equal and opposite to the terminal pressure. Since the machines I and II are similar, and have the same excitation and speed, it follows that their electromotive forces are the same. Hence, their terminal pressures are equal and the balance of pressure is perfect.

As a matter of fact, however, the result will be affected by the armature resistance, which is not negligible. The terminal pressure of the generator will be given by the formula

$$V = E - I_a.R_a,$$

and that of the motor by the formula

$$V = E + I_a.R_a.$$

As the generator is connected across the more heavily loaded side, this side will have the lower pressure. The difference can be diminished by exciting each machine of the balancer set off the opposite side of the system. The field coils of the generator would then be supplied from the pressure of the lightly loaded side, thus increasing its E.M.F., while the E.M.F. of the motor would be decreased. The same result can be obtained by passing the middle-wire current round a few series-turns on the fields of each machine.

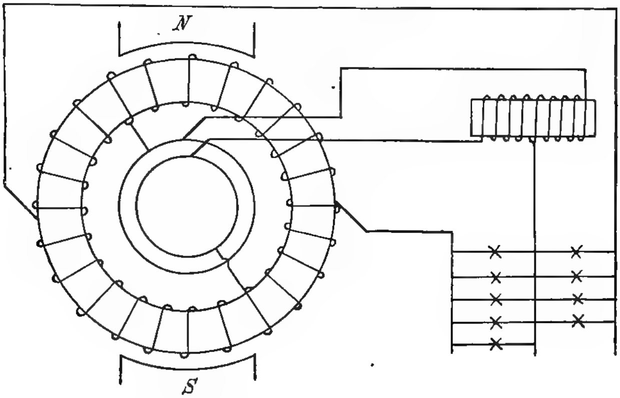

Fig. 145

A simple and interesting method of obtaining a middle point is that due to Dolivo-Dobrowolsky (Fig. 145). Two points of the winding, separated from each other by a pole-pitch, are connected to two slip-rings. By means of brushes these rings are connected to the terminals of a so-called choking coil, a coil of large self-induction wound on an iron core. An alternating current flows through this coil and produces lines of force, which cut through the turns of the coil and induce electromotive forces opposing the terminal voltage. The alternating current passing through the coil is therefore only large enough to produce this flux, and is practically negligible. The coil has a very small resistance to continuous currents. If now the middle wire is connected to the middle point of the coil, the pressure difference between it and each of the outer wires is a half of the terminal pressure of the dynamo. A more perfect arrangement was proposed by Sengel, who connected three points of the winding, separated by $\frac{2}{3}$ of the pole-pitch, to three slip-rings. The three brushes were connected to the terminals of three choking coils, the other terminals of which were connected together and to the middle wire (compare three-phase star connection).

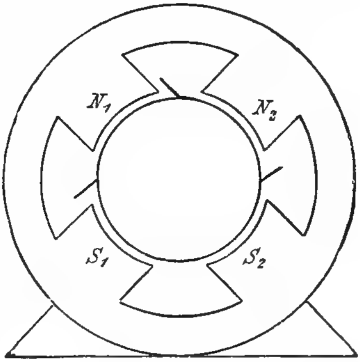

Fig. 146

Dettmar solved the problem in another manner by connecting the middle wire to an auxiliary brush placed on the commutator halfway between the main brushes (Fig. 146)*. As this auxiliary brush short-circuits two segments whenever it bridges the insulation between them, it is necessary that the coil thus short-circuited should be moving in a weak field. This is arranged by leaving a gap in the middle of the pole. The figure represents really a two-pole machine, the north pole of which is split into two parts N_1 and N_2 with a gap between them. In order that the pressure of either side of the system may be regulated without interfering with the other, the field coils of the poles N_1 and S_2 must be connected in series and controlled separately from the field coils of the poles N_2 and S_1, which are likewise in series, since diametrically opposite wires on the armature are connected in series.

It can be easily seen that the cross-magnetisation of the armature will lower the pressure of one half of the system and raise that of the other. As in the case of the balancer, it is advantageous to excite the field magnets of one side from the pressure of the other side of the system.

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* See *E. T. Z.*, 1897, pp. 55 and 230.

CHAPTER VIII

57. Characteristics of separately-excited dynamo.—58. Characteristics of series dynamo.—59. Characteristics of shunt dynamo.—60. Dynamo and battery in parallel.—61. Constant-current variable-speed dynamos.—62. Efficiency of dynamos.

57. Characteristics of Separately-excited Dynamo.

In this machine the field magnets are excited from an external independent source (Fig. 147). In the first place we shall consider the effect of varying the speed when the machine is running light, i.e. unloaded, with constant excitation. From equation (79), on page 124, the electromotive force is known to be

$$E = \frac{p}{a} . \Phi . n . Z . 10^{-8} \text{ volts.}$$

Since, under the assumed conditions, n is the only variable on the right-hand side of the equation, the electromotive force must be proportional to the speed. If, then, we plot the observed values of the speed as abscissae and the corresponding values of the electromotive force as ordinates, we shall obtain a straight line through the origin, as shown in Fig. 148.

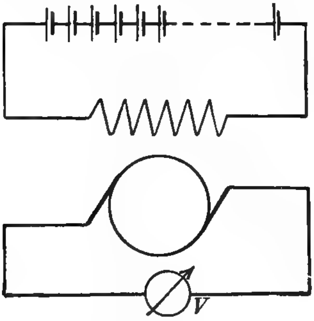

Fig. 147

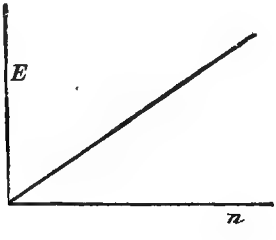

Fig. 148

If now we vary the experiment by keeping the speed constant and varying the exciting current by means of a resistance in series with the field winding (Fig. 149), and plot the observed electromotive forces against the corresponding values of the exciting current, we obtain the curve shown in Fig. 150. This curve is called the no-load characteristic, or magnetisation curve. The electromotive force increases proportionately with the magnetising current up to a certain point, beyond which, however, the electromotive force increases less rapidly owing to the increasing saturation of the iron.

The values of the ordinates are obtained by connecting a voltmeter across the dynamo terminals. At no-load the P.D. between the terminals is equal to the E.M.F. induced in the armature, which is proportional to the flux Φ. The ordinates in Fig. 150 therefore represent to a certain scale the values of Φ corresponding to the various values of the exciting current. Hence we are quite justified in calling the curve the magnetisation curve of the dynamo.

There is, however, an important difference between the magnetisation curves considered in a previous section and this no-load characteristic. There the abscissae represented the ampere-turns per cm and the ordinates the flux-density or lines per sq. cm, whereas here the abscissae represent the ampere-turns for the whole path, partly in iron and partly in air, and the ordinates represent the total flux.

We will now predetermine the no-load characteristic of a given machine, shown quarter-size in Fig. 151. The armature teeth and the air-gap are shown full-size in Fig. 152. We must first determine the cross-section of the magnetic path in the armature core, teeth, air-gap, pole-cores and yoke. We have

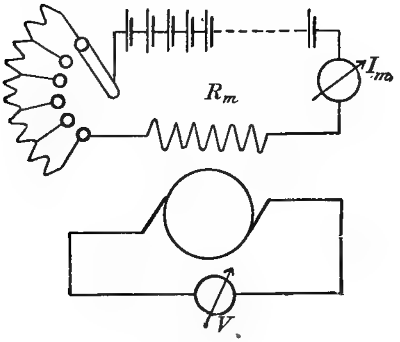

Fig. 149

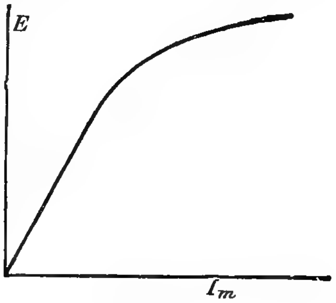

Fig. 150

diameter of armature ... ... $D = 15$ cm,
internal diameter of armature (in this case the diameter of the shaft) ... ... $D_0 = 3{\cdot}5$ cm,
axial length of armature ... ... $L = 11$ cm,
depth of slot ... ... $d = 2$ cm,
breadth of slot ... ... $b = 0{\cdot}5$ cm,
number of slots ... ... $S = 36$,
distance from pole to armature ... ... $= 0{\cdot}2$ cm,
angle subtended by pole ($\beta = 0{\cdot}778$) ... ... $= 70°$,
space lost in paper insulation between laminations $= 15$ %,
cross-section of magnet yoke ring ... ... $= 37$ sq. cm.

For the cross-section of the armature core normal to the flux, we have

$$A_a = 0{\cdot}85\,(D - D_0 - 2d).L = 70 \text{ sq. cm.}$$

To find the effective cross-section of the teeth we must subtract the sum of all the slot-widths from the periphery measured halfway down the slots and multiply the remainder by $\frac{L\beta}{4}$. Taking into account the factor 0·85 for the paper insulation, we have

$$A_t = 0{\cdot}85\,[(D-d)\,\pi - S\,.\,b]\,\frac{L\,.\,\beta}{4} = 41{\cdot}6 \text{ sq. cm.}$$

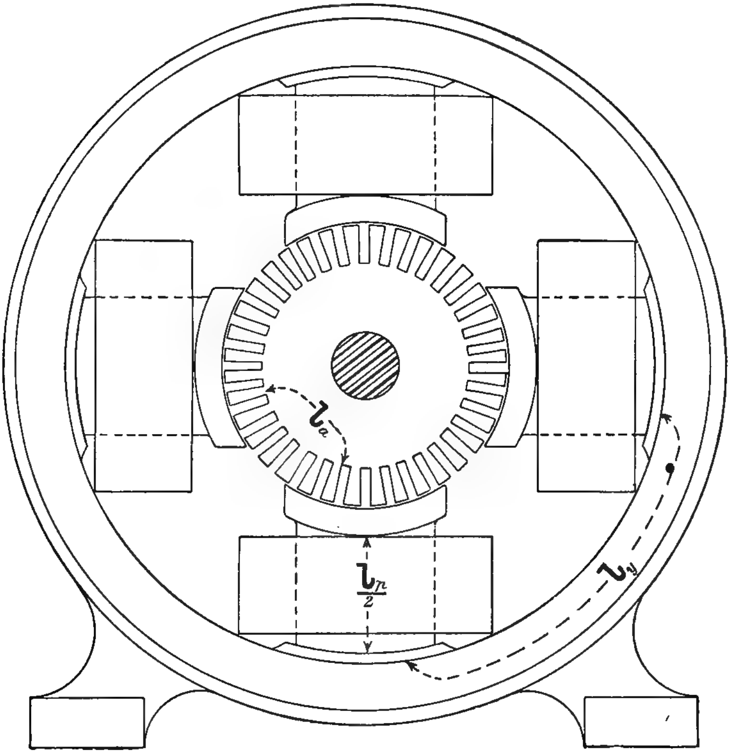

Fig. 151

If we make the simple but approximate assumption that the lines of force cross the air-gap in the manner shown in Fig. 152 the cross-section of the gap A_g is the mean of the area of the pole-face and the total area of the tops of the teeth facing the pole. If the double air-gap has a length

$$l_g = 0{\cdot}4 \text{ cm},$$

the area of the pole-face is

$$(D + l_g)\,\pi\,.\,\frac{\beta\,.\,L}{4} = 104 \text{ sq. cm.}$$

The tops of the teeth have an area

$$(D\pi - 36b)\,\frac{L\,.\,\beta}{4} = 62 \text{ sq. cm.}$$

As the mean of these two values, we get

$$A_g = \frac{104 + 62}{2} = 83 \text{ sq. cm.}$$

The cross-section of the round pole-cores can be seen from the drawing to be

$$A_p = 7{\cdot}4^2 \cdot \frac{\pi}{4} = 43 \text{ sq. cm.}$$

Since the flux of each pole divides into two equal parts in the yoke, the effective cross-section of the yoke is twice that of the actual magnet ring, that is,

$$A_y = 2 . 37 = 74 \text{ sq. cm.}$$

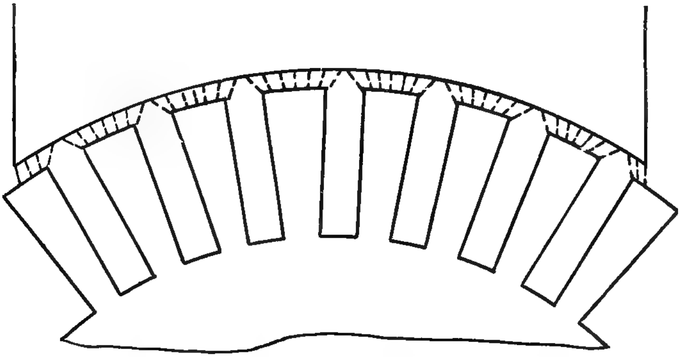

Fig. 152

The lengths of the paths in the different portions of the magnetic circuit can be found from the drawing. It is important to notice that the air-path l_g is twice the gap between armature and pole, while the path in the teeth is twice the depth of a tooth. Similarly l_p is twice the length of a pole, since the flux passes through both a north and a south pole. For l_y, however, we put the simple length through the yoke from pole to pole. The pole-shoes need not be separately considered. The values so obtained are collected in the following table:

Armature core (stampings)	Armature teeth (stampings)	Air-gap	Pole-core (W. I.)	Yoke (C. I.)
$A_a = 70$ $l_a = 7$	$A_t = 41{\cdot}5$ $l_t = 4$	$A_g = 83$ $l_g = 0{\cdot}4$	$A_p = 43$ $l_p = 12{\cdot}5$	$A_y = 74$ $l_y = 22$

We have now to determine the number of ampere-turns required to drive the flux through each individual part of the magnetic circuit. For this purpose we shall assume four different values of the flux, and work them out side by side. By dividing the flux by the cross-section of the part under consideration, we get the flux-density B, and by reference to the curves in Fig. 58, the necessary ampere-turns per cm $\left(\frac{X}{l}\right)$. This must be multiplied by the length of the path in order to get the total number of ampere-turns for the part of the magnetic circuit under consideration.

1. ARMATURE CORE.

$\Phi =$	0·2	0·4	0·5	$0·6 \, . \, 10^6$
$B_a = \frac{\Phi}{A_a} = \frac{\Phi}{70} =$	2850	5700	7150	8550
$\left(\frac{X}{l}\right)_a$ from the curve for stampings $=$	0·5	1·1	1·4	1·8
$X_a = l_a \left(\frac{X}{l}\right)_a = 7\left(\frac{X}{l}\right)_a =$	3·5	7·7	9·8	12·6

2. ARMATURE TEETH.

$\Phi =$	0·2	0·4	0·5	$0·6 \, . \, 10^6$
$B_t = \frac{\Phi}{A_t} = \frac{\Phi}{41·5} =$	4800	9650	12000	14400
$\left(\frac{X}{l}\right)_t$ from the curve for stampings $=$	0·8	2·3	4	9
$X_t = l_t \left(\frac{X}{l}\right)_t = 4\left(\frac{X}{l}\right)_t =$	3·2	9·2	16	36

3. AIR-GAP.

For the air-gap, we have the fundamental equation (43) on p. 61,

$$B_g = H_g = \frac{0·4\pi \, . \, X_g}{l_g},$$

or

$$X_g = \frac{B_g \, . \, l_g}{0·4\pi} = 0·8 \, B_g \, . \, l_g \, .$$

We have, therefore,

$\Phi =$	0·2	0·4	0·5	$0·6 \, . \, 10^6$
$B_g = \frac{\Phi}{A_g} = \frac{\Phi}{83} =$	2400	4820	6020	7220
$X_g = 0·8 B_g \, . \, l_g = 0·32 B_g =$	770	1540	1930	2310

4. POLE-CORES.

In calculating the ampere-turns for the poles and yoke we must take into account the fact that the flux in them is larger than the armature flux, on account of leakage. Assuming that the leakage coefficient

$$\nu = \frac{\Phi_m}{\Phi} = 1·2,$$

we have

$$\Phi_m = 1·2\Phi.$$

$\Phi =$	0·2	0·4	0·5	$0·6 \, . \, 10^6$
$\Phi_m = 1·2\Phi =$	0·24	0·48	0·6	$0·72 \, . \, 10^6$
$B_p = \frac{\Phi_m}{A_p} = \frac{\Phi_m}{43} =$	5600	11200	14000	16800

$\left(\frac{X}{l}\right)_p$ from the curve for wrought iron =	1·6	·5	14	100
$X_p = l_p\left(\frac{X}{l}\right)_p = 12{\cdot}5\left(\frac{X}{l}\right)_p =$	20	62·5	175	1250

5. YOKE.

$\Phi =$	0·2	0·4	0·5	$0{\cdot}6 \,.\, 10^6$
$\Phi_m = 1{\cdot}2\Phi =$	0·24	0·48	0·6	$0{\cdot}72 \,.\, 10^6$
$B_y = \frac{\Phi_m}{A_y} = \frac{\Phi_m}{74} =$	3250	6500	8100	9720
$\left(\frac{X}{l}\right)_y$ from the curve for cast-iron =	4·1	19	53	107
$X_y = \left(\frac{X}{l}\right)_y . l_y = 22\left(\frac{X}{l}\right)_y =$	90	418	1160	2350

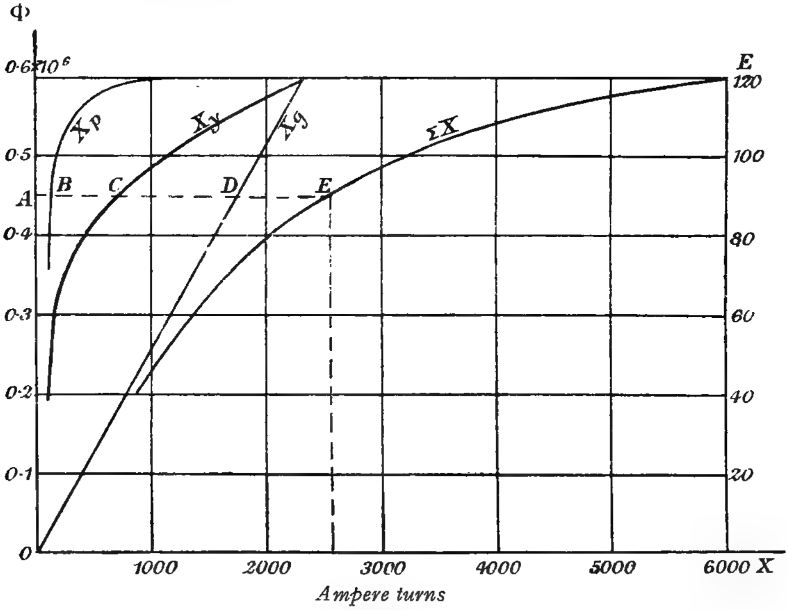

Fig. 153

The total ampere-turns can now be found by adding up the five components, thus

$\Phi =$	0·2	0·4	0·5	$0{\cdot}6 \,.\, 10^6$
$X_a =$	3·5	7·7	9·8	12·6
$X_t =$	3·2	9·2	16	36
$X_g =$	770	1540	1930	2310
$X_p =$	20	62·5	175	1250
$X_y =$	90	418	1160	2350
$\Sigma X = X_a + X_t + X_g + X_p + X_y =$	890	2040	3290	5960

In Fig. 153 the values of armature flux Φ are plotted as ordinates, and the values of X_p, X_y, X_g, and ΣX as abscissae. The curve X_g is naturally a straight line. The values of X_a and X_t are so small that they have been omitted in the figure. It is easy to see from the figure how the total ampere-turns ΣX are used up in the various parts of the machine. To produce the flux OA, for example, we have to employ an excitation AE. Of this, a part AB is required for the poles, a part AC for the yoke, and the larger part AD for the air-gap, and $AE = AB + AC + AD$.

Near the origin the curve for ΣX coincides very nearly with the straight line X_g, while as the saturation of the iron increases it becomes gradually flatter.

The electromotive force can be calculated directly from the flux Φ by means of equation (79) on p. 124,

$$E = \frac{p}{a} \cdot \Phi \cdot n \cdot Z \cdot 10^{-8} \text{ volts.}$$

If, for example, the speed is 1665 revs. per minute and the number of wires on the armature periphery is 720, we have for a parallel-wound armature

$$E = \Phi \cdot \frac{1665}{60} \cdot 720 \cdot 10^{-8} = 200 \cdot \Phi \cdot 10^{-6} \text{ volts.}$$

The ordinates of the curve ΣX give the electromotive force directly on the scale shown on the right-hand side of the figure.

It must be particularly noticed that ΣX is the number of ampere-turns per pair of poles, since the calculation was made for a complete magnetic circuit linking both a north and a south pole. The total number of field ampere-turns on the whole machine is $p . \Sigma X$.

Having thus fully investigated the effect on a separately excited dynamo of varying its field current, we must now consider the effect of loading the machine, that is, taking current from it. We assume that both its speed and excitation are kept constant while its terminals are connected by means of an external resistance, such as a bank of glow-lamps. The greater the number of lamps connected in parallel across the terminals the bigger is the load on the dynamo. This increase in the current taken from it causes a decrease in the P.D. between the terminals. When running on open circuit the terminal pressure is equal to the electromotive force, but when loaded it is less by the pressure drop due to the armature resistance. If the resistance of the armature be R_a, the terminal pressure will be given by the formula

$$V = E - I . R_a.$$

If we neglect the effect of armature reaction, the electromotive force will be constant since the exciting current is constant, and it can be represented by the horizontal line E in Fig. 154. The pressure drop in the armature is proportional to the current, that is, to the abscissae, so that the terminal pressure is given by the straight line V drawn at an angle α below the line E, where

$$\tan \alpha = \frac{I . R_a}{I} = R_a.$$

When setting out the angle α it must be carefully noticed that the scales for the ordinates and abscissae are probably different, so that we cannot draw simply the geometrical tangent. The difference between the ordinates of the curves E and V is equal to the ohmic pressure drop for a certain value of the current, while the ordinates of the curve V give us the corresponding terminal pressure. Experimentally the process is of course reversed, since we observe the terminal pressure and add to it the calculated armature drop to find the electromotive force.

The results can be plotted in another manner, namely, by setting out the external resistance R, i.e. the quotient $\frac{V}{I}$, as abscissae and plotting both V and I as ordinates (Fig. 155). Since the product $I\ (R + R_a)$ is equal to the constant electromotive force, the curve representing I must be a rectangular hyperbola. Its asymptotes are the horizontal axis and a vertical line through a point representing $- R_a$. The curve I cuts the vertical axis in the point A. This corresponds to no external resistance, or, in other words, the dynamo is short-circuited, and $OA = \frac{E}{R_a}$ is the short-circuit current. This point lies, naturally, quite outside the working limits, and could only be found by plotting the curve for a very small value of the excitation.

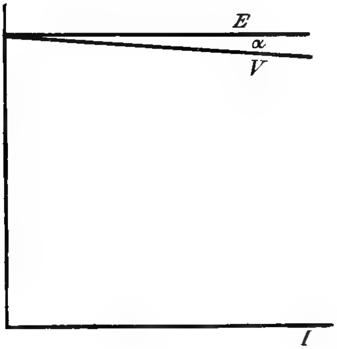

Fig. 154

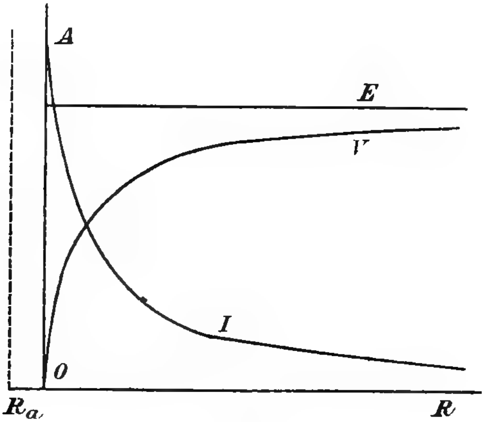

Fig. 155

The limit of the curve V in Fig. 155 can be found from the consideration that the terminal pressure is equal both to the product IR and to the difference $E - I.R_a$. On short-circuit it must be zero, since the terminals are electrically equivalent to a single point and can have no difference of potential. The whole electromotive force is then used up in driving the enormous short-circuit current through the armature resistance. On the other hand, the terminal pressure attains its maximum value and becomes equal to the electromotive force E when the external resistance becomes infinitely great, i.e. on open-circuit.

As we have already indicated, the curves would be modified in an actual case by the demagnetising effect of armature reaction. Moreover, the armature resistance is not constant, but increases with its temperature and therefore with the load. It is hardly possible, however, to take these points into consideration in a simple manner. Another disturbing influence, which we have neglected and which we shall continue to neglect, is the effect of remanent magnetism.

58. Characteristics of Series Dynamo.

Both the consideration and the actual working of a separately excited dynamo are extremely simple, since with constant speed and constant excitation the electromotive force is independent of changes in the external circuit. The behaviour of self-excited dynamos is, however, far more complicated, since every change in the external circuit affects the magnetising current, and also, therefore, the magnetic flux and the electromotive force.

The relations are, perhaps, simplest in the series dynamo, since the armature current is here identical with the magnetising current and the external current. The series-connection of armature, field and external circuit was that which first suggested itself. It has, however, become almost obsolete as far as generators are concerned. It was used for supplying a number of arc lamps connected in series, but is quite unsuitable when the lamps, as is now general, are connected in parallel between the mains. As motors, however, series machines are of enormous importance.

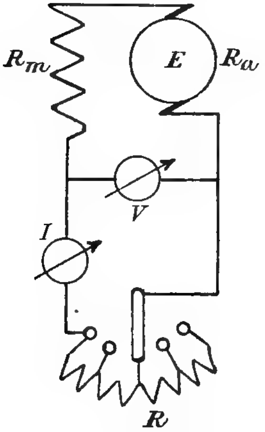

Fig. 156

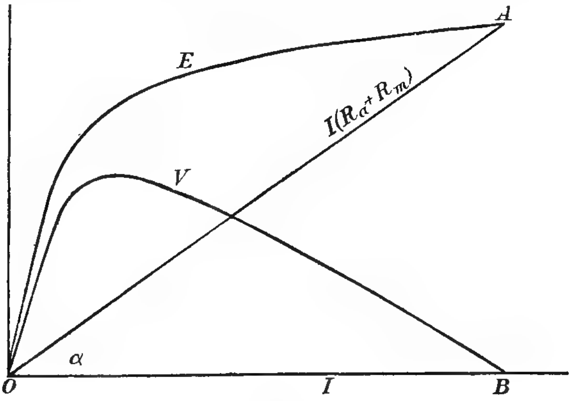

Fig. 157

We shall now consider the effect of varying the external resistance of a series dynamo running at a constant speed (Fig. 156). We take corresponding readings of the current and the terminal pressure, and plot a curve with abscissae equal to the current I and ordinates equal to the pressure V (Fig. 157). We then draw a straight line through the origin making an angle α with the base line, where

$$\tan \alpha = R_a + R_m,$$

R_m being the resistance of the field winding. The ordinates of this straight line represent the pressure drop in the dynamo itself, and by adding this to the corresponding ordinate of the curve V we obtain the curve E of the electromotive force. This is the characteristic curve of the series dynamo, and represents the relation between electromotive force and magnetising current. If, instead of the current I, we had plotted the product of current and the number of turns on the field, we should have obtained the same magnetisation curve as that found in the previous section (Fig. 153).

We see from the curve that the electromotive force vanishes on open-circuit, since when the magnetising current ceases the magnetic flux disappears. If the terminals be joined by means of a reasonably small resistance, the machine will excite and produce an electromotive force which will drive a current round the circuit. The smaller we make the external resistance, the larger, by Ohm's law, will be the current and therefore also the magnetic flux and the electromotive force. This reaches its limit when the machine is short-circuited. The terminal pressure then vanishes, and the whole electromotive force AB is used up in driving the short-circuit current OB through the internal resistance. This occurs at the point A, in which the straight line cuts the curve E.

With regard to the terminal pressure, we see that the curve reaches a maximum with a certain load and then decreases as the load is further increased. That this must be so is evident, for, as the iron becomes saturated, a point will be reached at which the increase of electromotive force, due to a given increase in the current, is not enough to balance the increased drop thereby produced.

This peculiarity was turned to useful account in the machines designed for lighting a string of arc lamps in series. It was necessary to make the increase of current, due to the short-circuiting of a lamp, as small as possible. The cross-section of the iron was therefore made so small that it was highly saturated at the normal working current. The machine worked beyond the knee of the characteristic curve E, where an increase in the current causes but a slight increase in the E.M.F. By making the armature reaction, which we have neglected, strong enough, an increase of current might even cause a decrease in the E.M.F. On short-circuiting a lamp the current increased slightly, but not so much as it would have done had the electromotive force increased with the current.

The peculiar behaviour of a self-exciting dynamo, in that the electromotive force depends on the current taken, is easier to comprehend if we fix our attention on the external circuit. The question arises as to whether the current is produced by the electromotive force, or the electromotive force by the current, through the medium of the magnetic flux. Both views must be looked upon as correct, and we then have a complete cycle, without being able to say which is cause and which is effect. In practice, however, it is the external resistance which is arbitrarily varied, and both the electromotive force and the current are thereby caused to change simultaneously.

59. Characteristics of Shunt Dynamo.

It is customary, in connection with shunt dynamos, to distinguish between an internal and an external characteristic. The internal or static characteristic is obtained by running the machine light at a constant speed and varying the resistance of the field circuit by means of a rheostat or variable resistance. By plotting the values of the magnetising current I_m as abscissae and the values of the terminal pressure V as ordinates we obtain a curve exactly similar to the characteristic of a series dynamo or to the no-load characteristic of a separately excited machine, and there is therefore no need to examine it further. This curve is called the internal or static characteristic of the shunt dynamo.

We must now consider the behaviour of a shunt dynamo when its load or external current is varied. In the series dynamo the field winding and the external circuit are connected in series, with the result that the electromotive force is largely dependent upon the external current. In the shunt dynamo, on the other hand, the field winding is in parallel with the external circuit, that is, it is connected directly to the brushes (Fig. 158). As the result of this arrangement variations in the external current do not react directly on the electromotive force. The external current has, in fact, only a small secondary effect on the excitation and E.M.F., and, within the practical working limits of the machine, we are dealing with a relatively constant electromotive force. The behaviour of a shunt dynamo in actual operation is consequently simple, but a clear understanding of its behaviour is usually a matter of some difficulty to the beginner.

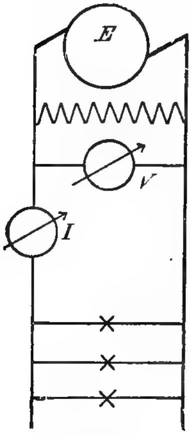

Fig. 158

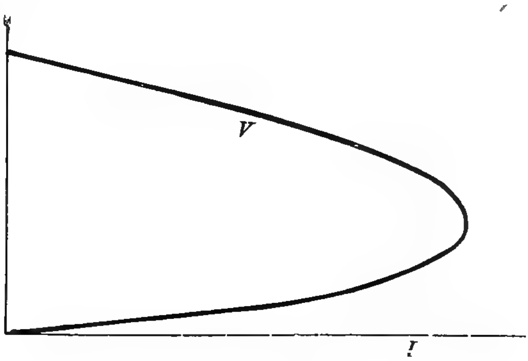

Fig. 159

In the first place, an increase in the external current causes an increase in the armature current. The armature current is the sum of the external and the shunt currents, thus

$$I_a = I + I_s.$$

In consequence of this increase the pressure drop in the armature is increased and the terminal pressure falls, in accordance with the equation

$$V = E - I_a R_a.$$

The terminal pressure is therefore greatest at no-load and gradually falls with increasing load. Since the field winding is connected directly across the brushes, the decrease in the terminal pressure causes a corresponding decrease in the shunt current. This leads to a diminished flux and a proportionately smaller E.M.F., in consequence of which the terminal pressure decreases still further. The variation of pressure with load is therefore greater in a shunt dynamo than in a similar separately excited machine. It is, however, very much smaller than the pressure variation of a series dynamo.

It is possible, however, to design the shunt dynamo so that the drop of pressure from no-load to full load is almost as small as that of a separately excited machine. This result is obtained by working some part of the magnetic circuit at a high saturation. The magnetic flux is then but little affected by considerable variation in the magnetising current, and the drop is almost entirely due to the ohmic resistance. Such machines are specially suitable for electric light stations, where constancy of pressure is an absolute necessity in the interests of steady illumination. The small unavoidable changes in the pressure which still occur are counteracted by means of variable resistances, or rheostats, in the shunt circuit.

Machines which are to be used for charging accumulators should, on the other hand, be designed with low flux-densities in the iron. When beginning to charge, resistance is put into the field circuit, which, as the charging proceeds, is gradually cut out. In this way the flux and E.M.F. are gradually increased, and the current maintained at a constant strength in spite of the increasing counter-electromotive force of the battery.

If now we determine the external characteristic of a given machine experimentally by switching lamps, one after the other, in parallel across the mains and so varying the external current, we find that the P.D. between the terminals gradually falls. This is exactly what we should expect from our above considerations, and is shown graphically by the upper branch of the curve in Fig. 159. If, however, we increase the number of lamps, and therefore decrease the external resistance, beyond a certain point, the current does not continue to increase, as we should expect, but decreases. The terminal pressure continues to decrease until the machine is short-circuited, when it naturally falls to zero. The magnetising current ceases at the same time, and with it the magnetic flux which it had produced. Were it not for remanent magnetism, there would be no E.M.F. and consequently no armature current. The lower part of the curve in Fig. 159 would then pass through the origin, whereas the remanent magnetism causes it to meet the base line a little to the right of the origin.

The action of the shunt dynamo will probably be made much clearer if we evolve the external or dynamic characteristic from the internal or static characteristic. We shall neglect the effect of armature reaction. We plot, in the first place, the magnetising current as abscissae and the electromotive force E as ordinates. This is the internal characteristic, and is shown in Fig. 160. Since, in our present experiment, we vary only the external current and leave the resistance of the field circuit unaltered, the terminal pressure must be proportional to the magnetising current, in accordance with the equation

$$V = I_s R_s.$$

For the curve of terminal pressure V we therefore get the straight line in Fig. 160. Now, the difference between the ordinates of the two curves E and V must be equal to the pressure drop $I_a . R_a = (I + I_s) . R_a$ in the armature, and is therefore proportional to the armature current. When there is no load on the dynamo, the only current through the armature is the field current, which is then equal to OC, and the pressure drop in the armature is then $OC \cdot R_a$. From this it follows that the difference between the ordinates of the two straight lines OA and OB is always equal to the product $I_s \cdot R_a$. The difference between the ordinates of the curve E and the straight line OA is therefore equal to the product $I \cdot R_a$, and is consequently proportional to the external current. By plotting the values of this current as ordinates to a different scale, the curve I in Fig. 160 is obtained. As the curve shows, there is one special load at which the current is a maximum.

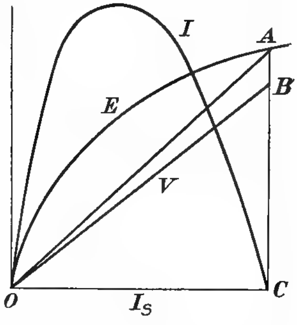

Fig. 160

Since the abscissae I_s in Fig. 160 are proportional to the terminal pressure, the curve I in Fig. 160 is identical, except as regards scale, with the curve V in Fig. 159. It is, however, turned through 90°, so that the left-hand part of the curve I in Fig. 160 corresponds to the lower curve in Fig. 159, where, owing to the large number of lamps connected in parallel, the machine is nearly short-circuited. The right-hand part of Fig. 160 corresponds to the upper curve in Fig. 159, that is, the working portion of the characteristic where considerable variation in the load causes but slight changes in the pressure.

Compound dynamos may be regarded as shunt dynamos, the fields of which are strengthened when the load comes on by means of a few series turns carrying the main current. In this way the drop due to armature resistance and armature reaction is neutralised and the terminal pressure maintained constant at all loads. If over-compounded, that is, provided with a greater number of series turns, the terminal pressure increases with the load, rising, for example, in many traction generators from 500 volts at no-load to 550 volts at full load, thus making it possible to maintain a pressure of 500 volts at distant points of the system.

Now, the maintenance of a constant pressure in spite of variations of the load is of the utmost importance for the supply of power to glow-lamps, since these are extremely sensitive to changes of pressure, and the effect of such changes on the eye is very disagreeable.

The compound dynamo appears therefore to be specially suitable for electric lighting. As a matter of fact, however, it is rarely used for this purpose, and the majority of lighting stations will be found to contain shunt dynamos only. This is due to the fact that sudden overloads cause a momentary drop of pressure even on a compound dynamo, since the engine governor cannot act instantaneously and the speed is consequently reduced. This causes the E.M.F. and the terminal pressure to decrease, which leads to a decrease in the shunt current and a further fall of pressure. A lighting station, moreover, is not subjected to the sudden, and often enormous, variations of load experienced in a traction station. The changes are usually gradual and allow the switch-board attendant ample time to counteract the drop by means of the field rheostat. The added complication of compound dynamos is therefore unnecessary in such a station.

Compound machines, like series machines, are quite unsuitable for charging accumulators. If they be used for this purpose and the speed of the engine falls on any account, the back E.M.F. of the battery may overcome the forward E.M.F. of the dynamo and drive a reverse current round the circuit. This current will pass through the series field winding and reverse the polarity of the magnets. The electromotive forces of dynamo and battery are no longer in opposition, but acting together in a circuit of very low resistance. The current will therefore rapidly rise to a dangerous value.

If, however, a shunt dynamo is used, the positive pole of the battery is always connected to the positive pole of the dynamo, and the shunt current in the dynamo field-coils is always in the same direction, even when the main current is reversed. The electromotive forces of the battery and dynamo are always in opposition. When the main current reverses, the battery drives the machine as a motor and the current cannot reach a dangerous limit. Trouble might be caused, however, by sparking at the commutator, since the correct position of the brushes is different in the case of a dynamo and that of a motor. This can be prevented by putting a minimum cut-out in the main circuit. This consists of an electromagnet excited by the main current and holding up an armature so long as the current exceeds a certain value; if, however, the current falls below this value, the armature is released and automatically opens a switch in the main circuit.

60. Dynamo and Battery in Parallel.

The practically constant supply pressure which we saw in the last section to be necessary for electric lighting can be obtained by the use of a battery of accumulators connected in parallel with the dynamo. The battery, moreover, constitutes a very desirable reserve, as the dynamo need only be large enough to deal with the average load instead of the maximum load, and may, at times of very light load, or in case of necessity, be shut down without interrupting the supply. Finally, the battery enables the dynamo to work at constant full load in spite of the variable load on the station (buffer batteries in traction stations). The steam consumption is thereby made uniform and considerable economy effected.

These advantages depend on two characteristic properties of the accumulator, viz. its constancy of E.M.F. and its low internal resistance. As a result of the latter the drop of pressure in the battery may be almost neglected, and the terminal pressure is practically equal to the electromotive force. The shunt dynamo may be looked upon as separately excited, since the field winding is virtually connected across the constant terminal pressure of the battery. We shall consider the effect of (1) Variation of load, (2) Variation of engine speed, (3) Variation of excitation.

(1) EFFECT OF VARIATION IN THE EXTERNAL LOAD.

If the speed and excitation be constant and the electromotive force be E, then the armature current will also be constant and will be given by the equation

$$I_a = \frac{E - V}{R_a},$$

where V is the terminal pressure, maintained constant, as we have seen, by the battery. We see, then, that neither the terminal pressure nor the dynamo current is affected by changes in the external load. If, for example, the battery is discharging (Fig. 161) and more glow-lamps are switched into the external circuit, the extra current required is supplied almost entirely by the battery discharging at a higher rate. If, on the other hand, the battery is being charged (Fig. 162) when the extra lamps are switched on, the current required will be drawn from the battery by taking a part of its charging current. We have here, therefore, an exceptional case in which the generator gives practically a constant current, which we divide, as we choose, between the battery and the external circuit.

The battery will pass from a condition of charging to one of discharging simply by changing the load in the external circuit. If, for example, the external current is smaller than the armature current, the excess current will pass into the battery and charge it. If, now, the external current increases beyond the armature current, the battery will give a discharging current. If the dynamo gives the average current, the battery will be continually changing from charge to discharge and *vice versa*. This change is only possible if the terminal pressure of the battery changes. When charging the terminal pressure must be greater than the E.M.F. in accordance with the equation

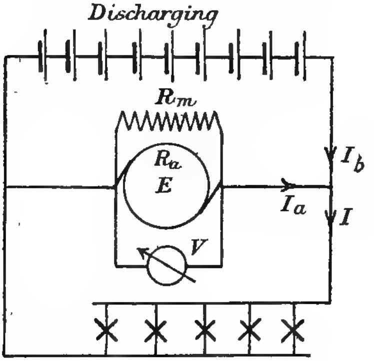

Fig. 161

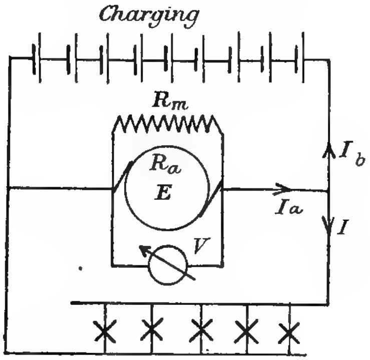

Fig. 162

$$V = E_b + I_b . R_b.$$

When discharging, on the other hand, the terminal pressure must be smaller than the E.M.F. of the battery, as shown by the equation

$$V = E_b - I_b . R_b.$$

These changes are small, and we have therefore neglected them in our consideration of the subject.

The actual relations can be seen more clearly from the following considerations. In Fig. 163 the battery and machine are represented by two parallel paths each containing a source of electromotive force and a resistance. Any external current must so distribute itself between the two branches that each has the same terminal P.D. The greater part of any increase in the external current must pass through the branch possessing the smaller resistance, i.e. the battery, whilst the smaller part flows through the machine. This is, of course, the object of connecting the battery in parallel with the machine. The smaller the resistance of the battery compared with the dynamo the better will be the buffer action of the battery.

A perfect buffer action would necessitate an infinitely small battery resistance. In practice this is impossible and a part of the fluctuations of the load come on to the machine. This can be prevented, however, by the use of a booster in series with the battery. This consists of a dynamo usually coupled directly to a shunt or differentially compounded motor. In the simplest type of booster for the purpose under consideration the dynamo has two field windings, so connected that their ampere-turns are in opposition, one a shunt winding connected across the mains and carrying therefore a constant current, the other a series winding carrying all or a definite fraction of the load on the station. When the load has its average value the two windings exactly neutralise each other and the dynamo, having no field, produces no E.M.F.; under this condition the only E.M.F. in the battery circuit is that of the battery itself which is equal and opposite to the P.D. of the bus-bars, and the battery current is consequently zero. When the load increases above the average the series field winding overcomes the shunt and the booster develops an E.M.F. in the same direction as that of the battery which is thus caused to discharge at a rate depending on the magnitude of the excess load. When the load is small the shunt winding overcomes the series winding and the booster E.M.F. acts in opposition to that of the battery which is consequently caused to take a charging current. The battery can thus be made to charge and discharge with variations of the load without any variation in the P.D. between the bus-bars or main terminals of the station and therefore without any variation in the load on the main generators.

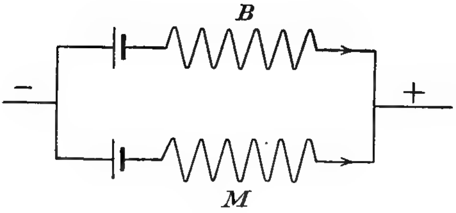

Fig. 163

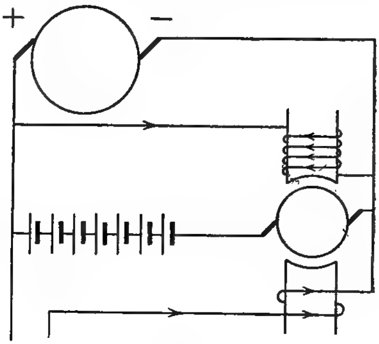

Fig. 164

(2) EFFECT OF SPEED VARIATION.

We have assumed in the foregoing that the electromotive force of the dynamo is constant, but we shall now consider the effect of speed variation on terminal pressure and armature current. We shall assume the external load to be constant. When running without a battery in parallel a change of speed has a large effect on the terminal pressure, but in the present case the terminal pressure is maintained constant by the battery. A change in the E.M.F. of the dynamo will, however, cause the armature current to change, in accordance with the formula

$$I_a = \frac{E - V}{R_a}.$$

If, for example, the speed suddenly drops, the E.M.F. decreases, and with it the armature current. Since the current in the external circuit is constant, the battery, if discharging, must give a greater current the smaller that given by the dynamo. If charging, a decrease in the dynamo current causes a decrease in the current passing into the battery. Hence, speed variations of the engine are converted into variations of battery current.

There are cases, such as in train lighting, where it is required to charge batteries by means of dynamos running at very variable speeds. For such purposes special types of dynamos have been developed; these are considered in Section 61.

(3) EFFECT OF VARIATION OF EXCITATION.

A change in the rheostat in the field circuit has, naturally, a similar effect to a change of speed. If, for example, we wish to charge the battery, we adjust the field current until the terminal pressure of the machine is slightly above that of the battery. The switch is then closed so that similar poles of the machine and battery are connected. Since the electromotive forces oppose each other, the current is practically nil, but can be brought up to the desired value by adjusting the field rheostat (compare the above formula for I_a). The reading on the voltmeter is only slightly changed, since the terminal pressures of both dynamo and battery are identical.

To stop charging, the field current is weakened until the armature current falls to the value of the external current. The battery is then carrying no current, but can remain across the bus-bars or terminals to act as a pressure regulator.

At times of exceptionally heavy load the shunt rheostat is used to distribute the load in a suitable manner between battery and machine. The excitation can be adjusted until the machine is working at full load, so that the battery is left to deal with the variable overload.

We have assumed that the battery pressure is constant, and we are justified in making this assumption, in so far as the small effect of the variable current is concerned. We have already seen, however, that the terminal pressure of the battery gradually changes as charging or discharging proceeds (Figs. 26 and 27). To maintain a constant pressure, battery switches are often employed, by means of which cells can be added to or removed from the battery. In order that the current may not be interrupted when switching another cell in or out, an arrangement is shown in Fig. 165, in which auxiliary contacts are fitted between the main contacts and connected to them by means of resistances. The moving contact is wide enough to span across the insulation between the segments. In the position shown the lamps are connected across the main cells only, and the first regulating cell is short-circuited through the resistance between the contacts. By turning the switch-arm to the right it is made to lie on the auxiliary contact only. The first regulating cell is then in series with the battery across the mains, and by suitably choosing the auxiliary resistance, which is now in the main circuit, the supply pressure may be made to rise about 1 volt, depending on the strength of the current. On moving the arm over on to the main contact, the first regulating cell is connected directly to the mains and the pressure increased by another volt. The pressure can thus be adjusted by steps of two volts, the auxiliary contacts being only used to pass from one step to the next.

Another device is illustrated in Fig. 166. Here the switch-arm consists of two parts rigidly fixed together, but insulated from each other. In the position shown the first regulating cell is short-circuited through the two arms and the auxiliary resistance. On turning the switch further to the right, the main arm leaves the end contact and thereby puts the first regulating cell in series with the battery across the mains. The auxiliary resistance is in circuit, however, and the increase of pressure is less than two volts by an amount depending on the current. On turning the switch still further to the right, both arms will lie simultaneously on the second contact block, and finally the auxiliary contact will leave the block and lie between it and the next block. The pressure on the mains has then been increased by two volts.

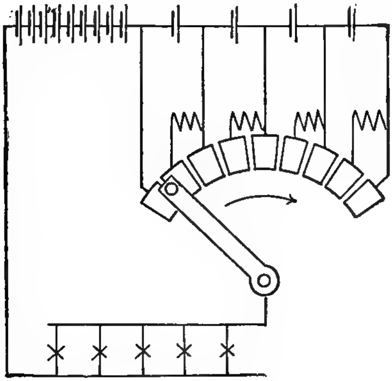

Fig. 165

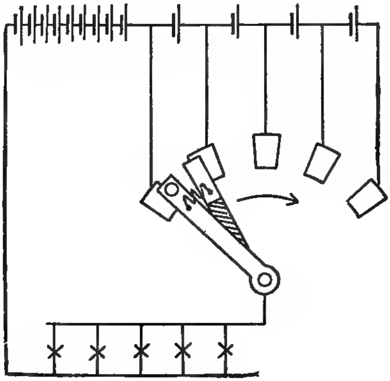

Fig. 166

The use of these simple battery switches is to be recommended where the charging is done when there is no external load (Fig. 167). The switch is then used during the morning as a charging switch, beginning with it on the right-hand stop and gradually moving it over to the left as the cells become charged. The right-hand regulating cells were probably only used towards the end of the previous evening, and are consequently fully charged long before the main battery. During the process of charging, therefore, the switch-arm is gradually moved from right to left. Since the battery pressure during the charge is greater than that for which the lamps are designed, it is necessary to keep the switch A open. When the cells are all fully charged the switch A_1 is also opened.

When lamps are to be lighted, the switch A is put on to such a contact that the volts across the mains on closing the switch A will be correct. The switch A is then closed and the dynamo run up and put in parallel as already described. The field current is adjusted until the dynamo is on full load, if possible, and the battery is not called upon to share the load on the station until it exceeds the full load of the dynamo. During the night the dynamo can be shut down and the battery left to take charge of the all-night load. To maintain the pressure on the lamps, the switch-arm will have to be moved over gradually from left to right.

The number of regulating cells required with this arrangement can be found from the fact that the pressure of a cell at the commencement of discharge is about 2 volts, while after discharging as far as permissible it is only 1·8 volts. If the supply pressure is 110 volts the number of newly charged cells must be 110/2 = 55. At the end of the discharge the number must be increased to 110/1·8 = 60. We see, therefore, that about 10 per cent. of the battery must be connected up to the battery switch.

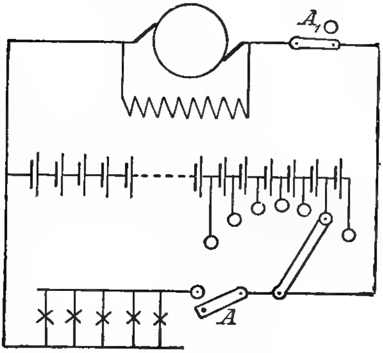

Fig. 167

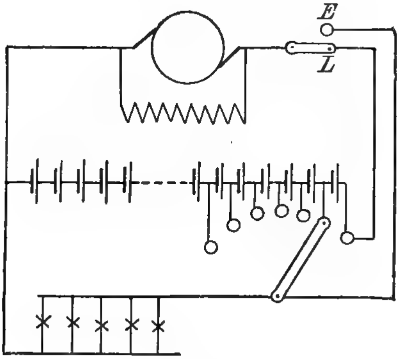

Fig. 168

The above plan of disconnecting the external circuit while charging the battery, will hardly be practicable in the majority of cases. If we have to maintain the supply during the time that the battery is being charged, we must use our single battery switch as a discharging switch to maintain constant supply pressure (Fig. 168). The dynamo must be connected to the last cell by means of the two-way switch EL. The dynamo current divides between the battery and the external circuit, except that the regulating cells to the right of the switch-arm carry the whole current. Hence, the very cells which need it least get the greatest charging current. When the charging is complete the dynamo must be disconnected and its pressure reduced by means of the field rheostat to that of the mains. The two-way switch is then put on to E, and the battery used simply as a pressure regulator, until the load increases to such an extent that battery and dynamo must work in parallel.

The disadvantages of this arrangement are evident. The regulating cells are not only a source of waste, but are rapidly destroyed through being continually overcharged at a high current density. Another disadvantage is that the dynamo must give the full charging pressure of the whole battery, which, with a supply pressure of 110 volts and the 60 cells calculated above, would be $60 . 2{\cdot}7 = 162$ volts. If, however, the end cells could be cut out as they became charged, we should only require a terminal pressure of 150 volts. The cost of the machine would therefore be unnecessarily high, since such a machine is always designed for the maximum pressure, i.e. for the corresponding flux in the field magnets. When the machine is working in parallel with the battery and giving a terminal pressure of 110 volts, the flux is reduced by means of the field rheostat, and the machine is larger than actually required for working under these conditions.

In order that the cells may be charged without interrupting the supply, the regulating cells must be connected up to a double battery-switch (Fig. 169). The supply mains are then permanently connected to the discharging side of the switch, which serves as a pressure regulator both when charging and when discharging. The charging switch-arm lies always to the right of the discharging arm, and should never be brought inside it. Both arms will be gradually moved from right to left as the charging proceeds until finally both are on the left-hand contacts, and dynamo, battery and mains have the same pressure. Although the end cells are switched out as they become charged, and are consequently saved from the continual overcharging which they suffered in the previous arrangement, the cells lying between the two switch-arms are still charged with a current equal to the sum of the battery current and that supplied to the lamps.

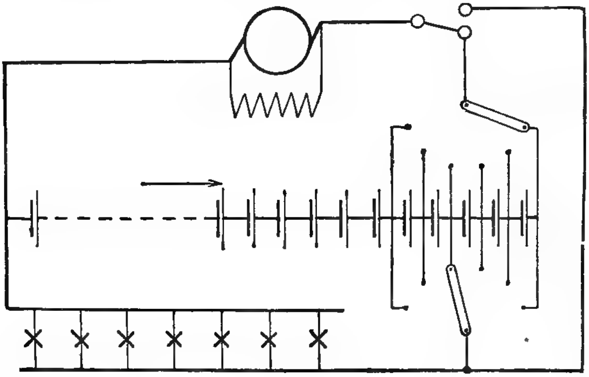

Fig. 169

The two-way switch is now moved over to the upper contact and the dynamo and battery are in parallel across the mains. By suitably adjusting the field rheostat the load can be subdivided as desired. As the battery discharges the lower switch-arm is gradually moved to the right, sometimes carrying the upper one with it by means of a stop, so as to ensure that the latter is not to the left of the former when charging is commenced.

As can be seen from the figure, the corresponding contacts of the charging and discharging switches are metallically connected and only a single row of contacts is actually necessary, over which two switch-arms can be independently moved. The arrangement shown in Fig. 169 is, however, more easily understood.

Although the use of a double battery-switch makes it possible to charge without interrupting the constant pressure supply, and also to switch off the regulating cells as they become fully charged, it does not obviate the necessity for making the dynamo capable of giving a pressure far in excess of the normal supply pressure. The disadvantage of requiring the dynamo to work normally at one pressure and yet be capable of giving another, much higher, pressure when necessary, is specially noticeable in large stations. The machines are compelled to run normally at an output far below their maximum, and the capital outlay is unnecessarily large. This disadvantage is completely removed by the use of a so-called booster, which is a comparatively small dynamo connected in series with the main dynamo. The latter is then designed for the normal supply pressure, without any provision for a large increase of pressure. The iron of the machine can be highly saturated, thus reducing both the size and cost of the machine. A further saving is effected in the field rheostat, which need not be nearly so large.

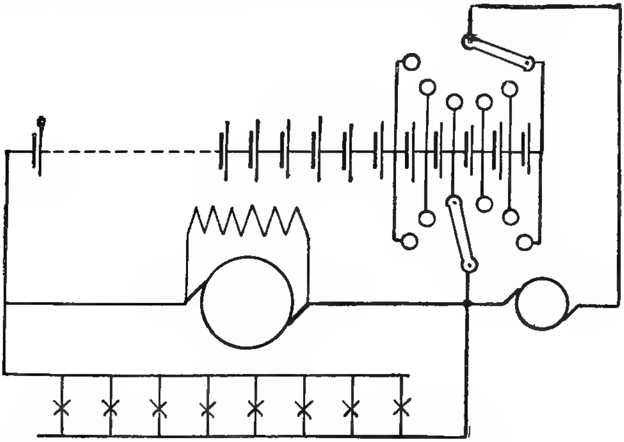

Fig. 170

An arrangement including both a double battery-switch and a booster is shown in Fig. 170. The main dynamo is connected permanently to the discharge switch-arm, while the booster is connected between the two switch-arms and charges the cells lying between them. If the excitation of the booster is adjusted until its current is equal to the charging current through the main battery, no current will flow through the discharge switch-arm. The latter will then resemble the galvanometer connection in the Wheatstone bridge in that it connects two points at the same potential. The battery will then act simply as a pressure regulator.

To determine the number of regulating cells required in this arrangement, we have, in the first place, to be able to obtain the supply pressure of, say, 110 volts towards the end of the charging, when each cell has a pressure of about 2·7 volts. The main battery must therefore contain no more than $\frac{110}{2\cdot7} = 40$ cells. In the second place, when the cells are almost discharged, the pressure of each will drop to 1·8 volts, and the number then required will be $\frac{110}{1\cdot8} = 60$. The whole battery must therefore consist of 60 cells, of which 20, i.e. about 30 per cent., are connected to the battery-regulating switch.

In many modern stations battery-regulating switches are more or less eliminated by the use of automatic boosters, in which the field is produced by the resultant effect of several distinct windings. One winding will cause the E.M.F. of the booster to vary with the load, while another will take into account the state of the battery. In this way the pressure of supply is maintained practically constant, and the battery caused to charge or discharge according as the load on the station is below or above the normal*.

61. Constant-current Variable-speed Dynamos.

In the great majority of cases dynamos are driven by prime-movers running at approximately constant speed. An important exception occurs in connection with train lighting. The dynamo is generally placed under the carriage and driven from one of the axles. The devices suggested for getting over the difficulties of the enormous speed variation and the reversal of rotation have been very numerous. Here, if anywhere, however, success will depend on the absence of complicated devices, and it is undoubtedly a great step in advance if we can obtain a dynamo which is independent of the direction of rotation and also of the speed, at least, within wide limits. A battery of accumulators is necessary, of course, to take the load when the train is at rest or moving very slowly, and this battery can be connected permanently across the mains. The dynamo should then automatically come into action when the speed reaches a certain low value and give a current sufficient to supply the lamps and charge the battery, which current must, however, not increase to any great extent as the speed increases.

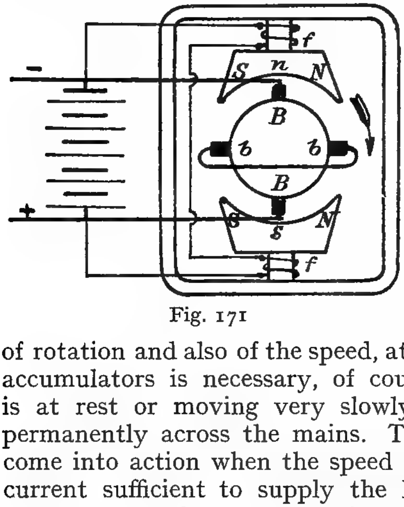

Fig. 171

In Rosenberg's train-lighting dynamo these peculiar characteristics are obtained in a very ingenious manner. Fig. 171 represents a 2-pole machine. The yoke and the pole-cores are very small, but the pole-shoes are abnormally large. The field coils *ff* are connected across the battery and thus carry a constant exciting current which makes the lower pole south and the upper one north. By rotation in this vertical field no P.D. will be produced between the brushes *BB*, but two other brushes *bb* are placed in the position of maximum P.D. and are short-circuited. The reason for making the pole-cores so small is now obvious, since a very small flux will suffice, even at a low speed, to produce the small E.M.F. required to drive the normal current through the short-circuited armature. This current will produce a field along a horizontal axis, that is, at right angles to the main or primary field. The return path for this secondary field is provided by the massive pole-shoes. Its direction depends on the direction of the short-circuit current through *bb*, and this depends on the direction of the E.M.F. induced by the rotation of the armature in the primary vertical field. Hence the horizontal field, due really to armature reaction, depends on the direction of rotation, and reverses when the latter reverses. The P.D. between the brushes *BB* due to the rotation of the armature in this horizontal field will thus be independent of the direction of rotation. These brushes *BB* are the main brushes and are joined to the battery terminals.

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* See *Storage Battery Engineering*, by **Lamar Lyndon.**

It is not difficult to see why large speed variations have such a small effect. The main current flowing through the armature from *B* to *B* will tend to demagnetise the primary field, and under normal conditions may neutralise 90 per cent. of the ampere-turns *ff*. The resultant vertical field will then be produced by the surplus 10 per cent. of *ff*. If now the speed increases the current will tend to increase, but this will increase the demagnetising effect and enormously weaken the resultant vertical field, on which the dynamo absolutely depends. It is obvious that if the speed increased infinitely the current could not increase more than 10 per cent. for a 10 per cent. increase would completely neutralise the coils *ff* and destroy the main or vertical field. There would then be no short-circuit current, and consequently no horizontal field. If the speed falls, say to 70 per cent. of its normal value, the main current will decrease, but if this decrease amounts to 10 per cent., the vertical field will be exactly doubled. Hence, although the speed is only 70 per cent., the short-circuit current will be increased 40 per cent., and with it the horizontal field. This horizontal field is almost inversely proportional to the speed, thus producing an almost constant E.M.F. in the direction *BB*. Fig. 172 shows the actual results obtained from such a dynamo working in parallel with a battery; the lower curves show the effect of putting a rheostat in series with the shunt-field *ff*. At a low speed—about 10 miles per hour—the dynamo begins to take the load, at 20 miles per hour it gives nearly its full current, which changes very little up to 60 miles per hour, or any higher

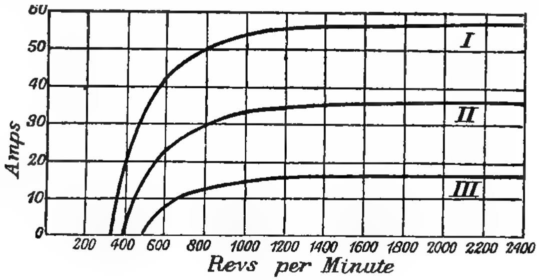

Fig. 172

speed which may be practicable. Means are provided for preventing the battery discharging through the dynamo at rest or at very low speeds. For further information respecting this dynamo, the student should consult the articles by Dr Rosenberg in the *Electrotechnische Zeitschrift*, 1905, p. 393; 1906, p. 1035; and by Osnos, 1907, p. 197.

A somewhat similar result is obtained by Osnos by the arrangement shown diagrammatically in Fig. 173. The field current of the main dynamo M is supplied by a direct coupled auxiliary dynamo D. The latter has two sets of brushes and two sets of field magnet windings. If the vertical field be Φ_y an electromotive force E_y will be induced between the brushes B_1B_2, and the current I_x will be due to the difference between this E.M.F. and that E of the auxiliary battery. Thus

$$I_x = c_1 (E - E_y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

The electromotive force E_y is proportional to the vertical flux Φ_y and to the speed, thus

$$E_y = c_2 . \Phi_y . n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

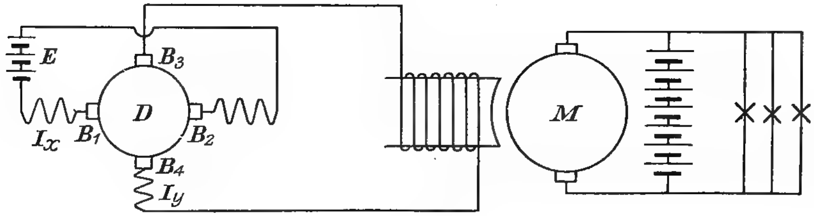

Fig. 173

The magnetic flux Φ_y is produced by the current I_y which flows through the armature between the brushes B_3 and B_4 and through the compensating field winding shown below the armature; neglecting saturation the flux is proportional to this current, thus

$$\Phi_y = c_3 . I_y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

The exciting current I_y is proportional to the electromotive force E_x induced in the armature between the brushes B_3 and B_4 by its rotation in the horizontal magnetic flux Φ_x, thus

$$I_y = c_4 E_x = c_5 \Phi_x n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

Finally, neglecting saturation, the flux Φ_x is proportional to the current I_x, thus

$$\Phi_x = c_6 I_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e).$$

Putting this value for Φ_x in equation (d), and substituting for I_x from equation (a) and for E_y from equations (b) and (c), one obtains for I_y the equation

$$I_y = \frac{n}{a + bn^2},$$

where a and b are constants.

Above a certain value of the speed n the constant a may be neglected in comparison with bn^2 and I_y, which is the field current of the main dynamo M, becomes inversely proportional to its speed. If, therefore this machine works below the knee of the magnetisation curve, so that its flux is proportional to the field current, its induced E.M.F. will not be affected to any great extent by variations in the speed, provided that this is above a certain value.

62. The Efficiency of Dynamos.

In considering the losses occurring in dynamos we shall first consider the no-load losses. In the first place we have mechanical friction at the bearings and brushes. To this must be added the air-friction, or so-called windage, which is very considerable in modern well-ventilated armatures. These frictional losses are independent of the excitation. We have also, however, the losses due to hysteresis and eddy currents, which depend on the excitation. The hysteresis loss occurs in the iron of the armature, and is approximately proportional to the 1·6th power of the flux-density. The eddy current loss occurs in the armature iron, the armature conductors and the pole-shoes. This loss is proportional to the square of the flux-density, since an increase in the number of lines causes an increase not only of the electromotive force but also of the eddy currents. The no-load loss for a given excitation can generally be looked upon as a constant loss.

To this no-load loss we must add:

the copper loss in the armature $I_a^2 . R_a$,

the copper loss in the series field winding $I_m^2 . R_m$,

the copper loss in the shunt field winding $I_s . V$.

The last one of these losses is dependent on the excitation, and will therefore vary somewhat with the load under which the machine is running. The variation is small, however, and this loss may be considered constant. We are assuming here that the terminal pressure is maintained constant.

The overall efficiency of the machine is the ratio of the useful output $V . I$ to the total power supplied to the machine. The latter is made up of the no-load losses and the total electrical power $E . I_a$ generated in the armature.

We have then for the efficiency

$$\eta = \frac{V . I}{E . I_a + P_0} \qquad \text{..............................(90)},$$

where P_0 is the power taken when running light, excluding the loss in the field windings, or, if we introduce the individual losses,

$$\eta = \frac{V . I}{V . I + I_a^2 . R_a + I_m^2 . R_m + V . I_s + P_0} \qquad \text{............(91)}.$$

The formula applies both to series, shunt and compound dynamos. If the compound machine has the short-shunt arrangement, that is, with the shunt field across the brushes instead of the terminals, the pressure across the shunt will be less than V, but the difference is so small that we may safely neglect it. In the simple shunt dynamo $I_m = 0$ and $I_a = I + I_s$.

We have then for a simple shunt dynamo

$$\eta = \frac{V.I}{V.I + I^2.R_a + 2I.I_s.R_a + I_s^2.R_a + V.I_s + P_0} \quad ......(92),$$

or

$$\eta = \frac{V}{V + 2I_s.R_a + I.R_a + \dfrac{P_0 + I_s^2.R_a + V.I_s}{I}}.$$

The efficiency will be a maximum when the denominator is a minimum. We must therefore differentiate this expression with respect to I, and put the result equal to 0. We have seen above that both V and I_s may be considered constant. We will also neglect the product $I_s^2.R_a$ on account of its smallness.

We get then
$$R_a - \frac{P_0 + V.I_s}{I^2} = 0,$$

or
$$I^2.R_a = P_0 + V.I_s.$$

Now, the external current is practically equal to the armature current, so that the left-hand side of this equation represents the armature copper loss. The right-hand side represents the losses which are practically constant, viz. the friction loss, the iron loss, and the shunt field loss. Hence, the efficiency is a maximum when the variable losses are equal to the constant losses.

Moreover, it is specially noteworthy that the efficiency curve, plotted to a base of variable output, is more or less parallel to the base line over a large range on either side of the maximum point. A large variation of load has consequently little effect on the efficiency.

As an example we shall take a machine for 110 volts and 100 amperes. Its no-load loss, that is, the power required to drive it when excited, but on open circuit, is 750 watts. Its full load efficiency is 0·88. The total power transmitted from the engine to the dynamo shaft is therefore $\frac{110 \,.\, 100}{0{\cdot}88} = 12{,}500$ watts. The total losses are $\frac{12}{100} \,.\, 12{,}500 = 1500$ watts. The dynamo has then its maximum efficiency at full load. The no-load loss of 750 watts is distributed between the field coils, which take 250 watts, and the friction and iron loss, which amount to 500 watts.

We have then

$$\begin{aligned} &V.I_s && = 250, \\ &I_s = \tfrac{250}{110} && = 2{\cdot}3, \\ &I_a = I + I_s = 100 + 2{\cdot}3 && = 102{\cdot}3, \\ &I_a^2.R_a && = 750, \\ &R_a = \frac{750}{I_a^2} = \frac{750}{102{\cdot}3^2} && = 0{\cdot}071. \end{aligned}$$

We now assume that the dynamo is run first on half-load and then on 100 per cent. overload. Neglecting any small change in the field current, we have

	Half-load		100 per cent. overload	
$I =$	50	and	200	respectively,
$I_a = I + I_s = 50 + 2{\cdot}3 =$	52·3	,,	202·3	,,
$I_a^2 . R_a = 52{\cdot}3^2 . 0{\cdot}071 =$	194	,,	2900	,,
$V . I_s =$	250	,,	250	,,
$V . I$ output =	5500	,,	22,000	,,
$P_0 =$	500	,,	500	,,
Input =	6444	,,	25,650	,,
$\eta = \frac{5500}{6444} =$	0·85	,,	$\frac{22,000}{25,650} = 0{\cdot}86.$	

Hence we see that the efficiency is very constant in the neighbourhood of its maximum, even for large variations in the load. The designer has a certain amount of freedom in the distribution of the total losses between constant losses and armature copper loss. The necessity for keeping down the pressure drop in the armature makes it impossible, as a rule, to design the machine so as to have its maximum efficiency at the normal load. In the above example the armature drop $I_a . R_a = 100 . 0{\cdot}071 = 7$ volts, the E.M.F. is $110 + 7 = 117$ volts, and the percentage drop in the armature is $7/117 = 6$ per cent. Moreover, if the armature resistance be made too great, the heat generated may be more than the armature surface can safely get rid of without reaching a dangerous temperature. In fact, the exact point of maximum efficiency is hardly considered when designing a dynamo, and the losses are distributed from entirely different practical considerations. As a rule the armature copper losses are less than the constant losses under normal running conditions, so that the efficiency increases with the load, even up to considerable overloads.

We shall now consider the experimental determination of the various individual losses which go to make up the no-load loss of a dynamo. The loss in the field windings is easily found from the ohmic resistance and the field current, so that we need consider only the remaining losses P_0. If we have another suitable machine which can be used as a motor to drive the dynamo we are testing, the two are coupled together and the power measured which the motor takes to run the dynamo at the normal speed with its field magnets unexcited. The amount by which this exceeds the power taken by the motor when uncoupled is almost exactly equal to the friction loss in the dynamo. If the experiment be repeated with the dynamo fields excited, the power taken by the motor will increase by an amount $P_h + P_e$ equal to the hysteresis and eddy current loss in the dynamo.

Having found in this way the loss due to friction and that due to hysteresis and eddy currents, the experiment is repeated at another speed, the excitation of the dynamo being the same as before.

The loss $P_h + P_e$ is divided by the speed n to which it corresponds, and the quotient plotted vertically on a base representing the speed n (Fig. 174).

Now the hysteresis loss is directly proportional to the speed, while the eddy current loss varies as the square of the speed, so that we may write

$$P_h + P_e = c_1 n + c_2 n^2 \qquad (93),$$

where c_1 and c_2 are constants.

From this it follows that

$$\frac{P_h + P_e}{n} = c_1 + c_2 n.$$

This shows that the points representing $\frac{P_h + P_e}{n}$ in Fig. 174 should lie on a straight line which cuts the vertical axis at a height c_1 equal to the hysteresis loss at a speed of one revolution per second. Multiplying this by the speed in revolutions per second, we get the actual hysteresis loss. We have thus separated the loss P_0 into its three components, friction, hysteresis and eddy current loss.

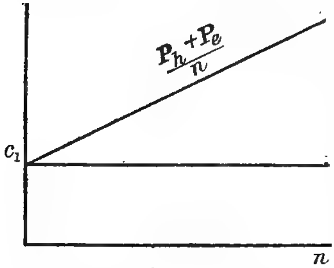

Fig. 174

In the absence of a second machine, the losses can be separated by the so-called retardation method. The machine is run light as a motor with a constant field excitation, but with an adjustable P.D. applied to its armature terminals. In this way the speed is varied and the power P_0 supplied to the armature is observed for each speed. In Fig. 175 the values of n are plotted as ordinates and the corresponding values of P_0 as abscissae.

The armature circuit is then opened suddenly without altering the field excitation, so that the armature gradually slows down and comes to rest. The speed is observed at frequent intervals during the slowing down and plotted to a base representing the time (Fig. 176). It is sometimes more convenient to read the armature P.D. on a voltmeter than the speed, to which it is proportional. The decrease in the kinetic energy of the armature during any interval is necessarily equal to the work done in the same time in friction, hysteresis and eddies. At a certain moment t seconds after opening the armature circuit, the speed $n = AC$, and the kinetic energy W in joules is decreasing at the rate of dW in the time dt. Energy is being dissipated, therefore, at the rate of $-\frac{dW}{dt}$ joules per second. The kinetic energy at any moment is proportional to the square of the speed. We have then

$$W = k.n^2,$$

$$dW = k.2n.dn,$$

or

$$\frac{dW}{dt} = 2k.n.\frac{dn}{dt},$$

where k is a coefficient depending on the moment of inertia of the armature. We have just seen, however, that the energy dissipated per second is equal to $-\frac{dW}{dt}$, and since the energy dissipated per second must be equal to the sum of the losses P_0, which is given by the abscissa OD in Fig. 175, we have

$$P_0 = -2k \, . \, n \, . \, \frac{dn}{dt} \quad \text{..........................(94)}.$$

If we draw both the tangent and the normal to the curve at the point A, we have (see Fig. 176)

$$\tan \alpha = -\frac{dn}{dt} = \frac{BC}{AC},$$

and since $AC = n$

$$-n \, . \, \frac{dn}{dt} = BC.$$

Hence

$$P_0 = 2k \, . \, BC.$$

We have already found $P_0 = OD$ for this speed in Fig. 175. It is evident that from a knowledge of P_0 and BC, we could calculate the moment of inertia of the armature, which is contained in the constant k. For our purpose, however, this is not necessary, since we now know that the length BC represents a certain power OD. If, instead of A, we take any other point on the curve in Fig. 176, and draw the normal and the ordinate, then the subnormal, i.e. the intercept on the base line between the ordinate and the normal, represents the power being dissipated at that speed.

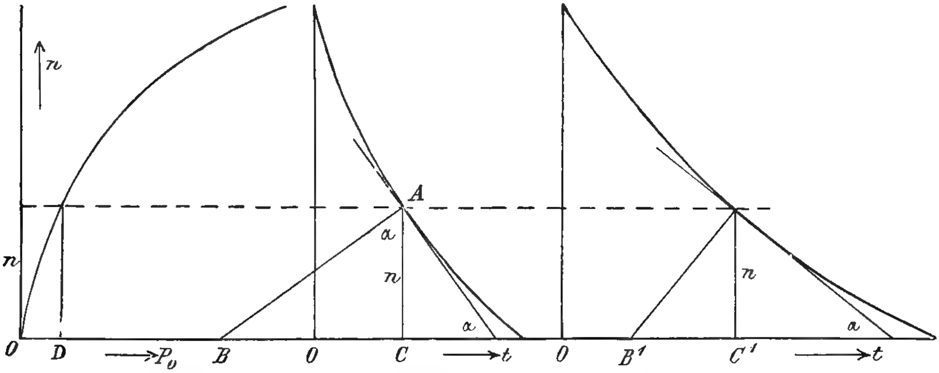

Fig. 175 Fig. 176 Fig. 177

The experiment is then repeated with the field magnets unexcited, in which case the time taken to slow down is much longer, since there is no hysteresis or eddy loss. In this way the curve in Fig. 177 is obtained, which has a subnormal $B'C'$ at the chosen speed n. In the same way as before, we could show that the power wasted in friction is given by the equation

$$P_f = 2k \, . \, B'C',$$

and is represented by the length $B'C'$ to the same scale that BC represents the power P_0; hence

$$\frac{P_f}{P_0} = \frac{B'C'}{BC}.$$

Up to the present we have tacitly assumed that for a constant speed and excitation, the hysteresis and eddy current losses are the same at full load as at no-load. This assumption is not strictly correct owing to the distortion of the field by the armature reaction, so that an accurate determination of the efficiency is only possible when the machine is actually running on full load. The supply of power required to test a large machine in this way would be very considerable. A great saving can be effected if we have two similar machines, by coupling them mechanically together and running one of them as a motor from an external source. This motor drives the other machine as a generator, and the latter supplies current to the motor. The external source of supply and the generator are thus connected in parallel to supply the motor. The power supplied from the external source is, of course, merely the total loss in the two machines, and can be easily measured. The excitation of each machine must be so adjusted that the speed and armature currents of the two machines correspond to the normal running conditions. This method of testing two similar machines was devised by Hopkinson, and is known as the Hopkinson test. At first sight, the exact conditions under which the two machines are running appear rather complicated. We must remember, however, that both machines have the same speed and the same terminal pressure. The E.M.F. of the generator is greater than the terminal pressure, whereas in the motor, as we shall see in Section 64, the terminal pressure exceeds the E.M.F. induced in the armature. Since the two machines have exactly the same speed, terminal pressure and number of wires on the armature, it follows that the generator must be more strongly excited than the motor, or, to put it more correctly, the machine with the weaker excitation will run as a motor and drive the other machine as a generator. If

Φ_1 is the flux per pole in the motor,
Φ_2 the flux per pole in the generator,
I_1 the armature current of the motor,
I_2 the armature current of the generator,
$I_1 - I_2 = I_0$ the current supplied from external source,
E_1 the back E.M.F. of the motor,
and E_2 the E.M.F. of the generator,

then for the motor (see Section 64) we have

$$V = E_1 + I_1 . R_a = \frac{p}{a} . \Phi_1 . \frac{n}{60} . Z . 10^{-8} + I_1 . R_a,$$

and for the generator

$$V = E_2 - I_2 . R_a = \frac{p}{a} . \Phi_2 . \frac{n}{60} . Z . 10^{-8} - I_2 . R_a$$

Adding these two equations, we get

$$2V = \frac{p}{a} . (\Phi_1 + \Phi_2) . \frac{n}{60} . Z . 10^{-8} + I_0 . R_a .$$

If we neglect the very small quantity $I_0 . R_a$, we have

$$\frac{p}{a} . \frac{n}{60} . Z . 10^{-8} = \frac{2V}{\Phi_1 + \Phi_2} .$$

The speed is thus inversely proportional to the sum of the fluxes Φ_1 and Φ_2. If we substitute this value of $\frac{p}{a} \cdot \frac{n}{60} \cdot Z \cdot 10^{-8}$ in the above equation for the generator, we get

$$V = \frac{\Phi_2 \cdot 2V}{\Phi_1 + \Phi_2} - I_2 \cdot R_a,$$

or

$$I_2 = \frac{\Phi_2 - \Phi_1}{\Phi_1 + \Phi_2} \cdot \frac{V}{R_a}.$$

Hence while the speed is proportional to $\frac{1}{\Phi_1 + \Phi_2}$, the generator current is proportional to $\frac{\Phi_2 - \Phi_1}{\Phi_1 + \Phi_2}$. Both the exact speed and the exact current strength at which we wish to test the machines can therefore be obtained by suitably adjusting the excitation of the two machines. The product of the terminal pressure V and the current I_0 supplied from the external source, that is, the product of two simple observations, gives us at once the total losses in the two machines. It is usual to assume that these are equally divided between the two machines, or, what is nearly the same thing, that the machines have the same efficiency. A still simpler way of considering the problem is as follows: The input to the motor is $V \cdot I_1$; the output of the dynamo $V \cdot I_2$, and therefore the combined efficiency $\frac{V \cdot I_2}{V\,I_1} = \frac{I_2}{I_1}$. If their efficiencies are the same, each must be equal to $\sqrt{\frac{I_2}{I_1}}$.

CHAPTER IX

63. Direction of rotation of motors.—64. Torque, speed and output of a D.C. motor.—65. Motor with constant excitation.—66. The starting and regulation of a shunt motor.—67. Principle of series motor.—68. Example of traction motor calculation.—69. The regulation of series motors.—70. Temperature rise of motors, etc.

63. Direction of Rotation of Motors.

Direct-current motors are identical in construction with direct-current generators. Since the same machine may act at one moment as a motor and at the next as a generator, much of what has been said concerning generators can be applied to motors. We must consider, however, the mechanical characteristics of motors, such as the direction of rotation, the speed, the torque and the output, and we have specially to investigate the manner in which these mechanical characteristics depend on the electrical and magnetic relations.

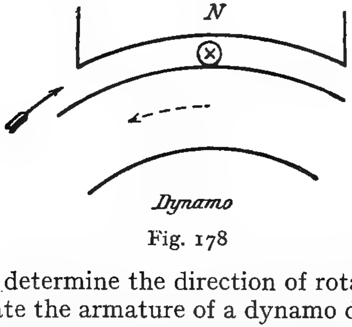

Fig. 178

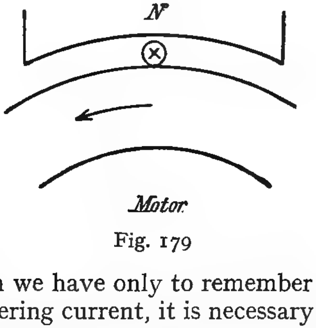

Fig. 179

To determine the direction of rotation we have only to remember that, to rotate the armature of a dynamo delivering current, it is necessary from the principle of the conservation of energy that mechanical work is done. The current induced by the motion tends to stop the motion, owing to the opposing force exerted on the conductor by the magnetic field. If the dynamo represented in Fig. 178 be rotated in a clockwise direction, the current will flow from front to back under a north pole and in the opposite direction under a south pole. The magnetic field exerts a force on the conductor, the direction of which can be found by the left-hand rule. Pointing the first finger of the left hand in the direction of the magnetic field and the second finger along the current, the thumb indicates the direction of the motoring force on the wire. In this case, however, this is unnecessary, as we know from Lenz's law that the force will be opposed to the motion which induced the current. This opposing force is shown by a dotted arrow. We may imagine that the machine is continually seeking to act as a motor, but never succeeds, as the driving force of the prime mover overcomes the motoring force.

This latter force has full play, however, if we shut down the engine and supply the armature current from some external source. We assume that the current in both field and armature is still maintained in the original direction (Fig. 179). It is evident, therefore, that the motor shown in Fig. 179 will rotate in an anti-clockwise direction as indicated by the arrow. We conclude, therefore, that **with similar magnetic polarity and similar direction of current in the armature, a machine runs as a motor in the opposite direction to that in which it was driven as a generator.** The same result is obtained by simultaneously reversing both the magnetic polarity and the armature current.

The student must be careful, however, not to make the mistake of assuming that any machine running as a generator will run, as a matter of course, in the opposite direction as a motor. We were careful to point out that both the magnetic polarity and the armature current were unchanged in our example in which the rotation was reversed. We must now turn to the special cases of shunt and series motors and see whether this supposition is correct.

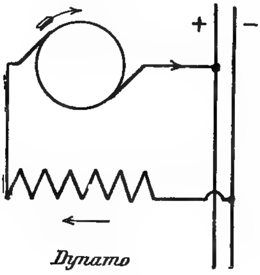

Fig. 180

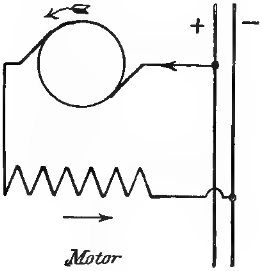

Fig. 181

In the series machine field windings and armature are connected in series. We assume that the series machine shown in Fig. 180 is driven in a clockwise direction as a generator and that the right-hand brush is positive. We now use the same machine as a motor without changing the connections in any way. From what has been said above it is evidently immaterial, as far as the direction of rotation is concerned, how we connect the motor terminals to the mains. The simpler plan, of course, is to leave the connections as they were (Fig. 181). The motor current is then reversed both in field and armature. The result is the same as if the current had remained unchanged in both. **Hence, if the connections of a series generator remain unchanged, it will run as a motor in the reverse direction, i.e. against the brushes.**

If, however, we wish the machine to run in the same direction, we must reverse the field connections. The direction of the current in the field windings will then be the same in the motor as it was in the generator, and the magnetic polarity will remain unchanged.

We shall now consider the case in which a machine which has bee running as a series motor is driven in the same direction and used as a seri generator. This case is of practical importance, since series motors a braked by being cut off from the supply mains and loaded as generato by means of a resistance connected between their terminals. The kinet

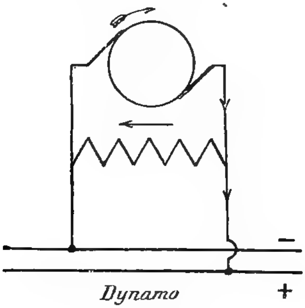

Fig. 182

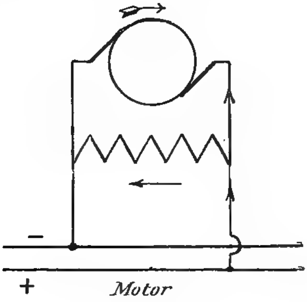

Fig. 183

energy of the tramcar, for example, is thereby converted into electric energy and dissipated as heat in the resistance, and the car is rapidl retarded. Now, the E.M.F. induced in the motor armature is opposed i direction to the motor current (Section 64). When used as a generator tl direction of rotation is unchanged, and the field is the remanent magnetisı of the motor field, and therefore in the same direction. The E.M.F. in tl generator is therefore in the same direction as in the motor, and produce a current in the opposite direction to the motor current. This current woul weaken the remanent magnetism instead of strengthening it, and tl machine would refuse to excite. Hence, it is necessary to reverse the fiel connections when using the motor as a brake.

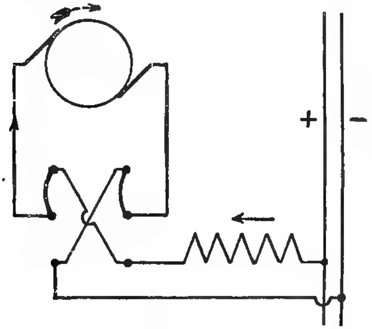

Fig. 184

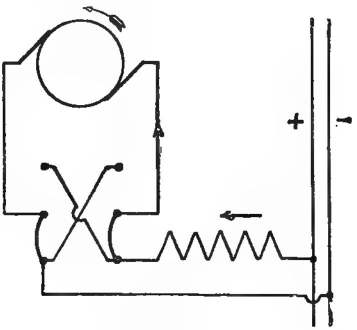

Fig. 185

In the shunt motor, however, the relations are very different. A glan at Figs. 182 and 183 shows us that the current in the field windings is in tl same direction, whether the machine is used as a motor or as a generatc if the connections remain unchanged. The armature current, on the oth hand, is reversed. Had the current remained the same in both armature ar field, the direction of rotation would have been changed, but as it has only reversed in the armature, the direction of rotation of the motor will be the same as that of the generator. Hence, a shunt dynamo will run as a motor in the same direction, i.e. with the brushes; similarly a shunt motor can be run as a generator in the same direction without altering the connections.

We have finally to consider the reversal of the direction of rotation of motors. Were we merely to change the connections with the supply mains, we should reverse the current in both armature and field windings, and the direction of rotation would be the same as before. We must evidently reverse the connections of either the field or armature, but not of both. As a rule it is the armature current which is reversed (Figs. 184, 185).

64. Torque, Speed and Output of a D.C. Motor.

An example was worked out in Section 25, showing how the turning-moment or torque could be calculated from the flux, the number of armature wires and the armature current. We must now find a general formula for the torque of a motor. Let

H be the strength of the field in the air-gap,
L the length of the armature in cm,
D the diameter ,, ,,
Z the number of wires on the armature periphery,
β the ratio of pole-arcs to pole-pitch,
I' the current in each wire in absolute units,
and I_a the external armature current in amperes.

Then, from equation (32) on page 54, the force exerted on each armature wire under a pole by the magnetic field is

$$H \, . \, I' \, . \, l \text{ dynes.}$$

The total number of wires under the $2p$ poles is $Z\beta$.
The total force is therefore

$$f = H \, . \, I' \, . \, Z \, . \, \beta \, . \, L \text{ dynes} \quad \ldots\ldots\ldots\ldots\ldots\ldots (95).$$

To convert this into kilogrammes we must divide it by 981,000. The radius of the armature is

$$r = \frac{D}{2 \, . \, 100} \text{ metres.}$$

For the torque in metre-kilogrammes we have, therefore,

$$M_t = \frac{H \, . \, I' \, . \, Z \, . \, \beta \, . \, L}{981{,}000} \cdot \frac{D}{200}.$$

Now, the total flux per pole Φ is equal to the product of the strength of the field in the gap and the area of the pole face, or

$$\Phi = \frac{\pi D}{2p} \, . \, \beta \, . \, L \, . \, H \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (96).$$

Substituting this value, we have

$$M_t = \frac{p \, . \, \Phi \, . \, Z \, . \, I'}{\pi \, . \, 9{\cdot}81} \, . \, 10^{-7} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(97).$$

This equation applies equally to all machines, bipolar or multipolar, series or parallel wound. If, as before, there are $2a$ parallel paths through the armature, we have

$$I' = \frac{I_a}{10 \, . \, 2a}.$$

Equation (97) then becomes

$$M_t = \frac{p}{a} \, . \, \frac{\Phi \, . \, Z \, . \, I_a}{2\pi \, . \, 9{\cdot}81} \, . \, 10^{-8} \text{ met.-kg} \quad \ldots\ldots\ldots\ldots\ldots\ldots(98).$$

For series winding $a = 1$ and for parallel winding $a = p$. This formula gives the total torque exerted on the armature. The useful torque is less than this by an amount necessary to overcome the friction.

If we call $2p\Phi$, "the total flux crossing the air-gap," and $\frac{ZI'}{10}$, i.e. the product of the number of wires or bars on the whole armature periphery and the current in amperes in each, "the ampere-bars on the armature," formula (98) may be expressed in the following convenient form,

$$\text{Torque} = \frac{\text{total flux} \times \text{ampere-bars}}{8{\cdot}54 \times 10^8} \text{ pound-feet.}$$

The full meaning of these formulae for the turning-moment only becomes apparent when we exchange the two sides of the equation. In their above form the equations state that the torque exerted by the motor is proportional to the product of the magnetic flux and the armature current. When a motor is working and has reached a steady condition, it exerts a torque exactly equal to the resisting torque due to the load on the motor. We are thus led to the important conclusion, that the product $\Phi \, . \, I_a$ automatically adjusts itself to the load on the motor, so as to exert the requisite torque. We do not send a current of an arbitrarily fixed strength through the motor and allow the motor to exert a corresponding torque, whatever it may be. This is only the case before the motor moves, when the current is not sufficient to exert the necessary starting torque. As soon as the motor reaches a uniform steady speed, equation (98) must be satisfied, and the current must adjust its strength to suit the torque given by the load.

We have yet to answer the question as to how it comes about that the current in the armature is just that corresponding to the load, and neither more nor less. There is no apparent regulator by means of which the current can be varied, since the armature has a negligible resistance. We have, however, in the motor a back electromotive force which, like the governor of a steam engine, is dependent on the speed. It is immaterial whether the machine is rotated as a generator by means of external forces, or as a motor by means of forces within the machine itself. In both cases the armature wires cut through the lines of force, and thereby induce electromotive forces in themselves. A motor is therefore not only similar to a generator in construction, enabling the same machine to be used first as the one and then as the other, but, while running as a motor, the machine is continually generating an E.M.F. and tending to act as a generator. By applying the right-hand rule it can be seen that this induced E.M.F. is in the opposite direction to the terminal pressure and current, and it is for this reason that it is often referred to as the back E.M.F. of the motor.

The back E.M.F. is, of course, proportional to the speed, and it is this fact which governs the adjustment of the current to the exact strength necessary for the torque. If, when the motor is running, its load is suddenly increased, the torque which it has been exerting is no longer sufficient to overcome the load. The speed of the motor drops, with the result that the lines of force are not cut so rapidly, and the back E.M.F. is therefore decreased. The terminal P.D. is then able to send a larger current through the armature. This decrease of speed and consequent increase of current goes on until the current is large enough to exert a torque corresponding to the new load.

Inversely, if the load be decreased, the torque exerted by the motor is too large. The motor therefore speeds up and increases the back E.M.F. until the current is choked down to a value corresponding to the torque required for the new load. When the motor has settled down again to a steady speed, the torque exerted by it is exactly equal to the resisting torque of the load, and not slightly larger than it, as many beginners are apt to imagine. Due allowance must be made, however, for the torque necessary to overcome friction at the motor bearings, etc.

The speed can be found from equation (79) on page 124, according to which

$$E = \frac{p}{a} \,.\, \Phi \,.\, n \,.\, Z \,.\, 10^{-8} \text{ volts.}$$

The back electromotive force E is determined by the armature current, for the applied terminal pressure V must exceed the back E.M.F. by an amount necessary to drive the current through the resistance of the armature, that is, by $I_a . R_a$ volts. We have then

$$V = E + I_a . R_a \qquad \text{(99).}$$

From these two equations, we have

$$n = \frac{E \,.\, 10^8}{p/a \,.\, \Phi \,.\, Z} = \frac{(V - I_a \,.\, R_a) \,.\, 10^8}{p/a \,.\, \Phi \,.\, Z} \qquad \text{(100).}$$

It is evident from this formula that a change of load, and therefore of armature current, does not cause merely a momentary change of speed, but a permanent change. The variation of speed with change of load is, however, very small in many cases.

We must now find the formula for the output or power of the motor, that is, the rate at which it does mechanical work. We have purposely postponed this until after the consideration of the torque, although in practice the capability of a motor is always expressed in terms of its output and not of its torque. To get a clear idea, however, of the action of a motor it is absolutely necessary to consider the torque first, since the output depends on the product of two variables, viz. the torque and the speed.

Now, if M_t be the turning-moment in metre-kg, and

$$\omega = 2\pi . n$$

be the angular velocity, we know from mechanics that mechanical work is being done at the rate of $M_t . \omega$ metre-kg per second. To obtain the power in watts we must multiply this by 9·81 (see Section 41); in this way we get

$$P = M_t . \omega . 9{\cdot}81 = M_t . 2\pi . n . 9{\cdot}81 \text{ watts} \quad \ldots\ldots\ldots\ldots (101).$$

If M_t be expressed in foot-pounds, the figure 9·81 becomes 1·36.

It is important to find the output as a function of the electrical quantities. For this purpose, we multiply equation (99) by I_a and get

$$V . I_a = E . I_a + I_a^2 . R_a \quad \ldots\ldots\ldots\ldots (102).$$

The product $V . I_a$ is the total power transmitted to the armature, while $I_a^2 . R_a$ is the power dissipated as heat due to the armature resistance. Hence, the remainder $E . I_a$ must represent the power transformed into a mechanical form. This will include both the useful output and the power wasted in the bearings and in the iron of the armature.

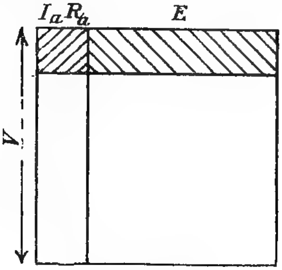

Fig. 186

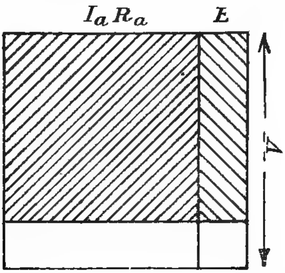

Fig. 187

It is possible for us now to compare the two expressions found for the output. Equating them, we have

$$E . I_a = M_t . 2\pi . n . 9{\cdot}81 \quad \ldots\ldots\ldots\ldots (103).$$

If we substitute for E and M_t the values found for them in equations (79) and (98), we shall find that both sides of our equation are identical.

When we consider the relation between the output and the armature current, we see at once that for a very small current, and consequently a very small torque, the output will be small. With increasing load the output increases, until with very large loads the speed drops so much that the output is decreased. To find when the output is a maximum, we must differentiate the equation

$$P = E . I_a = V . I_a - I_a^2 R_a,$$

and equate the result to 0. We have thus

$$\frac{dP}{dI_a} = V - 2I_a . R_a = 0,$$

or

$$I_a . R_a = \frac{V}{2}.$$

For this case we have also $I_a . R_a = E$.

The same result is obtained from Figs. 186 and 187, in which the side of a square is made to represent the terminal pressure V, and is divided into two parts, one equal to the ohmic drop $I_a . R_a$ and the other to the back E.M.F. R_a being constant, the area of the shaded square is proportional to the power $I_a^2 . R_a$ wasted in heating, while the area of the shaded rectangle is proportional to $I_a . E$ and represents the mechanical output. This shaded rectangle is small both for small (Fig. 186) and for large (Fig. 187) values of I_a. It has its maximum area when it is a square, i.e. when $I_a . R_a = E$. This is therefore the condition for maximum output, as we have already found. On account of the large armature current, this case lies far outside the limits of practical working. It would evidently be very bad practice to waste half of the energy in heating the armature wires.

65. Motor with Constant Excitation.

We shall consider in this section a motor in which the field winding is connected directly to the constant pressure supply mains. In the first place we shall also assume that the armature is connected directly across the mains (Fig. 183).

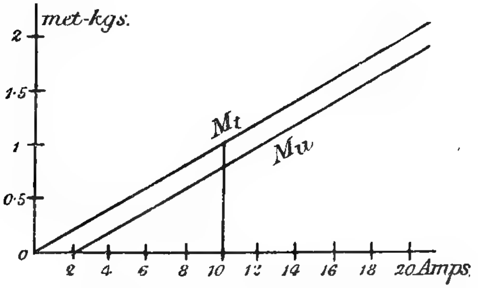

Fig. 188

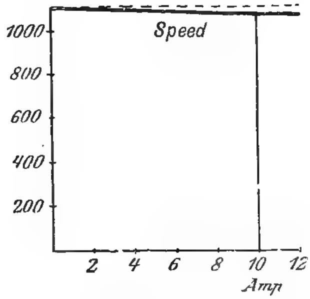

Fig. 189

The total torque exerted by the motor is given by equation (98) on page 200 as

$$M_t = \frac{p}{a} \cdot \frac{\Phi . Z . I_a}{2\pi . 9{\cdot}81} . 10^{-8} \text{ metre-kg.}$$

Since the magnetising current, and therefore the flux Φ, is constant in the present case, the total torque must be proportional to the armature current I_a. Plotting the armature current horizontally and the torque M_t vertically, we get a straight line passing through the origin (Fig. 188).

The useful turning-moment M_u is less than M_t by an amount necessary to overcome the resisting torque due to friction, hysteresis and eddy currents. If the no-load current in the armature is I_0, the useful torque is proportional to the difference $I_a - I_0$. If we plot the useful torque in Fig. 188, we get another straight line parallel to the first and cutting the horizontal axis at the point I_0.

For the speed, we have from equation (100) on page 201

$$n = \frac{(V - I_a . R_a) . 10^8}{p/a . \Phi . Z} \text{ revolutions per second } \dots\dots\dots(104).$$

Since the flux is still constant, the speed is proportional to the back E.M.F. $(V - I_a . R_a)$. The drop of speed from no-load is therefore proportional to the pressure drop $I_a . R_a$, and therefore also to the current I_a. If then we plot the armature current horizontally and the speed vertically, we get the slightly falling straight line in Fig. 189. Since the ohmic pressure drop in the armature is always small, the motor with constant excitation has almost a constant speed from no-load to full load.

The output in H.P. on the pulley, the so-called brake horse-power (B.H.P.), can either be found from the useful torque and the angular velocity, in which case we get $\frac{M_u . 2\pi . n}{76}$, or it can be found from the back E.M.F. and the useful current $I_a - I_0$, in which case we get $E . (I_a - I_0)/746$.

If I is the sum of the armature current I_a and the field current I_s, the total power supplied to the motor is $I . V$ watts, and the overall efficiency is

$$\eta = \frac{E . (I_a - I_0)}{V . I} = \frac{(V - I_a . R_a)(I_a - I_0)}{V . (I_a + I_s)} \quad \text{............(105).}$$

It is evident that at no-load, that is, when $I_a = I_0$, the efficiency must be nil. This is also the case when the load on the motor is so large that it cannot start even with the largest possible current

$$I_a = \frac{V}{R_a}.$$

The fact that such a current would burn out the armature does not concern us at the present moment. The important point to notice is that there must be some intermediate load for which the efficiency is a maximum. To find the most efficient armature current we multiply out the numerator of equation (105) and get

$$\eta = \frac{V . I_a - I_a^2 . R_a - V . I_0 + I_a . I_0 . R_a}{V . (I_a + I_s)}.$$

We now differentiate this with regard to I_a and put the result equal to 0. For the sake of simplicity we neglect the constant factor V in the denominator.

$$\frac{d\eta}{dI_a} = \frac{(V - 2I_a . R_a + I_0 . R_a)(I_a + I_s) - (V . I_a - I_a^2 . R_a - V . I_0 + I_a . I_0 . R_a)}{(I_a + I_s)^2}.$$

This expression will vanish when the numerator vanishes. If we multiply the numerator out and put it equal to 0, we get

$$I_a^2 . R_a - V . I_s - V . I_0 + I_s . R_a . (2I_a - I_0) = 0.$$

The meaning of this equation becomes clearer when we remember that, owing to the smallness of the magnetising current, the right-hand term is so small as to be practically negligible. We got then as the condition for maximum efficiency

$$I_a^2 . R_a = V . I_s + V . I_0 = V (I_s + I_0) \quad \text{...........(106).}$$

Now $I_s + I_0$ is the total motor current at no-load. Hence, the efficiency is a maximum when the ohmic heating of the armature conductors is equal to the constant losses both in armature and field, including the friction.

In other words, the efficiency is a maximum when the losses dependent or the load become equal to those independent of the load. The decrease o friction due to the small decrease of speed with increasing load is quite negligible.

In order to make the above considerations clearer, we shall apply then to a numerical example.

We shall consider a bipolar motor ($p = 1$, $a = 1$) having the following constants:

$$V = 110, \qquad \Phi = 2 . 10^6, \qquad Z = 300,$$
$$R_a = 0{\cdot}3, \qquad I_s = 1, \qquad I_0 = 2.$$

We have then

$$M_t = \frac{p}{a} . \frac{\Phi . Z . I_a}{2\pi . 9{\cdot}81} . 10^{-8} = 0{\cdot}1 . I_a,$$

$$M_u = 0{\cdot}1 \, (I_a - I_0) = 0{\cdot}1 I_a - 0{\cdot}2,$$

$$n = \frac{(V - I_a . R_a) . 10^8}{p/a . \Phi . Z} = \frac{110 - 0{\cdot}3\, I_a}{6} \text{ revs. per sec.} = 1100 - 3 I_a \text{ revs. per min}$$

If the normal current for which the armature is designed is 10 amperes we obtain a total full load turning-moment of 1 metre-kg of which 0·8 metre-kg is available at the pulley. The speed at full load is 1070. The drop of speed from full to no-load is 30 in 1100, or roughly 3 per cent.

For an armature current of 10 amperes, the overall efficiency will be

$$\eta = \frac{(V - I_a . R_a) . (I_a - I_0)}{V . (I_a + I_s)} = \frac{107 . 8}{110 . 11} = 0{\cdot}7.$$

To find the conditions at maximum efficiency, we must first find the constant losses. These are

$$V . I_0 + V . I_s = 330 \text{ watts.}$$

For maximum efficiency, the armature copper loss must also be equal to 330 watts, that is,

$$I_a^2 . R_a = 330 \text{ or } I_a = \sqrt{\frac{330}{0{\cdot}3}} = 33 \text{ amperes.}$$

Adding the field current of 1 ampere, we get a total current of 34 amperes and a motor input of

$$V . I = 110 . 34 = 3740 \text{ watts.}$$

If we subtract from this input the total losses which are 2 . 330 watts, we obtain the output 3740 − 660 = 3080 watts, and the overall efficiency is therefore

$$\eta_{\max} = \frac{3080}{3740} = 0{\cdot}825.$$

The output on the pulley is then $\frac{3080}{746} = 4{\cdot}1$ B.H.P. The percentage drop of speed from no-load is equal to the percentage pressure drop in the armature, that is 0·3 . 33 in 110 or 9 per cent. (see page 204).

It is interesting to determine how the efficiency varies when the load is varied over a large range. If we assume that the load is reduced to about half of that corresponding to maximum efficiency, so that the armature current is 17 amperes, then we have

$$E = V - I_a . R_a = 110 - 17 . 0{\cdot}3 = 105 \text{ volts,}$$

$$I = I_a + I_s = 17 + 1 = 18 \text{ amperes,}$$

$$I_a - I_0 = 17 - 2 = 15 \text{ amperes.}$$

The overall efficiency will then be

$$\eta = \frac{E . (I_a - I_0)}{V . I} = \frac{105 . 15}{110 . 18} = 0{\cdot}8.$$

In the same way, if the load is about doubled, so that the armature current is 66 amperes, we have

$$E = V - I_a . R_a = 110 - 66 . 0{\cdot}3 = 90 \text{ volts,}$$

$$I = I_a + I_s = 66 + 1 = 67 \text{ amperes,}$$

$$I_a - I_0 = 66 - 2 = 64 \text{ amperes.}$$

Under this heavy load, the efficiency will be

$$\eta = \frac{E . (I_a - I_0)}{V . I} = \frac{90 . 64}{110 . 67} = 0{\cdot}78.$$

Hence, we see that a large variation in the load has very little effect on the efficiency. It is therefore unimportant to design a motor to have its maximum efficiency at full load. To reduce the drop of speed as much as possible the normal load should be below the point of maximum efficiency, while a motor which is to work, as a rule, on very light loads, and only have its full load at intervals, should be designed to have its maximum efficiency at a small load.

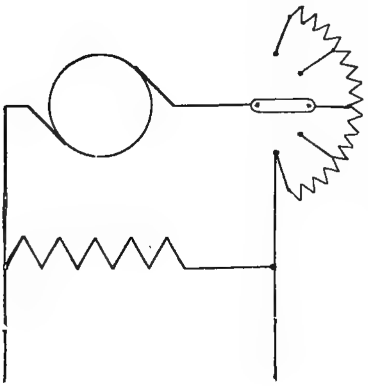

Fig. 190

We pass now to the consideration of a motor the field winding of which is connected directly across the constant pressure supply, but the armature of which has an adjustable resistance in series with it (Fig. 190). We make the striking discovery that, whether loaded or running light, the current is in no way affected by a change in the adjustable resistance. If, however, we turn to equation (98), on page 200, we see that the armature current must be determined entirely by the load, since the magnetic flux is constant. If the load on the motor remains constant, the armature current will remain constant, however much the series resistance may be varied. On suddenly decreasing the resistance there is, it is true, a momentary increase of the armature current. This causes the motor to exert a greater torque than that required to overcome the load, with the result that the armature is accelerated and runs permanently at a higher speed. This increase of speed, however, causes an increase in the back E.M.F., which reduces the armature current to its former value corresponding to the load. The only way to vary the armature current of a constantly excited motor is to vary the load.

The speed, on the other hand, is very dependent on the resistance R in series with the armature. The applied terminal pressure V has now to overcome the back E.M.F. and cover the pressure drop in both the armature and the resistance R. We have then

$$V = E + I_a . R_a + I_a . R,$$

or

$$E = V - I_a . R_a - I_a . R.$$

In this equation V and R_a are constant, and I_a is also constant so long as the load remains unchanged. It follows, therefore, that a change in the value of R will produce a large effect on E, and consequently on the speed.

This becomes much clearer if we neglect the small drop of pressure in the armature, and assume that the pressure across the brushes is exactly equal and opposite to the back E.M.F. On this assumption, a voltmeter connected across the brushes will indicate directly the back E.M.F. We find then that, on varying the value of R, the speed is almost directly proportional to the pressure across the brushes. The accuracy is greatest when the pressure drop in the armature is as small as possible, that is, at no-load. In the numerical example which we have worked out above, the speed, when running light with a P.D. of 110 volts between the brushes, is found from equation (104), on page 203, to be

$$n = \frac{(110 - 0{\cdot}3 . 2) . 10^8}{2 . 10^6 . 300} = 18{\cdot}2.$$

If a resistance of 27·5 ohms be connected now in series with the armature, the no-load current of 2 amperes will be unaltered. The drop of pressure in this resistance will therefore be 27·5 . 2 = 55 volts, and the pressure read on the voltmeter across the brushes will be reduced to 110 − 55 = 55 volts. The speed will therefore be

$$n = \frac{(55 - 0{\cdot}3 . 2) . 10^8}{2 . 10^6 . 300} = 9{\cdot}1.$$

The speed is thus exactly a half of what it was when the P.D. between the brushes was 110 volts. The speed is therefore practically proportional to the P.D. between the brushes when the motor is running light. The same relation is approximately true when the motor is loaded. We are thus led to the interesting conclusion that, in a motor with constant excitation, the current depends only on the load, when the speed depends only on the P.D. between the brushes. Hence the speed can be regulated to any desired extent by altering the P.D. applied to the brushes.

There are, however, two errors which we must be careful to avoid. In the first place, it is very important to notice that in the above experiments the adjustable resistance was in the armature circuit only and had no effect on the field. If the armature and field windings are connected directly to the same terminals, and the resistance connected in the common external circuit, an alteration in the resistance may, under some circumstances, have very little effect on the speed. If, for example, we double the terminal pressure of both armature and field, the magnetic flux may be nearly doubled, if the iron be only slightly saturated. In the formula

$$n = \frac{E \cdot 10^8}{p/a \cdot \Phi \cdot Z}$$

both the value of E in the numerator and the value of Φ in the denominator would then be doubled, leaving the speed n unchanged. If, on the other hand, the magnets are highly saturated, the motor will behave more like a constantly excited one, and doubling the terminal pressure will cause the speed to increase nearly 100 per cent.

The other point on which we must be quite clear is that speed regulation by means of a series resistance is only possible when an attendant can stand by the regulating switch, as on a tramcar, for example. Apart from the large loss of power in such a resistance, there is the disadvantage that every change of load causes a change of current and therefore a change in the amount of pressure lost in the resistance. The result is that the pressure across the armature terminals and the speed of the motor are subjected to large variations. It is possible, for instance, for a motor to be running light with so much resistance in series with its armature that it comes to rest when the load is put on.

66. The Starting and Regulation of a Shunt Motor.

The shunt motor is really no more than a constantly excited motor. It possesses, therefore, all the fundamental characteristics which we have found for this type of motor in the previous section. The current is proportional to the load, and the speed is practically constant from no-load up to full load. This latter property makes the shunt motor specially suitable for driving lines of shafting and machine tools, where constancy of speed is required. As we have already studied the principles of the motor, we need only consider here the more important points in connection with their starting and regulation.

The primary object of the starting resistance is to protect the armature from a dangerously large current. At the first moment of starting, the armature is at rest and there is consequently no back E.M.F. If the armature, with its comparatively small resistance, were connected directly across the full pressure of the supply mains there would be an enormous rush of current. A starting resistance must therefore be put in series with the armature, of such a value that the current is limited to a permissible strength with regard to its heating effect on the armature. There are also mechanical reasons for limiting the turning-moment exerted at starting, especially when there are heavy masses to set in motion. A further reason is the large pressure drop and consequent unsteadiness of the lamps in the neighbourhood, when the supply mains are suddenly called upon to carry a large current.

When the motor has been set in motion, the back E.M.F. increases with the speed and limits the current to a value corresponding to the load. The starting resistance can therefore be gradually cut out as the speed increases.

If it is required to regulate or vary the speed of a shunt motor, it must be done by altering the resistance of the shunt field circuit. This is clearly seen by applying equation (104) on page 203 to the case of a motor running on no-load. We have then

$$n = \frac{V \,.\, 10^8}{p/a \,.\, \Phi \,.\, Z}.$$

If resistance is put in the field circuit, and the magnetising current and the flux Φ thereby reduced, we see from this equation that the speed n must increase. This is also evident when we consider that the motor must run faster to produce the same back E.M.F. in the weaker field as it formerly did in the stronger field. The fundamental reason, however, for the increased speed is that on weakening the field, the back E.M.F. decreases, the speed not being able to change instantly. This decreased back E.M.F. enables the supply P.D. to drive an increased current through the armature, the increase being proportionately much greater than the decrease of field strength. The torque is therefore increased and the armature caused to accelerate.

If the load remains the same, this weakening of the field will naturally affect the armature current. Since the load or torque is proportional to the product $\Phi . I_a$ a weakening of the field must cause a corresponding increase in the armature current. To make this clearer we shall go back to our example in which the terminal pressure $V = 110$ volts, the armature resistance $R_a = 0{\cdot}3$ ohm, the number of wires $Z = 300$ and the flux $\Phi = 2 . 10^6$ lines. For an armature current I_a of 10 amperes, we have for a bipolar motor, from equation (104) on page 203,

$$n = \frac{(V - I_a \,.\, R_a) \,.\, 10^8}{p/a \,.\, \Phi \,.\, Z} = \frac{(110 - 10 \,.\, 0{\cdot}3) \,.\, 10^8}{2 \,.\, 10^6 \,.\, 300} = 17{\cdot}8.$$

If now, without changing the load, we reduce the magnetic flux 20 per cent., the armature current must increase in the same proportion, viz. 0·8 : 1. We shall then have

$$\Phi = 0{\cdot}8 . 2 . 10^6 = 1{\cdot}6 . 10^6,$$

$$I_a = \frac{10}{0{\cdot}8} = 12{\cdot}5 \text{ amperes.}$$

The product $\Phi . I_a$ has remained unchanged, but the armature current has increased. As this will cause an increase of pressure drop in the armature, the back E.M.F. will decrease. This will naturally affect the speed and make the actual increase less than we should otherwise expect. In the above example, for instance, we have weakened the field by 20 per cent. and should therefore expect the speed to increase in the ratio 0·8 : 1, that is from 17·8 to $\frac{17{\cdot}8}{0{\cdot}8} = 22{\cdot}3$. As an actual fact, the new speed attained on weakening the field is

$$n = \frac{(110 - 12{\cdot}5 \,.\, 0{\cdot}3) \,.\, 10^8}{1{\cdot}6 \,.\, 10^6 \,.\, 300} = 22{\cdot}2.$$

This indicates at the same time that there is a limit to the speed obtained by weakening the field. The effect of the pressure drop in the armature may become so great that, on further weakening the field, the speed is decreased. This is, of course, an absolute necessity, since we should otherwise be led to the conclusion that the loaded motor would run at an infinite speed when the field circuit was broken and the flux reduced to zero. As a matter of fact, the motor remains stationary in such a case, since one of the two factors that produce the torque has vanished. Hence, there must be a certain value of Φ or of I_a, for which the speed at a given load is a maximum. We can find this critical value of I_a from equation (103) on page 202, in which we equated the mechanical and the electrical power. We have then

$$E . I_a = V . I_a - I_a^2 . R_a = M_t . 2\pi . n . 9{\cdot}81.$$

As one of the conditions of our experiment is that the load be kept constant, M_t is a constant, and we can put

$$n = c . (V . I_a - I_a^2 R_a),$$

where c is merely a coefficient.

For n to be a maximum $\frac{dn}{dI_a}$ must vanish, or

$$\frac{dn}{dI_a} = c . (V - 2I_a . R_a) = 0.$$

Hence $$I_a . R_a = \frac{V}{2}.$$

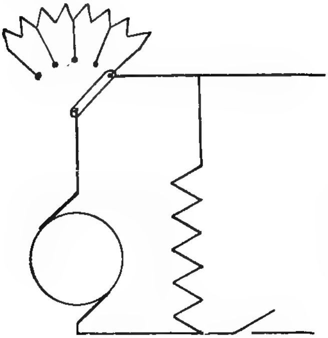

Fig. 191

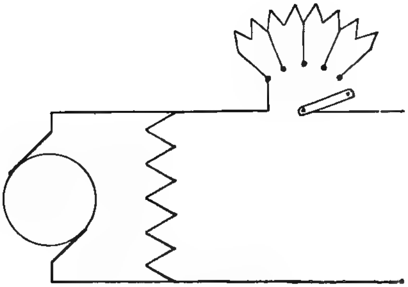

Fig. 192

The maximum value of the speed is therefore reached when the field is so far weakened that the pressure drop in the armature is equal to a half of the terminal pressure. The electrical efficiency of the armature would then be 50 per cent., but the conditions, if for no other reason than the enormous armature current, are far beyond the normal working range.

With regard to the arrangement of starting resistance in the armature circuit and field rheostat in the field circuit, care must be taken that the field circuit is not suddenly broken when the motor is being switched off. This would not only cause sparking at the point where the circuit was broken, but might break down the insulation of the field coils, owing to the large electromotive force induced by the sudden cessation of the current through the highly inductive circuit. The motor can be switched off sparklessly by opening the external circuit and leaving the field connected across the armature (Fig. 191). The motor will slow down slowly because of its kinetic energy, and will act as a dynamo sending a gradually decreasing current through the field windings in the same direction as before. The field will gradually diminish as the motor slows down.

It is, however, very dangerous to adopt the plan, shown in Fig. 191, of switching off in the external circuit, since the moving arm of the starting resistance is left in such a position that a dangerous short-circuit would result on again closing the switch. It is impossible to place the starting resistance in the external circuit, as shown in Fig. 192, since the P.D. between the brushes at the moment of starting would only be $I_a . R_a$, which is so small that there would be practically no field current and no magnetic flux. The motor would therefore not start.

The unbroken connection of field and armature, which we have seen to be necessary for sparkless switching, can, however, be arranged by connecting the field across the free ends of the armature and starting resistance as shown in Fig. 193. When the arm is in the middle of the row of contacts, the right-hand half of the resistance is acting as starting resistance, while the left-hand half is in series with the field winding, and is weakening the field by a small amount which, however, is almost negligible since the starting resistance is generally much smaller than the field resistance. This arrangement has the great advantage of the permanent connection of field, armature and starting resistance in series, thereby making it possible to switch off the motor without any dangerous sparking, especially if the switch-arm is moved rapidly from the running position, right across the contacts and off at the left-hand side before the motor slows down appreciably. We have the disadvantage, however, of the starting resistance being normally in series with the field winding and causing a small but unnecessary loss due to the heat continually being generated in the resistance.

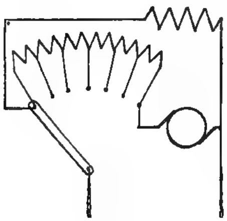

Fig. 193

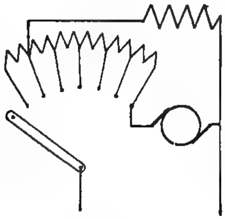

Fig. 194

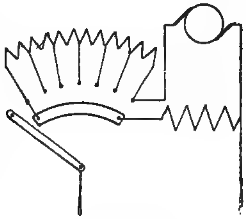

Fig. 195

The loss can be diminished by connecting the field to some intermediate point of the resistance, as shown in Fig. 194. This brings with it, however, the disadvantage of a weak field at the first moment of starting, and is really a compromise between Figs. 192 and 193. Since the self-induction of the field winding causes the growth of the field to be gradual, it is an advantage to apply the full P.D. of the mains at the first moment so as to hasten the growth of the flux as much as possible.

These disadvantages are overcome in the arrangement shown in Fig. 195, in which the field circuit is connected at once to the full P.D. of the mains and remains so connected by means of a short contact arm which moves over a segment. The longer contact arm, which is virtually a prolongation of the short one, moves over the contacts of the starting resistance. This arrangement does not give quite so sparkless a break as that shown in Fig. 193, since, at the moment of switching off, the field current is supplied by the armature and, having to pass through the starting resistance, is suddenly reduced. This sudden reduction of the current through the inductive field windings causes a spark.

This is even more pronounced in the arrangement shown in Fig. 196. The "off" position of the switch-arm is on the extreme left; on moving the arm to the right, the field is switched on to the full pressure of the mains, and the resistance R_1 is in series with the armature. As the motor runs up to speed the switch-arm is moved to the right, until it finally stands on the right-hand contact of R_1 and on the right-hand end of the left-hand segment. If now we wish to increase the speed, the arm is moved still further to the right, putting resistance R_2 into the field circuit and supplying the armature directly through the right-hand segment.

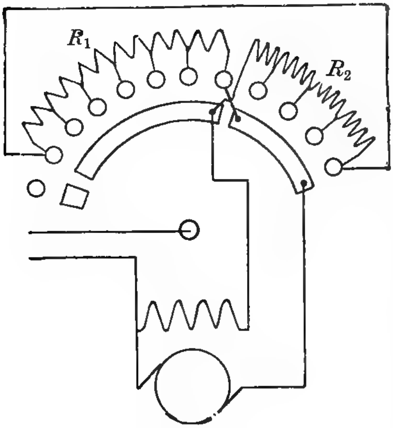

Fig. 196

The spark on breaking is diminished by connecting the left-hand end of the starting resistance with the right-hand end of the regulating resistance. When switched off, the motor will slow down while acting as a generator and sending current through armature fields, starter R_1 and field regulator R_2 all connected in series.

When the motor is running, the arm in any of these starters would be over to the right, so that the full P.D. of the supply mains is across the armature. If now, owing to any cause, the supply pressure failed for a few moments, the motor would come to rest, and on the pressure being re-established the armature would probably be burnt out. This possibility is avoided by means of a "no-voltage release." The arm is fitted with a spring which tends to move it to the left, i.e. to the off position. When running, the arm is held over to the right by means of a small electromagnet, the winding of which is either across the mains or in series with the field winding of the motor. If the supply pressure fails, the arm is released and flies back to the off position. Many motor starters are also provided with an "over-load release," by means of which the arm is released if the current taken by the motor exceeds a certain pre-arranged value.

Considerable difficulties arise when starting apparatus has to be designed for very large motors, and the amount of energy lost as heat in the starting resistance is by no means negligible when motors are continually being started, more especially when large masses have to be set in motion. For this reason some electrically operated hoisting plants for mines have been started by means of a battery of accumulators connected across the supply mains. By means of a battery switch, the battery is divided into groups and only a fraction of the total pressure applied to the motor armature at starting. As the speed increases the back E.M.F. of the motor also increases, and the applied P.D. is gradually increased by means of the battery switch. The battery acts also as a buffer and equalises the load on the supply station to a certain extent.

Another method of starting a motor and regulating its speed conveniently over a wide range is to employ a motor generator set. In the Ward-Leonard arrangement, for instance (Fig. 197), an auxiliary shunt-motor is driven off the mains, and coupled to it is a separately excited generator. This set runs at a practically constant speed, but the excitation of the generator can be varied over a wide range. It is made very weak when starting the main motor, the armature of which is supplied directly from the generator. By gradually increasing the excitation of the generator, the motor is brought up to full speed and can then be switched over on to the mains.

At first sight the employment of an auxiliary motor and starting dynamo may appear to be an avoidable complication. It must be remembered, however, that the power supply in large plants is generally three phase, and that it is necessary in any case to transform it into continuous current because of the difficulties in the construction of a slow running three phase main motor and in its speed regulation.

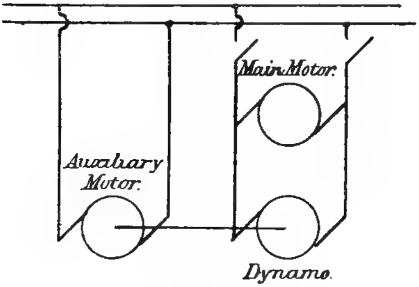

Fig. 197

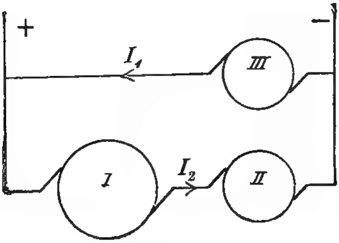

Fig. 198

It is specially advantageous to fix a heavy flywheel between the auxiliary motor and the generator in the above arrangement. This is generally known as the Ilgner system and has been largely adopted within the last few years. When the load is heavy, as when starting to hoist, the flywheel gives up a part of the energy stored in it, whereas at light loads or no-load energy is restored to the wheel by its acceleration. The flywheel, therefore, acts as a buffer and equalises the load on the station.

The auxiliary motor must, of course, be so constructed that the flywheel can act in this way. Its speed is therefore made to drop considerably as the load increases. In an induction motor this can be attained by increasing the rotor resistance, and in a direct-current motor, by compounding it with a few series turns so arranged that they strengthen the shunt field (see Section 67 for the effect of the series turns on the speed).

The action of the set is improved if resistance is automatically inserted in series with the auxiliary motor as the main motor is started. The increasing output of the latter increases the load on the motor-generator set. The auxiliary motor tries to draw the necessary power from the mains but is prevented from doing so by the inserted resistance which keeps the current constant. The motor-generator set must therefore slow down and the flywheel is thus enabled to supply the increased demand for power, that taken from the mains by the motor being kept constant. The reverse process takes place when the main motor is brought to rest. To do this, the field of the generator is weakened so that its E.M.F. is less than that of the main motor, which therefore sends current in the reverse direction. This retards the motor and accelerates the generator and flywheel. To maintain the auxiliary motor current constant, resistance must be automatically switched out of its circuit. With such an arrangement the power station is unaffected by variations in the load and delivers practically the same current to the auxiliary motor whether the hoisting motor is working or not, provided that the interval between two successive hoists is not too long, in which case the demand for power would naturally decrease. When the main motor is being started, the auxiliary motor and flywheel act together on the generator, and when the main motor is being brought to rest, the auxiliary motor and the generator, acting now as a motor, operate together to accelerate the flywheel. In an actual hoisting plant the main motor current varied between + 2000 amperes at starting and – 1000 amperes when braking. The current taken from the mains remained almost constant at about 400 amperes. The speed of the flywheel varied from 12 to 15 per cent.

One of the principal advantages of starting the hoisting motor by means of a motor-generator set is that, under all conditions, the hoisting motor has a definite speed for each position of the controller handle irrespective of the load on the motor. This is not so when starting by means of a series resistance since the drop of potential in this resistance varies with the load.

Instead of varying the resistance in the armature circuit of the auxiliary motor one can vary its field current in order to keep its speed constant. When the load on the hoisting motor increases, the field of the auxiliary motor is strengthened so as to keep its back E.M.F. constant in spite of the falling speed. The applied P.D. and the back E.M.F. being both constant, the current taken from the mains is necessarily constant.

One method of starting large motors, which is of considerable interest, depends on the use of an auxiliary generator, the E.M.F. of which is at first opposed to the supply pressure, but afterwards acts with and is added to it. The machine I in Fig. 198 is the hoisting motor, constructed for a pressure of 1000 volts. Machines II and III constitute a motor-generator set, both machines are for 500 volts, and the set runs at an approximately constant speed. The supply pressure is also 500 volts. We shall confine ourselves to a description of the method of starting the hoisting motor; a complete understanding of the various changes involved can only be obtained by applying Kirchhoff's rules to the circuit.

Before starting, motor I is at rest and merely acts as a supply lead to motor II. The latter drives the machine III as an unloaded dynamo, since the field of III is adjusted until its E.M.F. exactly counterbalances the pressure of the supply mains. By slightly weakening the field of motor II its speed is increased by a small amount. This increase of speed increases the E.M.F. of machine III and enables it to act as a loaded generator and supply a current I_1. This increases the load on motor II and causes its current to increase to I_2. This is exactly similar to the Hopkinson method of testing two similar machines, which was described on pages 194 and 195. The current I_1 flows round to motor II, and the mains supply the losses.

The field of motor II is weakened until the current I_2 reaches such a value that the main hoisting motor I, which is separately excited, starts. Every further decrease in the field of motor II causes a decrease in its back E.M.F., and thereby increases the fraction of the supply pressure available for the main motor I, the speed of which will consequently increase. Beyond a certain point the speed of the motor-generator set II and III will decrease, because the input of the motor II will decrease owing to the smaller P.D. across its terminals; the output of the dynamo III must therefore decrease, which it can only do by a decrease of speed, its field being constant.

When the field of motor II is reduced to zero, its back E.M.F. will have disappeared, and, except for the resistance of its armature, the full P.D. of 500 volts will be across the main motor I. The next step is to reverse the field of machine II and gradually strengthen it. Its E.M.F. will no longer oppose the pressure of the supply mains, but will now increase the P.D. across the main motor and therefore also its speed. Machine II is now a dynamo driven by machine III, the speed of which has now fallen so low that the current I_1 is reversed and drives it as a motor. When the field of machine II reaches its full value, its E.M.F. will be 500 volts. In this way the P.D. applied to the main motor I is gradually increased to 1000 volts. It has then reached its full speed, and one half of its input is supplied directly from the supply mains, and the other half indirectly through the motor III and the generator II. The whole power must, of course, come ultimately from the supply mains.

67. Principle of the Series Motor.

The behaviour of the series motor is not nearly so easy to understand as that of the shunt motor, since the magnetic field of the former is not constant but varies continually with the load The field winding is in series with the armature, and the armature current, which in conjunction with the field produces the torque, is at the same time the magnetising current which produces the field. If the load on a series motor be increased, both the armature and field currents are increased. The larger torque is seen from equation (98) on page 200 to be due to the growth of both of the active factors Φ and I_a.

If the motor is working below the knee of the magnetisation curve, that is, with unsaturated iron, doubling the current causes the field to be doubled and the torque, in accordance with the above equation, to be quadrupled. Inversely, if the load be quadrupled, the current is only doubled.

The speed of a series motor is given by equation (104) on page 203, viz.

$$n = \frac{(V - I_a . R_a) . 10^8}{\frac{p}{a} . \Phi . Z}.$$

To avoid complicating the formula, we shall let R_a represent here the combined resistance of armature and field. If the load and therefore also the armature current be increased, the numerator of the above expression will be but slightly altered, since the combined resistance of armature and field is always very small. The denominator, on the other hand, will be considerably changed, since the flux varies with the current. The speed will therefore decrease as the load is increased, since with the strengthened field a smaller speed will suffice to produce a back E.M.F. nearly equal to the supply pressure.

The rapid increase of torque with increasing current, and the variable speed, make the series motor specially suitable for cranes and traction purposes. When starting, the torque exerted by the motor must exceed the resisting torque due to the apparent load, in order to accelerate the masses which have to be brought up to speed. It is the ability of the series motor to provide this heavy starting torque with a comparatively small current that makes it so invaluable for the above purposes.

The series motor possesses, moreover, special advantages in respect of the variations of load which the continually changing slope of the track puts upon the tramcar motors. The large torque required when running up an incline is produced with a relatively small demand on the power station.

This is only possible, of course, because of the corresponding decrease of speed mentioned above. The matter is quite clear, apart from the above considerations, if looked at from a mechanical point of view. Power is the product of force and speed. If then the series motor exerts, say, double the normal pull with less than double the normal current, and consequently with less than double the power supply, it is evident that its speed must decrease. The generators and motors may therefore be constructed for a reasonably small output, and the load on the station will not be so irregular as it would be with shunt motors on the cars. The variable speed of a series motor must therefore be considered a great advantage, and it is an important point in the interests of the power station that a heavily loaded car goes very slowly uphill.

We must now investigate more fully the behaviour of a series motor under various conditions. For this purpose we first determine the magnetisation curve by running the motor as a generator by means of another motor. If the speed be maintained at a constant value n_0, and the generator be loaded by means of different external resistances, we shall obtain a set of corresponding values of the current I_a and the terminal pressure V_0. From these we can calculate the E.M.F. of the series generator by means of the equation

$$E_0 = V_0 + I_a . R_a,$$

where R_a includes both armature and field. As in Section 58, the E.M.F. is plotted as ordinates and the external current, which is, at the same time, both armature and field current, as abscissae. This curve is the characteristic of the series machine. By dividing the values of E_0 by the constant $\frac{p \,.\, n_0 \,.\, Z \,.\, 10^{-8}}{a}$ we get the magnetic flux Φ which is produced by the corresponding magnetising current I_a. Curve Φ only differs from curve E_0 in the matter of scale (Fig. 199), but is, unlike curve E_0, independent of speed.

If it is impossible to make the experiment in this manner by running the motor as a generator, it can be run as a motor at a constant terminal pressure with variable mechanical load on the brake. The flux can be calculated from the observed speed and current by means of equation (104) on page 203:

$$\Phi = \frac{(V - I_a \,.\, R_a) \,.\, 10^8}{\frac{p}{a} \,.\, Z \,.\, n}.$$

When we have found the flux corresponding to each value of the current I_a, the two are multiplied together and the product $\Phi \,.\, I_a$ set up as ordinates.

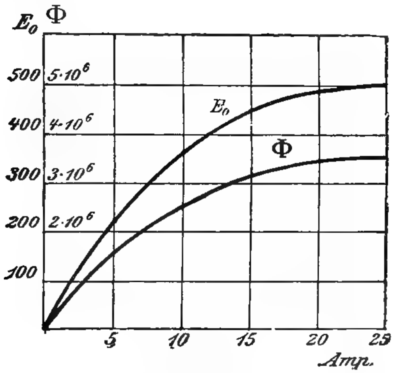

Fig. 199

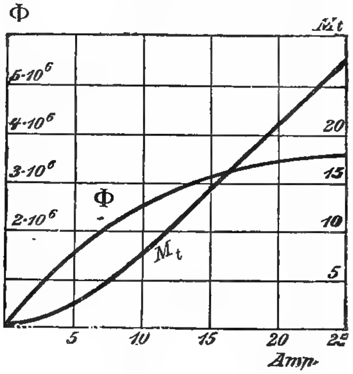

Fig. 200

In this way we obtain the curve M_t in Fig. 200, which represents the torque to some scale not yet determined. At the beginning the curve M_t is convex to the base, showing that the torque increases more rapidly than the current. As saturation sets in, however, the curve straightens out and gradually approximates to a straight line through the origin, corresponding to an increase of torque proportional to the increase of current.

The curve of magnetic flux Φ enables us, moreover, to find the speed corresponding to any condition of load at any terminal pressure. For this purpose we draw a horizontal line at a height above the base line equal to the terminal pressure V. In Fig. 201 this pressure is 500 volts. Through the point where this line cuts the vertical axis we draw a straight line sloping down from the horizontal at an angle α, where

$$\tan \alpha = R_a.$$

In measuring the tangent we must, of course, measure the ordinate on the scale of volts and the abscissa on the scale of amperes, and we must also remember that R_a includes both armature and field winding. The difference between the horizontal and the inclined line at any point is equal to $I_a \cdot \tan \alpha$, that is to $I_a \cdot R_a$, which is the ohmic pressure drop in the motor. The ordinates of the inclined line are therefore equal to the back E.M.F., which we denote by E. If now we divide each value of E by the corresponding value of Φ, we obtain a figure proportional to the speed (see equation (104) on page 203). In this way we obtain the curve r.p.m. in Fig. 201.

It is evident from this curve that a series motor will race if the load be removed. The speed, in fact, may reach such a value that the motor will fly to pieces owing to excessive centrifugal forces. The speed drops rapidly as the load is increased so long as the iron is comparatively unsaturated, but as saturation sets in the speed approaches an approximately constant value.

To find the total efficiency, we must subtract the losses due to heating of conductors, hysteresis, eddies and frictions from the motor input and divide the remainder, which is the output, by the input. In the shunt motor the losses, other than copper loss, were determined in a simple manner by finding the power required to run the motor at no-load and assuming that these losses were the same at all loads. This was justified to a certain extent because of the constant excitation and speed. In the series motor, however, such an assumption would be palpably incorrect, since both magnetisation and speed are variable.

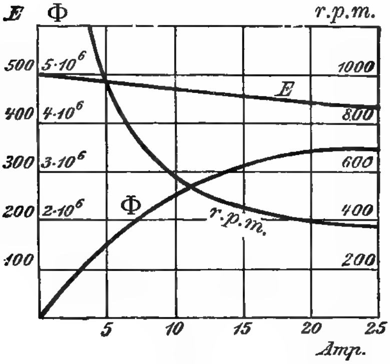

Fig. 201

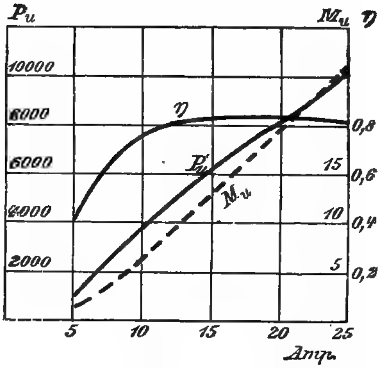

Fig. 202

The efficiency can be determined approximately by assuming that the torque M_0 required to overcome the friction and iron losses is constant at all loads. This assumption is not very incorrect if we consider the efficiency of the motor alone and exclude any reduction gear. By subtracting this constant torque M_0 from the values of M_t we obtain the dotted curve M_u of useful torque in Fig. 202.

The output P_u is given to a certain scale by the product of the useful torque and the speed. To obtain the efficiency this output must be divided by the input $V \cdot I_a$. In this way we obtain the curve η, the ordinates of which are equal to the efficiency. It is seen that the efficiency is small at low loads, but increases with increasing load until it reaches a maximum, beyond which it decreases owing to the large copper loss in armature and field windings.

68. Example of Traction Motor Calculation.

We shall now determine the speed, torque and efficiency of an actual motor, the details of which are given in Kapp's well-known *Dynamo Construction.* The motor is a 4-pole tramway motor made at Oerlikon in Switzerland. The armature has a series or 2-circuit winding arranged for a terminal pressure of 500 volts. The internal resistance of the motor is 2·75 ohms, and the number of wires on the armature is 944, so that $R_a = 2{\cdot}75$ and $Z = 944$. The motor was driven as a series dynamo at a constant speed of 450 r.p.m. and the characteristic so obtained is given by the first two columns in the table below.

The magnetic flux corresponding to each value of the current is found from equation (79) on page 124, as follows:

$$\Phi = \frac{E_0 \,.\, 10^8}{p/a \,.\, n_0 \,.\, Z} = \frac{E_0 \,.\, 60 \,.\, 10^8}{2 \,.\, 450 \,.\, 944} = 7070\, E_0.$$

From these values of the flux and the corresponding values of the current, the torque exerted on the armature can be found from equation (98) on page 200, and we get

$$M_t = \frac{p \,.\, \Phi \,.\, I_a \,.\, Z}{a \,.\, 2\pi \,.\, 9{\cdot}81} \,.\, 10^{-8} = \frac{2 \,.\, \Phi \,.\, I_a \,.\, 944}{2\pi \,.\, 9{\cdot}81} \,.\, 10^{-8} = 0{\cdot}307 \,.\, 10^{-6} \,.\, \Phi \,.\, I_a \text{ met.-kg.}$$

To calculate the speed of the motor we must know its back electromotive force E. This is given by the formula

$$E = V - I_a.R_a = 500 - 2{\cdot}75.I_a.$$

The speed can be found from equation (100) on page 201, viz.

$$n = \frac{E \,.\, 10^8}{\frac{p}{a} \,.\, \Phi \,.\, Z} = \frac{E \,.\, 10^8}{2 \,.\, \Phi \,.\, 944} = \frac{3{\cdot}18 \,.\, 10^6}{60} \,.\, \frac{E}{\Phi}.$$

The results of these calculations are given very clearly in the following table, and they are represented graphically by the curves in the previous section:

Generator		Motor and Generator	
I_a	E_0	$\Phi = 7070E_0$	$M_t = 0{\cdot}307.10^{-6}.\Phi.I_a$
5	225	$1{\cdot}59.10^6$	2·45
10	362	$2{\cdot}56.10^6$	7·85
15	450	$3{\cdot}18.10^6$	14·6
20	490	$3{\cdot}46.10^6$	21·3
25	505	$3{\cdot}56.10^6$	27·3

Motor		
I_a	$E = 500 - 2{\cdot}75 \,.\, I_a$	r.p.m. $= 3{\cdot}18 \,.\, 10^6 \,.\, \frac{E}{\Phi}$
5	486·2	970
10	472·5	590
15	458·7	460
20	445	410
25	431·2	390

By means of these figures we can determine the speed at which a tramcar of given weight will ascend a given incline, as well as the current taken in doing so.

Let W be the weight of the car in metric tons (1 metric ton = 1000 kg*),
f the tractive coefficient, i.e. the force in kg required to move one ton at a steady speed on the level,
and s the slope or gradient in parts per thousand;

then the pull required on the level will be $W.f$. On the slope, the weight AB (Fig. 203) acting vertically downwards can be resolved into two components, one normal to the surface and the other AO, which has to be directly overcome by the tractive pull. We have from the figure

$$AO = AB.\sin\vartheta.$$

If we substitute for $\sin\vartheta$ its value $\frac{s}{1000}$ and for AB the weight of the car in kilogrammes, we have

$$AO = 1000\,W.\frac{s}{1000} = W.s.$$

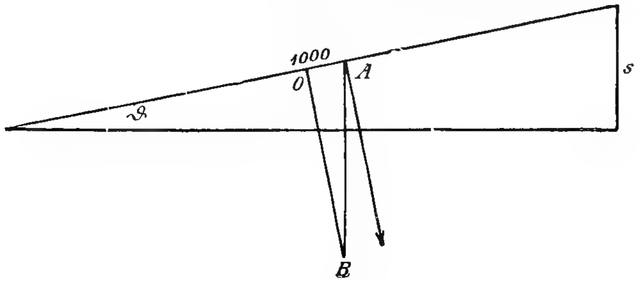

Fig. 203

The pull on the car must therefore be given by the formula

$$F = W.f + W.s = W\,(f+s)\ \text{kg*}.$$

If the radius of the car-wheels be r metre, the torque on the axles must be

$$F.r = W\,(f+s).r\ \text{met.-kg}.$$

If the gear is such that the motor runs k times as fast as the car axles, the torque of the motor, by the conservation of energy, must be a kth part of that on the axles. If we assume, further, that the gearing has an efficiency η, we obtain the following formula for the total torque of all the motors on the car:

$$\Sigma M_t = \frac{F.r}{k.\eta} = \frac{W.r.(f+s)}{k.\eta}.$$

In our present example the radius of the wheels was 0·39 metre and the ratio of the gearing 4·9. If we assume that the weight of the car is 8 tons and the tractive coefficient 12 kg per ton, we have

$$\Sigma M_t = \frac{8.0{\cdot}39.(12+s)}{4{\cdot}9.\eta} = \frac{0{\cdot}635.(12+s)}{\eta}.$$

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* See footnote on page 44.

In our case, as is almost universal, each car was provided with two motors, so that each motor had to produce half the required torque. Hence, for each motor, we have

$$M_t = \frac{\Sigma M_t}{2} = \frac{0\cdot318\,(12+s)}{\eta},$$

or

$$s = 3\cdot14.\eta\,.\,M_t - 12.$$

This gives us the slope up which the car can steadily run for the given torque on each motor.

As a rule, the efficiency η of the gearing, including the friction and iron losses of the motor, can be taken as 0·8, but in the present case it was accurately determined in each case and is given in the table below. In the same table we have the calculated slopes up which the car can run with the given strengths of current. The torque exerted by the motor has been taken from the table on page 219. A negative value of the slope denotes running downhill.

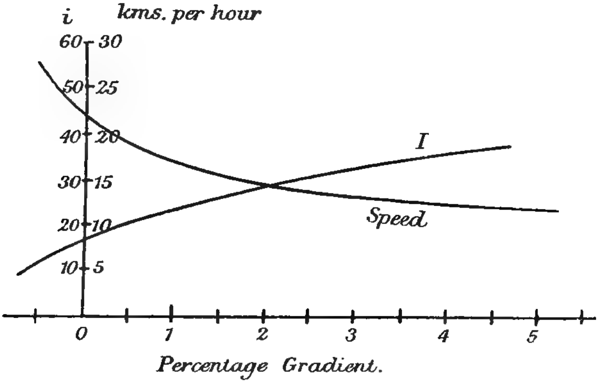

Fig. 204

We have yet to calculate the speed at which the car runs in any given case. To obtain the revolutions per hour of the car wheels we must multiply the revolutions of the motor per minute by 60 and divide by the gear ratio k. If the revolutions per hour made by the car wheels be multiplied by the circumference of the wheels in metres and divided by 1000, we get the speed of the car in kilometres per hour. Calling this speed S, we have

$$S = \frac{\text{r.p.m.}\,.\,60}{k}\,.\,\frac{2r\,.\,\pi}{1000}\ \text{km per hour.}$$

If we assume the radius of the car wheels to be 0·39 metre and the gear ratio k to be 4·9, the above equation becomes

$$S = 0\cdot03\,.\,\text{r.p.m. km per hour.}$$

This formula has been employed to calculate the last column but one in the following table. The values of r.p.m. have been taken from the table on page 219. The current I in the last column is the total current taken by the car, which, as the two motors are in parallel, is equal to twice I_a.

I_a	η	$s = 3{\cdot}14 . \eta . M_t - 12$	$S = 0{\cdot}03$. r.p.m.	$I = 2I_a$
5	0·75	$-6{\cdot}2\ {}^0/_{00}$	29 km per hour	10
10	0·775	+ 7 ,,	17·7 ,, ,,	20
15	0·805	+ 25 ,,	13·8 ,, ,,	30
20	0·835	+ 44 ,,	12·3 ,, ,,	40
25	0·85	+ 61 ,,	11·7 ,, ,,	50

In Fig. 204 the total current taken by the car, and the speed at which it runs, are plotted as ordinates on a base representing the slope or gradient. It is more customary in England to refer to a slope of 50 $^0/_{00}$ or 5 per cent. as a rise of 1 in 20*.

69. The Regulation of Series Motors.

The speed of a motor is given by equation (100) on page 201, viz.

$$n = \frac{(V - I_a . R_a) . 10^8}{p/a . \Phi . Z} \text{ revolutions per second.}$$

In this equation V represents the terminal pressure of the motor. A method of regulating the speed of a series motor, which suggests itself at once, is by regulating the terminal pressure by means of the starting resistance. The current corresponding to the load on the motor causes a fall of potential in the resistance, thus reducing the P.D. between the armature terminals. We may look at the matter from another point of view, and let V represent the supply pressure of the mains. Putting in an external resistance R_1 increases the total resistance from R_a to $R_a + R_1$, and the above equation becomes

$$n = \frac{\{V - I_a . (R_a + R_1)\} . 10^8}{p/a . \Phi . Z} .$$

It is quite immaterial which of these two ideas we adopt. In the one case we imagine the positive term in the numerator to be decreased, while in the other case we imagine the negative term to be increased. Both lead to the same result, viz. a fall of speed.

Although this method of regulating the speed is the one generally adopted, it is very wasteful, since the power saved at the motor by decreasing its speed is all dissipated in the regulating resistance. Another method which is occasionally resorted to is that of altering the excitation. As we have already seen when considering the shunt motor, the speed is increased by decreasing the value of Φ in the denominator of the above equation. We cannot here, however, weaken the field by putting resistance in series with the field winding, since such resistance would be in series with the armature and would merely lower the P.D. between the brushes. By putting a resistance in parallel with the field winding, as shown in Fig. 205, a part of the ature current is shunted and takes no part in the excitation.

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* For further information on this subject, the student should refer to *Electric Traction* by Wilson and Lydall.

If we assume, for the sake of simplicity, that the resistance connected in parallel with the field winding has the same resistance as the field winding, and that the magnetic flux is proportional to the magnetising current, it appears at first sight that the magnetic flux has been halved, and the speed consequently doubled. This, however, is not the case, for we have seen, when considering the shunt motor, that with a constant load a weakening of the field leads to an increase in the armature current. To determine the strength of the armature current that flows when the field winding is shunted, we must first draw the characteristic curves of the motor without the resistance (Fig. 206). The ordinates of the curve M_t are equal to the product of the corresponding values of the flux Φ and the current I_a. In Fig. 207 we have the same curves for the motor when the field winding is shunted. The flux curve is exactly the same as before, but the numbers along the base have been doubled, since the same flux is still produced by the same magnetising current, and therefore by double the armature current. The curve M_t

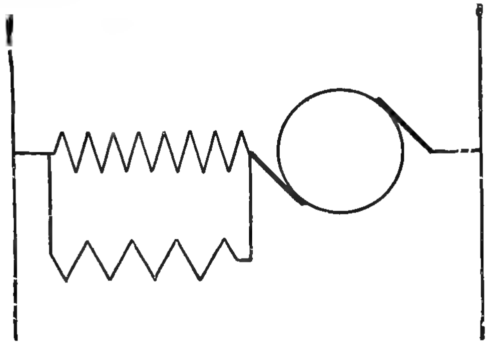

Fig. 205

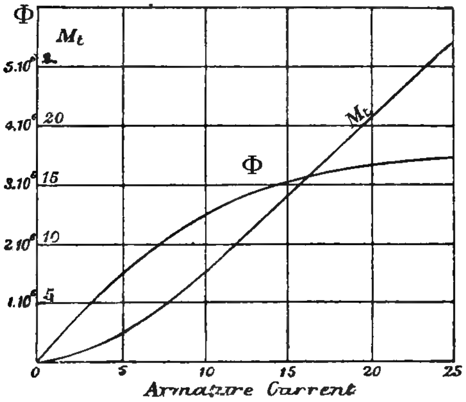

Fig. 206

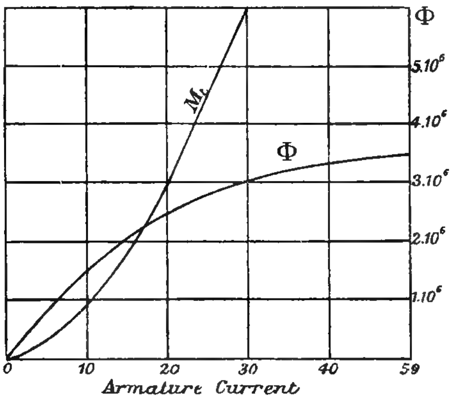

Fig. 207

in Fig. 207 is obtained, as before, by multiplying the flux Φ and the current I_a. We can then see from these two figures the values of the flux and current for the same turning-moment or load. From Fig. 206, for example, we find that a torque of 14 met.-kg requires a current of 15 amperes and a flux of $3{\cdot}1 . 10^6$ lines. For the same load, after the field has been weakened, Fig. 207 shows a current of about 19 amperes and a flux of $2{\cdot}5 . 10^6$ lines. If we neglect the small ohmic pressure drop, the speed is inversely proportional to the flux. In the present case, therefore, the speed has been increased in the ratio of 3·1 to 2·5, or 24 per cent., by putting an equivalent resistance in parallel with the field winding.

This method of varying the speed by shunting the field current necessitates a certain loss of energy in the shunt. This loss is small, but can be avoided by adopting a method due to Sprague, in which the field coils are all connected in series at starting. At full speed they are connected in parallel, so that each magnet coil carries only a part of the armature current. The speed can be economically varied over a considerable range by this method. Two steps in such an arrangement are shown in Figs. 208 and 209.

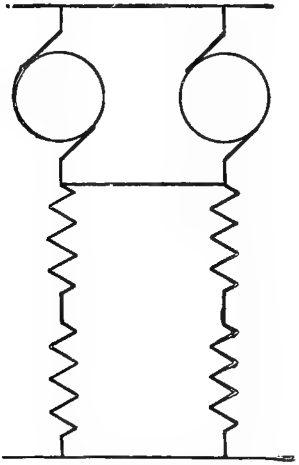

Fig. 208

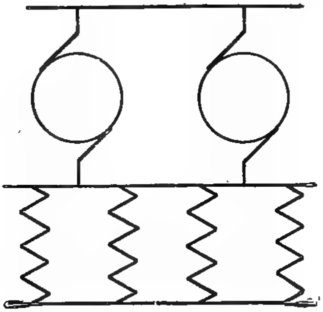

Fig. 209

The regulation of the speed of series motors by weakening the field is now seldom resorted to. From what we saw in Section 54 concerning the effect of cross-magnetisation, it is evident that an undue weakening of the main field must give rise to sparking at the commutator. Moreover, there is the danger of the magnetic flux not being the same in each of the several motors on a car. This may result from differences between the magnetic reluctances of the motors, or from faulty contacts due to the continual operation of the switching gear, which would cause the current to divide unequally between the various parallel paths. The effect of such differences can be seen clearly from the following example. A tramcar has two motors connected in parallel, the pressure V *across the brushes* is 500 volts and the armature resistance of each motor is 1 ohm. If the current in one motor is 15 amperes, the back electromotive force can be found as follows:

$$E = 500 - 15 . 1 = 485 \text{ volts.}$$

We shall now assume that the magnetic flux of the second motor is 5 per cent. less than that of the first motor. As the two motors are on the same car they must necessarily run at the same speed. The back E.M.F. of the second motor is therefore 5 per cent. smaller than that of the first, and we have in this case

$$E = 485 . 0{\cdot}95 = 460 \text{ volts.}$$

To find the current in the second motor, we have

$$I_a = \frac{V - E}{R_a} = \frac{500 - 460}{1} = 40 \text{ amperes.}$$

The total current taken by the car is $40 + 15 = 55$ amperes, and of this only 15 amperes flow through the more strongly magnetised motor, while 40 amperes pass through the weakly magnetised motor. The latter does almost three times as much work as the former. It is not difficult to calculate the conditions under which the back E.M.F. of one motor becomes equal to the terminal pressure, so that its current and input are both zero. It is possible, however, to go still further, and find a speed at which the E.M.F. of the strongly excited machine is greater than the supply pressure, and the machine works as a generator supplying power to the other motor. The latter has then not only to drive the car, but also to drive the other machine as a loaded generator. The result is that the armature of the under-excited motor becomes over-heated and is rapidly destroyed.

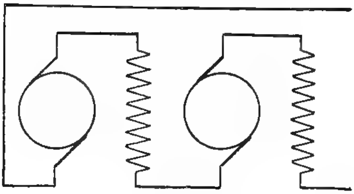

Fig. 210

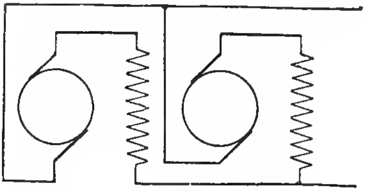

Fig. 211

It may also occur that the armature resistances are unequal. This would also be very disadvantageous in the Sprague arrangement.

As a result of these difficulties, speed regulation of series motors is rarely carried out by shunting or rearranging the field windings. The general method now adopted is the simple resistance in series with the motor. The armature is always in series with its own field winding, so that the armature current of any motor is identical with the magnetising current of that motor. At starting, both the current and power taken by the car can be greatly reduced by connecting the motors in series. Each motor then acts as a resistance in series with the other (Fig. 210), and the pressure across each motor is only half that of the mains. The current taken by the car is only that of a single motor; this is a most important point, since the current taken to start a car is very great. When a certain speed has been reached, the motors are connected in parallel (Fig. 211). It would appear at first sight as if we had again encountered the difficulty of an unequal distribution of current and work, if by any chance the internal resistances or field strengths were not exactly equal. If, for example, one of the armatures in Fig. 211 had a larger resistance than the other, its current

$$I_a = \frac{V - E}{R_a}$$

would apparently be smaller than that of the other motor. Since, however, its flux and back E.M.F. would be thereby simultaneously reduced, the inequality is almost entirely neutralised. Suppose, for example, that the following figures refer to one of the motors:

$$V = 500, \quad \Phi = 3 . 10^6, \quad I_a = 20, \quad R_a = 2,$$

while the resistance of the other is so different that its current is 18 amperes and the magnetic flux, found from the characteristic for a magnetising current of 18 amperes, is 2·9 . 10⁶ lines. We shall now calculate the resistance x of the second motor. Since both motors run at the same speed, we have from equation (100) on page 201:

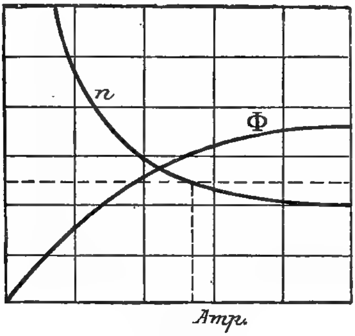

Fig. 212

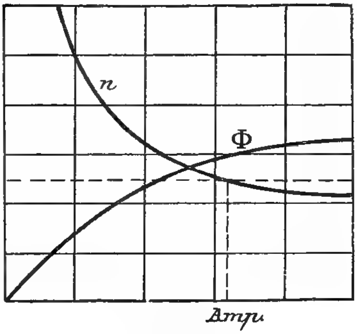

Fig. 213

$$n = \frac{(500 - 20 . 2) . 10^8}{\frac{p}{a} . 3 . 10^6 . Z} = \frac{(500 - 18 . x) . 10^8}{\frac{p}{a} . 2{\cdot}9 . 10^6 . Z} .$$

From this we get $x = 3{\cdot}05$ ohms.

Hence we see that an increase in the resistance of 50 per cent. merely causes the current to decrease from 20 to 18 amperes. Similarly, a difference in the magnetic flux will cause a very small difference in current (Figs. 212 and 213). We may assume that the gap between armature and poles is greater in the second motor, thus causing its characteristic curve, which shows the relation between magnetising current and flux, to lie lower than the corresponding curve of the first motor. For the sake of simplicity the values of the flux in Fig. 213 are taken 10 per cent. less than those in Fig. 212. If now we draw the speed characteristics under the assumption that the terminal pressure is the same and the internal resistance is negligible, we find that it lies higher in the second case. For any given speed, such as that shown by the dotted horizontal line, we see that the second motor takes a larger current. The difference is, however, very small and never reaches the values obtained above for the Sprague arrangement. The motor having the larger magnetic reluctance develops a smaller E.M.F., and thereby enables a larger current to flow through it. This larger current, however, passes round the magnets of the same motor and makes up to some extent for its greater reluctance, so that the difference in the flux cannot become very considerable.

70. Temperature Rise of Motors, etc.

In dynamos and motors, and in all electrical machinery, a certain fraction of the power supplied to the machine is unavoidably dissipated as heat. In the interests of economy this fraction should be kept as small as possible, consistent with compactness and cheapness. The heat will be conducted from hot to cold parts of the machine, and will ultimately be carried off by radiation and conduction combined. The allowable temperature rise will depend on the materials entering into the construction of the machine, such as paper, cotton, micanite, etc., and in some cases on mechanical considerations, such as the elongation of the commutator bars and consequent inequalities in the commutator surface. Experience has shown that the temperature of coils of cotton-covered copper wire, such as are used on the armatures and fields of motors, should not exceed the ordinary atmospheric temperature by more than about 40° C. This allows a certain margin, which is very necessary, for some parts of the winding will be hotter than others, and machines are liable to be overloaded. When a temperature rise is specified, the way in which it is to be measured should also be specified, for very different results may be obtained by applying a thermometer instead of measuring the increase in resistance of a coil and calculating therefrom the temperature rise. The temperature will depend on the time the machine has been running, and may, in large machines, increase for many hours. The final temperature may be predicted from the curve showing the gradual rise from the beginning of the test. The accuracy of the prediction is affected, however, by the exchanges of heat going on within the machine itself. In the majority of cases a six-hour run on full-load is quite sufficient to determine the heating of a machine. In many cases, such as occur in tramway and railway work, it is almost impossible to test the machines under running conditions. In such cases a test is made under certain prearranged conditions, generally a run of one hour with the covers open, the load being kept at a constant value to give the nominal H.P., and the temperature rise determined. It is usually specified not to exceed 75° C. under these conditions.

In many cases the load on a motor is perfectly definite but intermittent. Such cases occur in connection with cranes and pit hoisting gear. A heating test at the works would be rather inconvenient under such conditions, but quite apart from this it is very important to be able to predict from the steady heating curve the probable temperature rise under any given conditions of intermittent loading. The same problem is met with in starting resistances for motors which are repeatedly stopped and restarted.

We will now consider the question of temperature rise a little more closely. The rate at which heat is radiated by any body depends on the nature and extent of its surface, and on its excess temperature above the surroundings. The temperature will continue to rise until the rate of heat radiation is equal to the rate of heat production. The final excess of temperature will be directly proportional to the power dissipated as heat in the body and inversely proportional to its surface. In any actual example, such as the armature of a motor or dynamo, this simple relation is complicated by the effect of ventilation. It is difficult to know what allowance to make for such surfaces as the walls of ventilating ducts, and also for the effect of speed of rotation. The formulae used in dynamo design are more or less empirical, being based on the results of numerous tests of actual machines. They are generally of the form

$$T = \frac{k\,.\,\text{watts per sq. cm}}{1 + \dfrac{v}{k_1}},$$

where T is the final rise of temperature,

v is the peripheral speed in metres per second,

and k and k_1 are constants.

k depends on the nature of the surface, but also very largely on the method of computing the effective surface. k_1 depends on the cooling effects of rotation, and for ordinary armatures is usually taken as 10.

The curve representing the temperature at various times will start as a straight line from the origin and gradually bend round until it becomes horizontal at the final temperature. We will now investigate this curve.

Let M be the mass of the body,

A the area of its surface,

T its excess temperature at the given moment,

s its specific heat,

ϵ the emissivity of the surface, and

P the power dissipated in the body.

The energy developed as heat in a short interval dt will be $P.dt$. The energy radiated away in the same time will be $T.A.\epsilon.dt$. If these two quantities are equal the temperature has reached a steady value, but if the former exceeds the latter, the temperature will increase by an amount dT and the heat stored in the body by an amount $M.s.dT$. Hence

$$P.dt - T.A.\epsilon.dt = M.s.dT \quad \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

If T_m be the final excess temperature which would be reached if the power supply P were maintained long enough, we have

$$P = T_m.A.\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

The ratio $\frac{M \cdot s}{A \cdot \epsilon}$ is a constant for the body. It is the ratio of the heat stored to the rate of radiation. We will call this the time constant τ of the body.

Substituting (*b*) in (*a*) and introducing τ, we have

$$T_m . A . \epsilon . dt - T . A . \epsilon . dt = A . \epsilon . \tau . dT,$$

or

$$(T_m - T)\, dt = \tau . dT.$$

If we write this $dt = \tau . \frac{dT}{T_m - T}$ and integrate both sides, we have

$$t = -\tau \log (T_m - T) + \text{a constant.}$$

When $t = 0$, we know that $T = 0$, so that the constant must be $\tau \log T_m$, and

$$t = \tau . \log \frac{T_m}{T_m - T} \quad \text{..............................} (c).$$

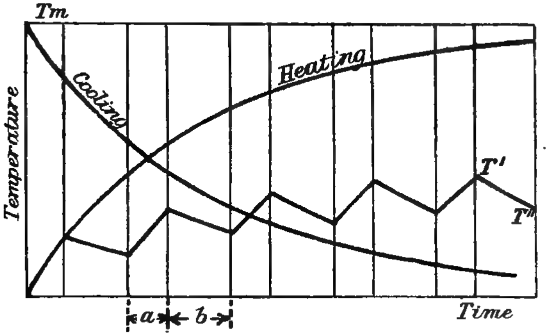

Fig. 214

This is very similar to the formula for the rate of growth of current in an inductive circuit, when a P.D. is suddenly applied, in which case $\frac{L}{R}$ is called the time constant of the circuit.

If we put $t = \tau$, we have

$$\log \frac{T_m}{T_m - T} = 1 \quad \text{or} \quad T = 0{\cdot}633 T_m .$$

Hence the time constant is the time taken to reach 0·633 of the final temperature rise.

By using formula (*c*) the heating curve can be plotted, and similarly the cooling curve (Fig. 214). If, now, a load is applied intermittently, the duration of the load being *a* and the intervening pause *b*, the temperature will rise irregularly for some time, but will finally oscillate between a maximum T' and a minimum T''. This is shown in the figure by the zigzag line, which is always drawn parallel to the heating or cooling curve at the proper temperature. It is obvious that the temperature will rise until a point is reached at which the fall of temperature during the interval *b* is exactly equal to the rise during the interval a. The permissible overload due to the heating being intermittent is obviously equal to the ratio of T_m to T'. This depends not only on $\frac{a}{b}$, but also on $\frac{a}{\tau}$, that is, on the duration of the load with regard to the time constant. It is obvious that no overload can be allowed for intermittent working, if a and b each represent several hours.

For further information on this interesting subject, students are referred to the *Elektrotechnische Zeitschrift*, 1900, p. 1058, and 1905, p. 346.

CHAPTER X

71. The instantaneous value of the induced E.M.F.—72. The electrolytic mean or average value of an alternating current.—73. Power in A.C. circuits and R.M.S. current.—74 Vector diagrams.—75. The E.M.F. of self-induction.—76. Ohm's law applied to alternating currents.—77. Resistance and inductance in series.—78. Resistance and inductance in parallel.—79. Effect of phase difference on A.C. power.—80. Effect of capacity.—81. Resistance and capacity in series.—82. Circuit containing resistance, inductance and capacity.—83. Self-induction and capacity in parallel.

71. The Instantaneous Value of the Induced E.M.F.

When a coil of wire is rotated about a diameter in a uniform magnetic field, electromotive forces are induced in it which change periodically both in strength and direction. The axis about which the coil rotates should be at right angles to the direction of the field, as shown in Figs. 215 and 216, where a rectangular coil is wound on a non-magnetic cylinder. The dotted lines represent the lines of force, the large circle the non-magnetic cylinder, and the smaller circles the sides of the coil in section. In Fig. 215 the coil is at right angles to the lines and the magnetic flux threading the coil is a maximum. The motion of the conductors on the cylindrical surface is, at the given moment, parallel to the lines of force, and no E.M.F. is induced in them. We shall take this position, in which there is no induced E.M.F., as our starting point or origin. In Fig. 216 the coil has moved through an angle α and is cutting obliquely through the lines of force, so that an E.M.F. is being induced in the coil. If we assume that the lines of force pass up the page and that the cylinder is rotated in a clockwise direction, an application

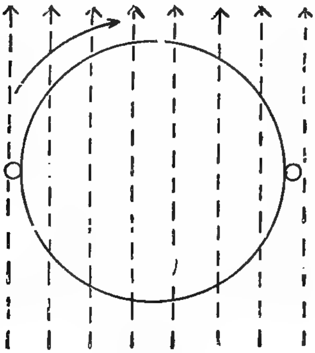

Fig. 215

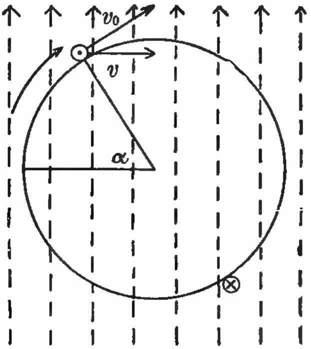

Fig. 216

of the right-hand rule shows the direction of the induced E.M.F. to be as indicated in the figure.

The E.M.F. induced in the coil is greater the more rapidly the lines of force are cut, that is, the less obliquely the coil cuts through them. The E.M.F. will therefore reach its maximum value when the lines are cut at right angles, that is, when the coil is vertical and is threaded by no line of force. On rotating the coil still further the instantaneous value of the E.M.F. decreases, and reaches zero when the coil is once more horizontal. At this moment the direction of the induced E.M.F. in the coil is reversed. So long as the wire is on the upper half of the drum it cuts the lines from left to right, but when it passes over to the under side of the drum it cuts them from right to left. Hence, the induced E.M.F. varies both in magnitude and in direction.

If H be the strength of field in lines per sq. cm,

Φ the maximum flux through the coil,

v_0 the constant peripheral speed in cm per sec.,

n the revolutions per second,

D the diameter of the drum in cm,

L the induced length of the coil-side,

and z' the number of wires connected in series (if the coil consist of a single loop $z' = 2$),

then for the instantaneous value of the induced E.M.F. we have equation (62) on page 81, according to which

$$E = H.l.v.10^{-8} \text{ volts.}$$

For l, which is the total length of induced wire, we must put $L.z'$. With regard to the speed v with which the lines are cut, it is evident from Fig. 216 that it is merely the horizontal component of the peripheral speed v_0. The same lines would be cut if the conductor moved horizontally with a speed v equal to $v_0 \sin \alpha$. Now

$$v_0 = \pi.D.n.$$

Making these substitutions in the above equation, we have

$$e = H.L.z'.D.\pi.n.\sin\alpha.10^{-8},$$

where e is the instantaneous value of the induced E.M.F.

The product $D.L$ is equal to the area of the coil in square centimetres, and H is the number of lines per sq. cm. Hence the product $H.D.L$ is equal to the total flux Φ passing through the coil when it is perpendicular to the field.

We have, therefore,

$$e = \pi.\Phi.n.z'.10^{-8}.\sin\alpha \quad \dots\dots\dots\dots\dots\dots (107).$$

The instantaneous value of the E.M.F. is therefore proportional to the sine of the angle which the plane of the coil makes with the initial position. When $\alpha = 0$, $e = 0$, and when $\alpha = 90°$, $\sin\alpha = 1$, and the E.M.F. has its maximum value, as we have already seen. The value of this maximum E.M.F. is seen from equation (107) to be

$$\hat{e} = \pi . \Phi . n . z' . 10^{-8} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (108).$$

Equation (107) can now be written in the form

$$e = \hat{e} \sin \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (109).$$

For values of α between 0 and 180°, $\sin \alpha$ is positive, whereas for values between 180° and 360° it is negative. This is in agreement with the fact already mentioned, that the E.M.F. changes its direction in the horizontal position of the coil, corresponding to $\alpha = 180°$. If a curve be drawn, having the values of the angle α for abscissae and the corresponding instantaneous values of the induced E.M.F. for ordinates, the well-known sine curve is obtained. We can, however, introduce the time t as the variable and express the E.M.F. as a function of the time. If ω be the angular velocity of the coil and t the time taken to reach its present momentary position, the angle $\omega . t$ is identical with the angle α, and we may write equation (109) in the form

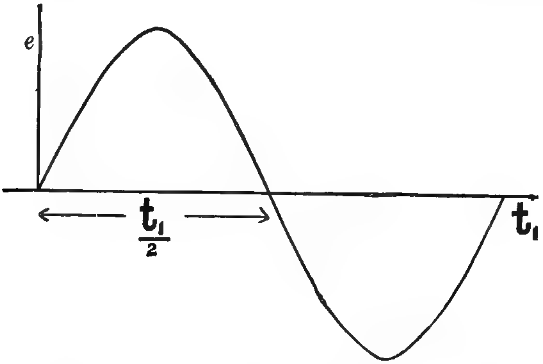

Fig. 217

$$e = \hat{e} . \sin (\omega . t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (110).$$

If the abscissae of our curve be made to represent the time t instead of the angle α, we merely alter the scale and obtain the curve shown in Fig. 217. This shows clearly how the E.M.F. varies from moment to moment. The time t_1 of a complete revolution corresponds to the angle 2π. During this time the E.M.F. passes through both a positive and a negative maximum, and returns to its initial value ready to pass again through the same cycle. The time thus taken to go through a complete cycle is called the period, or the periodic time. A common value of the periodic time in alternating current practice is a fiftieth of a second, so that the current or E.M.F. passes through 50 periods or cycles per second. It is then said to have a frequency or periodicity of 50. Some American engineers call a half-period an alternation, and speak of a current with a frequency of 50 as having 6000 alternations per minute.

If the drum on which the coil is wound be made of iron and it be rotated between the poles of an electromagnet in the usual way (Fig. 218), the induced E.M.F. will not be a sine function of the time. The lines of force now pass perpendicularly across the gap, so that they are no longer cut obliquely according to a fixed trigonometrical rule. It is possible, however, by suitably shaping the pole-tips, to make the gradation from the neutral zone to the strong field under the pole very gradual, so that we may assume that the field strength is distributed round the armature according to the sine law. Its maximum value will be under the centre of the pole and it will gradually decrease on either side towards the neutral zone, in the centre of which it will be zero. The induced E.M.F. will again be a sine function of the time, and will be given by equation (107). As before, Φ represents the maximum flux passing through the coil. This is here the same thing as the flux leaving the north pole of the field magnet.

For multipolar machines with p pairs of poles, the equation must be slightly modified. As in direct current armatures each coil will span a pole-pitch, the wire which passes under the centre of a north pole crossing at the end and coming up under the next south pole (Fig. 219). The number of lines of force cut in one revolution will be p times as many as in a bipolar machine with the same value of Φ. The E.M.F. induced in the coil will be p times as great, and we get the following general formula:

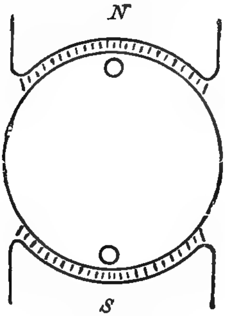

Fig. 218

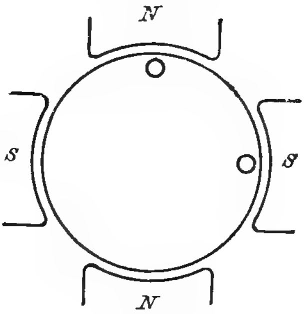

Fig. 219

$$e = \pi . \Phi . p . n . z' . 10^{-8} . \sin \alpha.$$

The product $p.n$ is equal to the number of cycles per second. This is sometimes represented by the sign ~, which is derived from the sine wave but we shall employ the letter f. If we substitute the frequency f for $p.n$ in the above equation, we get

$$e = \pi . \Phi . f . z' . 10^{-8} . \sin \alpha \quad \text{....................} (111).$$

Similarly $$\hat{e} = \pi . \Phi . f . z' . 10^{-8} \quad \text{..........................} (112).$$

These formulae are equally applicable to two-pole or multipolar machines. The angle α has not, however, the same meaning in both cases. In the four-pole machine in Fig. 219, a movement of the wire from the neutral axis to the middle of the pole represents a geometrical angle of 45°. The E.M.F. has increased, however, from 0 to its maximum value, so that sin α must have its maximum value 1 for the position under the middle of the pole. It is evident that we cannot, in such a case, put sin α equal to sin 45°. The difficulty could be avoided by putting sin $(p.\alpha)$ instead of sin α in the equation, in which case α would always represent the geometrical angle. This is, however, an unnecessary complication. We shall assume in all that follows that a degree is not the 360th part of a complete revolution, but the 360th part of a complete period or cycle. A complete cycle thus corresponds to an angle 2π. The angle π represents half a period, that is, the time taken for a wire to pass from the middle of a north pole to the middle of the adjacent south pole. A rigid adherence to this assumption will make a mistake in applying the above, or similar, equations impossible.

72. The Electrolytic Mean or Average Value of an Alternating Current.

We have seen in the previous section that the induced electromotive force varies as a sine function of the time. If we assume that the circuit contains only ohmic resistance and is free from self-induction, we can determine the current at any moment by dividing the electromotive force at that moment by the constant resistance. Hence, by simply altering the ordinate scale, the sine curve of E.M.F. may represent the alternating current. The instantaneous value of the current is therefore proportional to the sine of the angle α which the plane of the coil makes with its initial position, and we may write

$$i = \hat{i} \sin \alpha.$$

In order to determine the mean value of the current experimentally, the ends of the coil are connected to a two-part commutator, in the manner already described in Section 43. The current in the external circuit is then pulsating, but always in one direction, and we are said to have rectified the alternating current. If this current is passed through a copper voltameter, the quantity of copper deposited in a given time is a measure of the quantity of electricity which has passed through the circuit. To find the mean value of the current, we must divide the mass w of the deposit in milligrammes by the time t and by the electrochemical equivalent of copper, 0·328. We thus have

$$i_{\text{av}} = \frac{w}{0{\cdot}328 \,.\, t}\,.$$

The average value of the current may thus be called the electrolytic mean. To determine it graphically, we plot the time t as abscissa and the current i as ordinate. After a time t, reckoned from the initial position, the value of the current is given by the equation

$$i = \hat{i} \sin \alpha = \hat{i} \sin (\omega \,.\, t).$$

For an infinitely short interval of time dt, the current may be assumed to be constant. Since the current is defined as the quantity of electricity passing per second, the product $i\,.\,dt$, which is represented by the narrow vertical strip in Fig. 220, is equal to the quantity of electricity passing in the time dt. The total quantity passing during the whole period t_1 is equal to the sum of all such elemental strips, that is, to the shaded area in the figure.

The average value of the current is found by dividing this area by the time t_1. If a rectangle of equal area be drawn on the same base, the height of this rectangle gives the required mean current.

An approximate value can be obtained by finding the mean of the values of sin 0, sin 10°, sin 20°, and so on up to sin 90°, and multiplying it by $\hat{i}$. To obtain an accurate value of the mean current, we must determine the area of the shaded area (Fig. 221) corresponding to half a period. The integration is simplified by plotting angles as abscissae. The area of the narrow vertical strip in Fig. 221 is

$$i\,.\,d\alpha = \hat{i}\,.\sin\alpha\,.\,d\alpha.$$

To find the whole area we must integrate, or add up all the strips, between the limits 0 and π. This gives us

$$\sum_0^\pi i\,.\,d\alpha = \hat{i}\int_0^\pi \sin\alpha\,.\,d\alpha = 2\hat{i}.$$

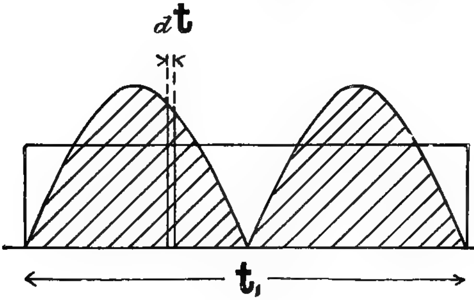

Fig. 220

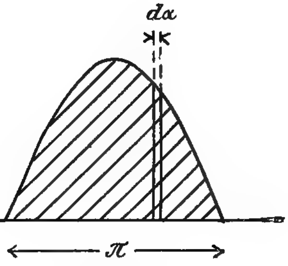

Fig. 221

By dividing this area by the base π we get the average height or average current. Hence

$$i_{\text{av}} = \frac{2}{\pi}\,.\,\hat{i} = 0{\cdot}636\,.\,\hat{i} \quad\text{..........................}(113).$$

Hence, on the assumption that the current is a simple sine function of the time, its average value is to the maximum value as $2:\pi$ or as $7:11$.

Similarly for the electromotive force, we have

$$e_{\text{av}} = \frac{2}{\pi}\,.\,\hat{e} = 0{\cdot}636\,.\,\hat{e} \quad\text{.......................}(114).$$

The average values of the alternating current and pressure are, however, little used in electrical engineering, since they cannot be used to calculate the power. The reason for this will be clear after reading the next section.

73. Power in A.C. Circuits and Root-mean-square Current.

As we have just mentioned, the power cannot be obtained by multiplying together the mean values of the current and pressure. We must find the momentary value of the power by multiplying the momentary value of the current by the value of the pressure at the same instant. The same result is obtained by squaring the momentary value of the current and multiplying by the resistance. Hence, for the power at any instant, we have

$$P = \hat{e}.\sin\alpha.\hat{i}.\sin\alpha = (\hat{i}.\sin\alpha)^2.R.$$

These values of P are found for various moments during the period, and plotted as ordinates in Fig. 222. As power is the work done per second, the area of the small strip $P.dt$ must represent the work done during the interval dt. The whole shaded area must represent the total work done during a period. If this be divided by the time t_1 in which the work is done, we get the mean work per second, that is, the mean power. This is given by the height of a rectangle of the same area and on the same base as the shaded figures. This height is found to be exactly half of the total height of the curves.

The same result is obtained by calculation. This is simplified, as before, by plotting the angle α as abscissa (Fig. 223). The area of the narrow strip in Fig. 223 is equal to $P.d\alpha$. Substituting the value of P found above, we find the total area of the figure as follows:

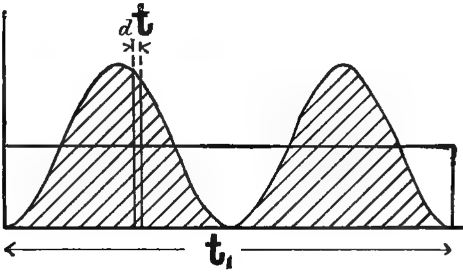

Fig. 222

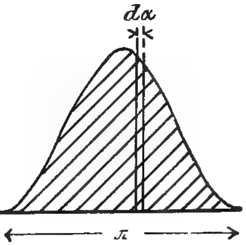

Fig. 223

$$\int_0^\pi P.d\alpha = \int_0^\pi \hat{i}^2.R.\sin^2\alpha.d\alpha;$$

but $$\int_0^\pi \sin^2\alpha.d\alpha = \left(\frac{\alpha}{2} - \frac{1}{2}\sin\alpha.\cos\alpha\right)_0^\pi = \frac{\pi}{2},$$

hence, the shaded area in Fig. 223 is equal to

$$\hat{i}^2.R\int_0^\pi \sin^2\alpha.d\alpha = \hat{i}^2.R.\frac{\pi}{2}.$$

To obtain the mean power we must divide this area by the base π. We thus get

$$P = \frac{\hat{i}^2.R}{2} \quad \text{................................}(115).$$

If the circuit contain no self-induction, we can put

$$\hat{e} = \hat{i}.R,$$

and equation (115) becomes

$$P = \frac{\hat{e}.\hat{i}}{2} \quad \text{................................}(116).$$

It is interesting to obtain this result in another way, which requires no higher mathematics, but which, at the same time, does not give one such an insight into the reasons for the result as is obtained from a careful study of the above. We assume that a two-pole machine has a drum containing two independent windings at right angles to each other. Both windings are connected to equal external resistances. When the plane of one coil makes an angle α with the zero position, the other will make an angle $90^\circ + \alpha$ with the same position, and the momentary value of the total power of the two windings together is

$$\hat{e}.\hat{i}.[\sin^2\alpha + \sin^2(90^\circ + \alpha)] = \hat{e}.\hat{i}.(\sin^2\alpha + \cos^2\alpha).$$

Since $\sin^2\alpha + \cos^2\alpha$ is always equal to 1, it follows that the power of the two coils together is always constant and equal to $\hat{e}.\hat{i}$. This is the mean power of the two coils, and that of each will therefore be a half of this, viz.

$$P = \frac{\hat{e}.\hat{i}}{2},$$

as found above. Had we calculated the power by multiplying together the average values of the current and pressure, the result would have been

$$\frac{2}{\pi}\hat{e}.\frac{2}{\pi}.\hat{i} = 0{\cdot}405\,\hat{e}.\hat{i}.$$

This is 20 per cent. smaller than the real average power.

We must now define and calculate the values of current and pressure, which can be applied directly to the calculation of the power. For this purpose we write equation (115) in the following form:

$$P = \frac{\hat{i}}{\sqrt{2}}.\frac{\hat{i}}{\sqrt{2}}.R.$$

The expression $\frac{\hat{i}}{\sqrt{2}}$ in this equation represents a certain strength of current which, if squared and multiplied by the resistance, gives the mean power. This strength of current is called the effective or root-mean-square value of the alternating current. This is the current strength which is meant when speaking generally of the strength of an alternating current. We shall represent it by the capital letter I. We have therefore

$$I = \frac{\hat{i}}{\sqrt{2}} = 0{\cdot}707\,.\,\hat{i} \quad\text{..........................}(117).$$

We can therefore define the effective value of an alternating current in the three following ways:

(1) as that value which, squared and multiplied by the resistance, gives the mean power;

(2) as the root of the mean value of the squares of the momentary values of the current;

(3) in true sine waves, as 0·707 of the maximum value of the current.

Since we found the electrolytic mean to be $\frac{7}{11}$ of the maximum, the effective value is to the average value as 11 : 10, or more accurately as $\pi : 2\sqrt{2}$.

If, therefore, we wish to calibrate an ammeter for alternating currents by rectifying the alternating current and passing it through a copper voltameter, it will be necessary to multiply the average current, as calculated from the deposit in the voltameter, by 1·1 in order to obtain the correct value of the effective current which should be indicated on the scale of the ammeter. Those instruments, however, in which the deflecting force is proportional to the square of the current at any moment, will give the same deflection with direct current as with an alternating current of the same effective value. They take up a position corresponding to the mean value of the squares of the alternating current, and, if calibrated with direct current, indicate directly the effective value of the alternating current.

To this class belong those instruments in which a current-carrying coil attracts or deflects a small piece of soft iron. The force exerted at any moment is proportional to the product of the momentary value of the current and that of the magnetism induced in the soft iron. For low values of the saturation the force will be therefore proportional to the square of the current, and we should expect the instrument to read correctly both direct and alternating currents. As, however, no iron is entirely free from hysteresis and remanent magnetism, these instruments are not suitable for very accurate work even with direct currents. With alternating currents the hysteresis causes the reading to be lower than that obtained with a direct current of the same effective value.

Hot-wire instruments, on the other hand, have exactly the same scale for all currents. In this type of instrument a platinum or platinum-silver wire is heated by the passage of the current and thereby caused to elongate. This elongation is magnified, and transmitted to the pointer of the instrument. Since the wire possesses a certain thermal capacity and the pointer, etc., a certain inertia, the instrument is unable to follow the rapidly pulsating current, and consequently takes up a position corresponding to the mean power dissipated as heat in the wire. The readings on the scale correspond to effective currents, that is, the currents which, if passed steadily through the wire, would produce the same heating as the alternating current. It follows that the instrument must be equally applicable to both direct and alternating currents.

The same holds true for dynamometers, in which the current flows through a fixed and a moving coil connected in series. It was seen in Section 32 that currents flowing in the same direction are attracted, while currents in opposite directions are repelled from each other. Since the current is reversed simultaneously in both coils, the turning moment exerted by the fixed on the moving coil will act always in one direction. The deflection, being proportional to the product of the currents in the two coils, is proportional to the square of the current. In consequence of the inertia of the moving coil, the pointer takes up a position corresponding to the mean torque, and therefore to the mean of the squares of the current. The scale can be marked so as to read directly the effective current. This is, as before, that continuous current which gives the same deflection as the periodically varying current. The same scale can evidently be used both for direct and alternating currents. The same is naturally true for voltmeters.

74. Vector Diagrams.

The sine curve represents very clearly the changes occurring in the current or in the electromotive force during a period, and we shall often make use of it in our further studies of alternating current phenomena. The vector diagram is to be preferred, however, when we wish to determine rapidly the relations between the various currents and electromotive forces in an alternating current circuit. In Fig. 225 the line OE represents $\hat{e}$ to a certain scale, and rotates about the point O with a constant angular velocity $\omega = 2\pi f$. If we are considering a bipolar machine, the vector will rotate at the same rate as the armature. The direction of rotation is immaterial, but we shall always assume it to be anti-clockwise. In its initial position the vector will lie horizontally to the right of O. The angle α which it makes with its initial position at any moment is equal to the angle made by the plane of the coil in Fig. 224 with its zero position. The projection of OE on the vertical axis is evidently equal to $OE . \sin \alpha$, that is, to $\hat{e} . \sin \alpha$.

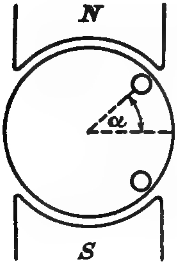

Fig. 224

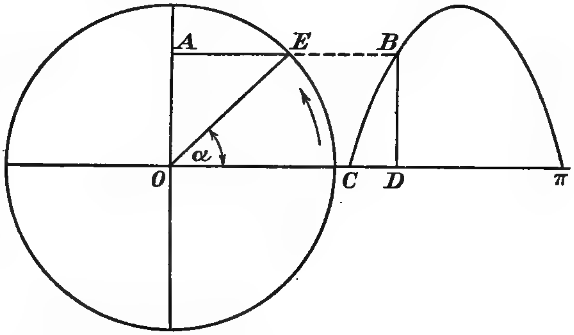

Fig. 225

Hence, the instantaneous value of the electromotive force is equal at every moment to the projection of the radius vector upon the vertical axis.

To make the matter still clearer, a sine curve representing half a period has been added in Fig. 225 to show the connection between it and the vector diagram. The radius of the circle described about point O as centre is equal to the maximum ordinate of the sine curve. The instantaneous value OA in the vector diagram is equal to the ordinate BD in the sine curve. Finally, the arc subtending the angle α at unit radius in the vector diagram is equal to the abscissa CD in the sine curve.

Electromotive forces of different magnitude and of different phase can be very conveniently compounded and their resultant determined by means of the vector diagram. When we speak of electromotive forces being of different phase we mean that they are out of step, as it were, and do not attain their maximum values at the same moment. On the armature in Fig. 226, for example, there are two coils consisting of a different number of turns and displaced by a certain angle from each other. Coil I possesses the greater number of turns and will therefore have a greater maximum E.M.F. than coil II. Two circles are drawn about the centre O (Fig. 227), one having a radius $OE_1 = \hat{e}_1$ and the other a radius $OE_2 = \hat{e}_2$, and the vectors OE_1 and OE_2 are drawn in directions corresponding to the positions of the coils I and II on the armature. As in the case of forces, we find the resultant OR by completing the parallelogram OE_1RE_2. At the given moment

OA is the instantaneous value of the E.M.F. in coil I,
OB ,, ,, ,, ,, ,, II,

and OC is the projection on the vertical of the resultant OR. From the similarity and equality of the shaded triangles, it follows that

$$OC = OA + OB.$$

Now, OC is the instantaneous value of an E.M.F. represented both in magnitude and phase by the vector OR. Thus the sum of the instantaneous values of the electromotive forces in the two coils is equal at every moment to the instantaneous value of the resultant. It follows that electromotive forces can be compounded by the parallelogram law in exactly the same way as mechanical forces.

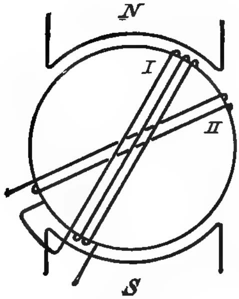

Fig. 226

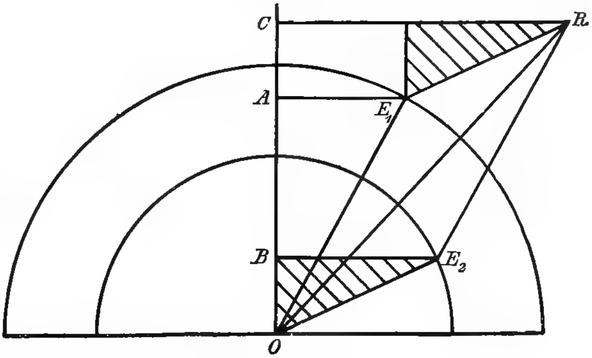

Fig. 227

To make this important result still clearer, we have represented in Fig. 228 the moment when the resultant E.M.F. has its maximum value. Its vector lies vertically, while the vector of E_1 lies a little to the left and that of E_2 a little to the right of the vertical. If we consider the wires constituting the adjacent coil-sides on one side of the armature, we can see that the maximum resultant E.M.F. will be induced when the wires as a whole lie under the middle of the pole. In this position, however, the wires of coil I lie just past the pole middle, while the wires of coil II lie just behind it, exactly as we have found from the vector diagram.

In a similar manner we may consider the moment when the coils lie in the neutral zone and the resultant E.M.F. is nil (Fig. 229). One coil-side is under the influence of the north pole, while the other is under the influence of the south pole, and the opposing electromotive forces induced in the two coils exactly neutralise each other. We arrive at the same conclusion from a consideration of the vector diagram in which the resultant OR is horizontal and its projection on the vertical axis is O. The vector OE_1 lies above the abscissa axis, while the vector OE_2 lies below it. The projection of OE_1 is therefore positive and the projection of OE_2 negative, and since they oppose each other, their resultant vanishes. Here again, the directions of the vectors correspond with the geometrical positions of the coils. Such considerations as these diminish, to a certain extent, the difficulty experienced by the student in grasping the idea of difference of phase and in the application of the parallelogram of forces to electromotive forces.

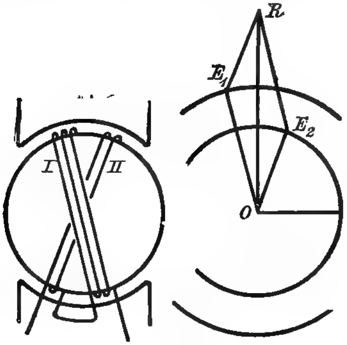

Fig. 228

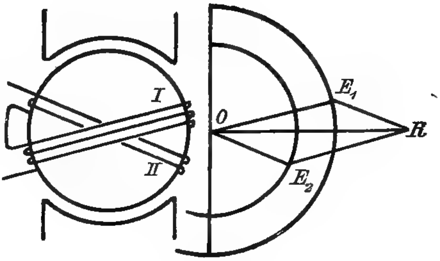

Fig. 229

We can, however, go a step further and substitute effective values of the electromotive forces instead of maximum values. This will simply alter the scale of the vector diagram without altering the angles. The length OR will then represent the effective value of the resultant. In a similar way we can find the resultant of two currents flowing in parallel paths, but differing in magnitude and phase. Such a compounding of electromotive forces or currents is called a vectorial or geometrical addition.

We have in the foregoing drawn all vectors from the origin and have compounded them by means of the parallelogram. Although this is to be recommended to beginners, the reader should make himself familiar with a simpler method in which the vectors are not necessarily drawn from the origin, and in which the triangle and polygon of forces are used instead of the parallelogram. The direction of the vector must be indicated by an arrow, and in referring to it by its terminal letters, these must be placed in the correct order. For example, the vector a in Fig. 230 is equivalent to the vector OD and must be referred to as AB and not BA. Similarly the vector b is equivalent to the vector OC and must be referred to as BO and not OB.

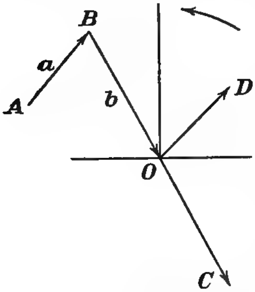

Fig. 230

In Fig. 231 the two vectors OA and OB are added by means of the parallelogram; in Fig. 232 the vector BC which corresponds to OA both in direction and magnitude is added to the vector *OB*, giving the same resultant *OC*. The principle of geometrical addition is shown more clearly in Fig. 232 than in Fig. 231. Fig. 233 shows the addition of the vectors *AO* and *OB* to give the resultant *AB*; its direction is from *A* to *B* because in carrying out the addition one starts at *A* and going always in the direction of the arrows arrives at *B*. Finally in Fig. 234 the vectors *BA* and *AC* give the resultant *BC*. The student should check both these examples by drawing the vectors from the origin and completing the parallelogram.

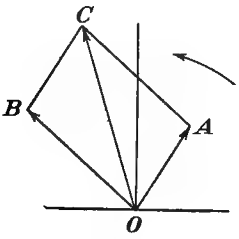

Fig. 231

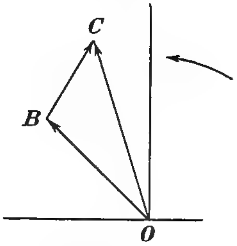

Fig. 232

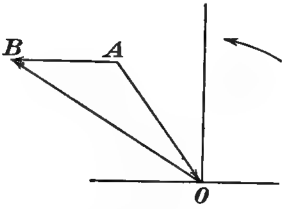

Fig. 233

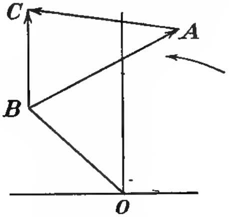

Fig. 234

Kirchhoff's second rule for any closed circuit can now be expressed as follows: The vector sum of the electromotive forces is equal to the vector sum of the products *IR*. This product has the same phase as the current. In Fig. 235 the vector *OA* represents the electromotive force induced in the armature of an alternating current generator by the magnetic field of the pole system, the vector *I* represents the current flowing through the armature and round the external circuit; this current is assumed to lag 45° behind *OA*. *OB* represents the electromotive force induced in the armature by the flux produced by the current *I* flowing through the armature winding; as will be shown in the following section, this E.M.F. lags 90 degrees behind the current, as seen in Fig. 235. *OD* represents the

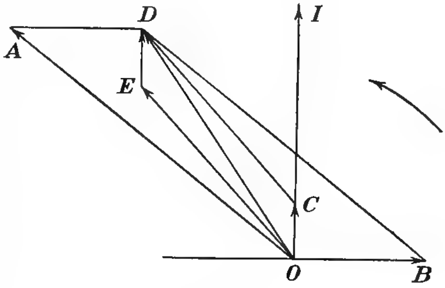

Fig. 235

resultant E.M.F. induced in the armature. This E.M.F. will be used up to some extent in overcoming the internal resistance of the armature; the product IR_a will be in phase with the current and is shown in Fig. 235 as the vector OC or ED. The other component of OD, viz. OE, represents the terminal P.D. of the generator.

Fig. 236 is the same construction but without the parallelograms. The electromotive force AD is added to OA giving the resultant OD, which must be also the resultant of the internal ohmic drop OC and the terminal voltage CD. It will be noted that in the closed figure $OADCO$ in Fig. 236, all the electromotive forces are directed one way around the figure, while the ohmic drops and the terminal P.D. are directed in the reverse direction. Since the figure closes, the vector sums in the two directions must be equal and opposite. For the same conditions different diagrams are obtained depending on the order in which the vectors are drawn; thus, in Fig. 237 the two electromotive forces OA and AD are added and then the line DE is set out equal and opposite to the ohmic drop vector; the figure is then completed by the line EO which is equal and opposite to the terminal P.D.

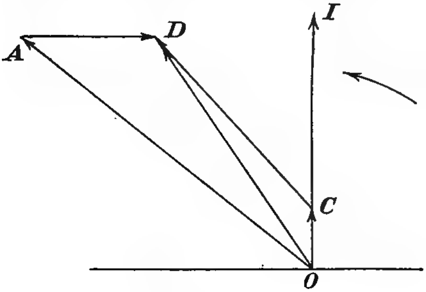

Fig. 236

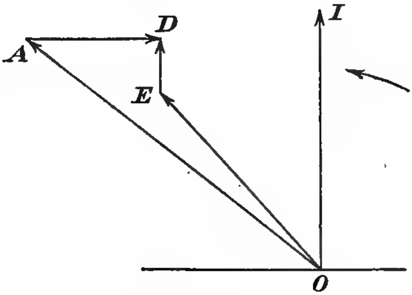

Fig. 237

75. The Electromotive Force of Self-induction.

It was seen in Section 35 that the electromotive force e_s, induced by a change of current in a coil of self-induction L, is given by the equation

$$e_s = -L \cdot \frac{di}{dt} \text{ volts,}$$

where
$$L = \frac{0{\cdot}4\pi \cdot T^2 \cdot \mu \cdot A}{l} \cdot 10^{-8} \text{ henries.}$$

For the angular velocity of the radius vector we have

$$\omega = 2\pi \cdot f,$$

so that
$$i = \hat{i} \cdot \sin \alpha = \hat{i} \cdot \sin (\omega t).$$

Differentiating, we have

$$\frac{di}{dt} = \hat{i} \cdot \cos (\omega t) \cdot \omega = \hat{i} \cdot \omega \cdot \cos \alpha.$$

Introducing this into the above formula for e_s, we get

$$e_s = -L\omega . \hat{i} . \cos\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (118).$$

The E.M.F. of self-induction is thus a cosine function and by plotting it as ordinates with either the angle α or the time t as abscissae we obtain a cosine curve. This is identical in form with the sine curve, but is displaced 90° from it. The relations found for the sine function can therefore be applied here, so that the effective value of the E.M.F. of self-induction is 0·707 of the maximum value.

From equation (118) it can be seen that the maximum value is reached when $\cos\alpha = -1$, that is, when $\alpha = 180°$. Its value will therefore be

$$\hat{e}_s = L\omega . \hat{i}.$$

If both sides of this equation be multiplied by 0·707, the maximum values of E.M.F. and current will be converted into effective values, and we can write

$$E_s = L\omega . I.$$

Hence, the effective value of the E.M.F. of self-induction is given by the product of the coefficient of self-induction, the angular velocity of the vector and the effective current.

A coil purposely made so as to have a large self-induction is called a choking coil. Such a coil is shown in Fig. 242, and consists of an iron core in the form of a ring with a small air-gap and wound with insulated copper wire. If the magnetic flux-density is kept small, the reluctance of the iron may be neglected and only the air-gap need be considered in calculating the self-induction.

We shall assume the following data for the choking coil:

Length of magnetic path in air-gap ...	$l = 0{\cdot}6$ cm.
Cross-section of path normal to the flux	$A = 12$ sq. cm.
Number of turns in coil	$T = 200$.
Effective value of current	$I = 10$ amps.
Frequency	$f = 50$.

Then, since the permeability of the air is 1, we have

$$L = \frac{0{\cdot}4\pi . T^2 . \mu . A}{l} . 10^{-8} = 1{\cdot}25 \frac{200^2 . 1 . 12}{0{\cdot}6} . 10^{-8} = 0{\cdot}01 \text{ henry}.$$

We have also

$$\omega = 2\pi f = 314.$$

Hence, for the electromotive force of self-induction, we have

$$E_s = L\omega . I = 0{\cdot}01 . 314 . 10 = 31{\cdot}4 \text{ volts}.$$

We shall see presently that a voltmeter connected across the terminals of the choking coil will read almost exactly 31·4 volts, showing that the coil acts like a resistance in causing this drop of pressure.

In order to design a choking coil, however, the formula needs to be modified, as we might otherwise choose a value of cross-section for the iron or a number of turns for the coil which would lead to a very unsuitable flux-density in the core. If Φ be the total flux at the moment of maximum current, we have

$$\Phi = \mathbf{B} . A.$$

For the value of the flux at any moment we have

$$\Phi \sin \alpha = \Phi \sin (\omega t).$$

Hence the rate of change of the flux will be

$$\Phi \frac{d \,.\, \sin \alpha}{dt} = \Phi \,.\, \omega \,.\, \cos (\omega t).$$

If the coil have T turns, the momentary value of the E.M.F. will be

$$e_s = - \Phi \,.\, \omega \,.\, T \,.\, \cos (\omega t) \,.\, 10^{-8}.$$

The only variable here is the cosine, which has a maximum value of 1. Hence

$$\hat{e}_s = \Phi \,.\, \omega \,.\, T \,.\, 10^{-8} = 2\pi f \,.\, \Phi \,.\, T \,.\, 10^{-8}.$$

If we divide the maximum value by $\sqrt{2}$ we get for the effective or R.M.S. value

$$E_s = 4{\cdot}44 f \,.\, \Phi \,.\, T \,.\, 10^{-8} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (119).$$

If in the formula

$$\mathbf{B} = \mu \,.\, \mathbf{H} = \frac{0{\cdot}4\pi \,.\, T \,.\, \hat{i} \,.\, \mu}{l},$$

we put $\hat{i} = \sqrt{2} \,.\, I$, we get

$$I = \frac{\mathbf{B} \,.\, l}{1{\cdot}78 \,.\, T \,.\, \mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (120).$$

This equation is very convenient for calculating the magnetising current of a transformer.

From the two equations (119) and (120) a choking coil can be designed when we have chosen a suitable value for the magnetic flux-density. If, for example, an E.M.F. of 30 volts is to be induced by a current of 10 amperes at a frequency of 50, we may proceed in the following manner. The flux-density is taken as 5000 lines per sq. cm and the path in the iron is neglected, so that we need only consider the length l of the air-path. The cross-section of the iron is assumed to be 15 sq. cm. We have then

$$\Phi = \mathbf{B} \,.\, A = 5000 \,.\, 15 = 75{,}000.$$

From equation (119) we have

$$T = \frac{E \,.\, 10^8}{4{\cdot}44 \,.\, \Phi \,.\, f} = \frac{30 \,.\, 10^8}{4{\cdot}44 \,.\, 75{,}000 \,.\, 50} = 180.$$

Putting the number of turns T equal to 180 in equation (120) we get

$$l = \frac{1{\cdot}78 \,.\, T \,.\, \mu \,.\, I}{\mathbf{B}} = \frac{1{\cdot}78 \,.\, 180 \,.\, 1 \,.\, 10}{5000} = 0{\cdot}64 \text{ cm}.$$

This is the length of the air-gap. The cross-section of the path across the air-gap has been taken as equal to the cross-section of the iron. As a matter of fact, however, the lines of force do not pass straight across the gap, but curve round from pole to pole, being driven outwards by the lateral pressure which we have seen to exist between the lines. The cross-section of the path across the gap is therefore considerably greater than the cross-section of the iron. The presence of the paper or varnish insulation between the sheet-iron stampings of which the core is built up will still further increase this effect. The above coil will consequently cause a greater drop than 30 volts when carrying 10 amperes.

Having considered the magnitude of this E.M.F. of self-induction, we must now turn to its phase. We saw above that the curve is a cosine curve, and is consequently displaced 90° from the sine curve of the current. This will be made plainer if we determine the magnitude and direction of the E.M.F. of self-induction at several critical moments during the period.

The current curve is represented by the thick black line in Fig. 238. When $\alpha = 0$, $\sin \alpha = 0$, but $\cos \alpha = 1$. This is the moment when the current passes through zero and the E.M.F. of self-induction has its maximum value, as seen from equation (118) on page 245. The current curve is steepest at this point, which means that, although there is no current at the moment, yet the rate at which the current, and with it the flux, is changing is then a maximum. That the E.M.F. induced in the coil is negative follows at once from the fact that it opposes the growing current and tends to keep up the dying current.

When the angle α is equal to 90° we have $\sin \alpha = 1$ and $\cos \alpha = 0$. The current is then a maximum and the induced E.M.F. is equal to 0. At the moment when the current is a maximum, the current curve in Fig. 238 is parallel to the base, and for an instant both current and flux are constant. There is therefore no cutting of lines at this moment, and the dotted curve must pass through the base line at the point representing $\alpha = 90°$.

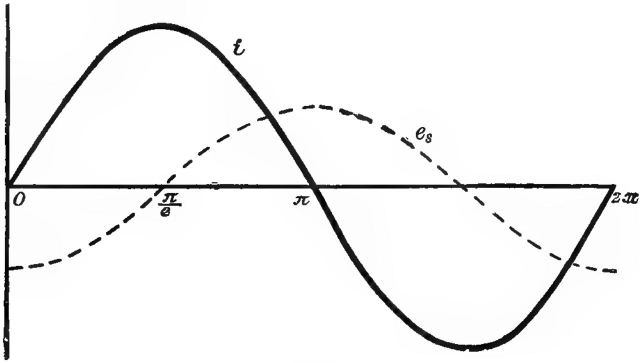

Fig. 238

By finding the various points on the curve e_s in this manner, the whole curve can be drawn, with the result that the E.M.F. of self-induction is found to lag 90° behind the current. This point should be carefully noticed, since one might easily come to the conclusion that, as the curve e_s in Fig. 238 is further to the right than the curve i, the E.M.F. is therefore ahead of the current. This conclusion is quite wrong, for we see that the curve e_s reaches its maximum value at a time when the curve i has already passed its maximum. Hence the E.M.F. of self-induction lags behind the current by a quarter of a period or by an angle of 90°.

This is made clearer by a consideration of the vector diagram in combination with the laws of self-induction already found in Section 35. The diagram is shown in Figs. 239, 240 and 241 for three successive moments. When the current vector passes through the horizontal position (Fig. 240), its projection on the vertical, and therefore also the instantaneous value of the current, is equal to 0. The E.M.F. of self-induction is a maximum at this moment, and its vector must consequently be vertical; moreover, since it lags 90° behind the current, it must be vertically downwards.

We shall now consider the relations a moment earlier and a moment later. In Fig. 239 the current is decreasing and its vector lies below the horizontal, so that its projection is in the same direction as that of the electromotive force E_s. The E.M.F. is thus in the same direction as the decreasing current, which agrees exactly with what we found in Section 35.

In Fig. 241 the current has passed through its zero value and is increasing. The projection of its vector is now above the horizontal, and therefore in the opposite direction to the projection of the E.M.F., which has hardly changed. The self-induced E.M.F. is thus opposed to the growing current, as was found in Section 35.

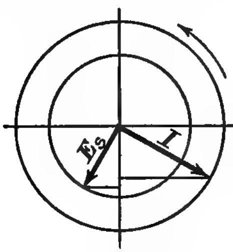

Fig. 239

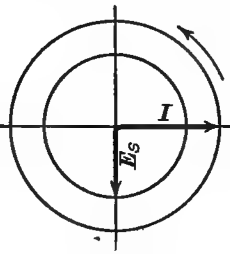

Fig. 240

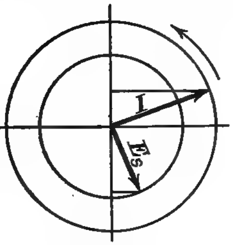

Fig. 241

If now we agree to consider the magnetic flux as having the same sign as the current producing it, there will be no difference of phase between the magnetising current and the flux. The current curve will then represent to a certain scale the number of lines threading the winding at any moment. Hence, the self-induced E.M.F. lags not only 90° behind the current, but also 90° behind the magnetic flux, through the variation of which the E.M.F. is induced.

76. Ohm's Law applied to Alternating Currents.

We shall now consider the circuit represented in Fig. 242, in which an ohmic resistance R is connected in series with a choking coil, the resistance of which is so small compared with its self-induction that it can be neglected. We have to determine the magnitude of the current which flows round the circuit under the joint influence of the terminal pressure V of the generator and the self-induced E.M.F. of the choking coil. In addition to the magnitude of the current, its phase relation with regard to the terminal pressure should be considered.

To simplify the problem we shall first consider it in the reverse form, viz. what terminal pressure will be required to drive a given current through the external circuit? In the first place we determine the momentary value of the ohmic pressure drop by multiplying the momentary value of the current by the resistance. This ohmic pressure drop is plotted as ordinates and gives the heavy black curve in Fig. 243. Since it must have its maximum value at the same moment as the current, it must be in phase with it. The curve $i.R$ differs from the current curve merely in the matter of scale.

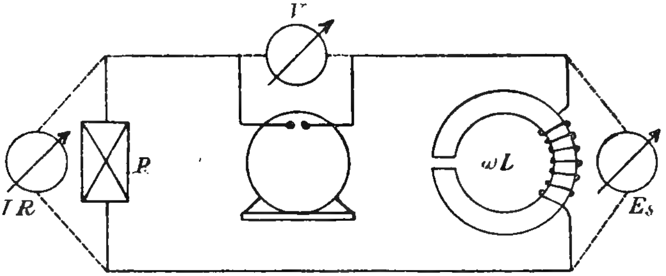

Fig. 242

The ohmic pressure drop must be provided by the combined action or resultant of the terminal pressure, which is at present unknown, and the electromotive force e_s, the curve of which lags 90° behind the curve $i.R$. We proceed now to determine the value of the terminal pressure for a few characteristic points during the period.

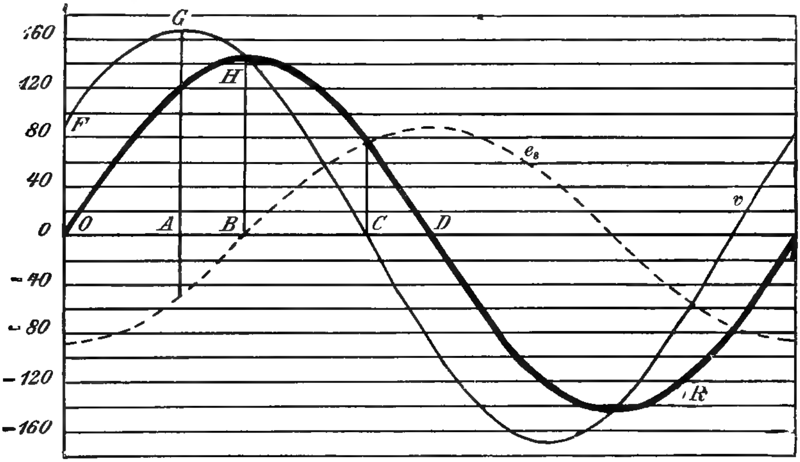

Fig. 243

At the origin O there is momentarily no current, although the E.M.F. of self-induction has at this very moment its maximum negative value. It is evident, therefore, that its effect is exactly neutralised by an equal and opposite terminal pressure OF. The current can only vanish when forward and back pressures exactly neutralise each other.

At the moment A, on the other hand, the terminal pressure has not only to overcome a considerable E.M.F. of self-induction, but has now to cover a large ohmic pressure drop. The terminal pressure must therefore be equal to the sum of these two, and will be represented by GA.

At the moment B there is no E.M.F. of self-induction, and the terminal pressure has merely to provide for the ohmic pressure drop HB. The curves of terminal pressure and ohmic drop will therefore cross at the point H.

Finally, at the moment C, the self-induced E.M.F. is just sufficient to drive the diminishing current through the resistance without assistance from the generator, the terminal pressure of which is therefore zero. The curve v cuts the horizontal axis at the point C. In this way we obtain the thin full-line curve in Fig. 243, the data for which are as follows:

$$I = 10 \text{ amperes}, \quad R = 10 \text{ ohms}, \quad \omega L = 6{\cdot}28.$$

The maximum value of the ohmic pressure drop is found as follows:

$$\hat{i}.R = \sqrt{2}.10.10 = 141{\cdot}4 \text{ volts},$$

while for the maximum value of the self-induced E.M.F. we have

$$\hat{e}_s = \omega L.\hat{i} = 6{\cdot}28.\sqrt{2}.10 = 88{\cdot}8 \text{ volts}.$$

The maximum value of the terminal pressure of the generator is found from Fig. 243 to be 167 volts.

An examination of the figure leads to the following important conclusions:

1. The curve of ohmic pressure drop does not reach such a high value as the terminal pressure curve, that is, the current is smaller than one would expect from the terminal pressure and the given resistance. The formula $V = I.R$ is therefore not applicable to alternating current circuits containing self-induction.

2. The sum of the maximum values of the ohmic pressure drop and the self-induced E.M.F. is greater than the maximum value of the terminal pressure. The sum of the first two is 230·2 volts, whereas the latter is given by the figure as 167 volts. This fact is illustrated in a striking manner by connecting three voltmeters as shown in Fig. 242. One is across the machine terminals, one across the resistance, and the other across the choking coil. The following effective values will be read on the three instruments:

$$V = \frac{\hat{v}}{\sqrt{2}} = \frac{167}{\sqrt{2}} = 118,$$

$$I.R = 10.10 = 100,$$

$$E_s = \omega L.I = 6{\cdot}28.10 = 62{\cdot}8.$$

Thus, the sum of the two separate pressures $I.R$ and E_s is greater than the total pressure V. This result seems quite contradictory to the ideas which we have formed in connection with direct currents. It must be carefully borne in mind, however, that this is only true of the effective or maximum values, and not of the instantaneous values at any moment. For the latter the total pressure is the algebraic sum of the separate pressures.

3. The curve representing the ohmic pressure drop or the current lags behind the curve of terminal pressure by the angle ϕ, which is represented in Fig. 243 by the distance CD. The current reaches its maximum value a short time after the pressure has passed through its maximum value. The

angle corresponding to this interval is known as the angle of lag. When the current is ahead of the pressure, that is, when the current reaches its maximum a short time before the pressure, the current is said to lead by a certain angle which we call the angle of lead.

The striking phenomena occurring in a circuit containing self-induction are brought out very clearly in the vector diagram. Instead of the current vector we may consider the vector of the ohmic pressure drop $I.R$, which is in phase with, and proportional to, the current. In Fig. 244 it is passing through the zero position, and the vector of the E.M.F. of self-induction points vertically downwards. Since the current is equal to 0 at this moment, the E.M.F. of self-induction must be exactly neutralised by an equal and opposite component of the applied terminal pressure. This component $-E_s$ is therefore drawn vertically upwards from O. The resultant terminal pressure V is given by the hypotenuse of a right-angled triangle, the sides of which are equal to the E.M.F. of self-induction and the ohmic pressure drop respectively.

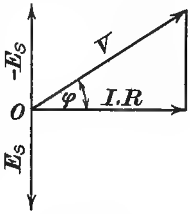

Fig. 244

It is seen that the terminal pressure has a double duty to perform, viz. to overcome the ohmic resistance on the one hand, and counterbalance the E.M.F. of self-induction on the other. Although the E.M.F. of self-induction is sometimes acting with the current and sometimes against it, we must always look upon it as a back E.M.F. which must be overcome by a component of the applied pressure. This is specially evident in the case shown in the figure, for the whole terminal pressure is momentarily engaged in counterbalancing this back E.M.F. of self-induction.

Several of the characteristic points noticed in connection with Fig. 243 are even more evident in the vector diagram. The terminal pressure V is greater than the pressure $I.R$ required to overcome the ohmic resistance, while the sum of the component pressures $I.R$ and E_s exceeds the total pressure V. As before, the current lags behind the terminal pressure by an angle ϕ. It is to be observed that this angle must be measured between the hypotenuse of the right-angled triangle and the side representing the ohmic pressure drop.

For the vector diagram we can easily determine the equation expressing the relation between current and pressure in an inductive circuit. It is evident from Fig. 244 that

$$V^2 = (I.R)^2 + E_s^2.$$

By substituting for E_s its value $L\omega.I$ and transposing, we get

$$I = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \quad \text{..........................(121).}$$

This may be called the Ohm's law for alternating current. It emphasizes the fact that self-induction causes an apparent increase in the resistance of a circuit, since the denominator is increased from R to $\sqrt{R^2 + (\omega L)^2}$.

This increased resistance is represented graphically by the hypotenuse of a right-angled triangle, the sides of which are equal to the resistance R and the quantity ωL respectively. Since this triangle (Fig. 245) is similar to the corresponding pressure triangle in Fig. 244, it follows that the angle between its hypotenuse and the side R is equal to the phase difference or angle of lag ϕ, and we have therefore

$$\tan\phi = \frac{\omega L}{R} \quad \text{..................................(122).}$$

The quantity $\sqrt{R^2 + (\omega L)^2}$ is called the apparent resistance or, more generally, the impedance, while the quantity ωL is known as the reactance or inductive resistance. The quantity ωL can only be looked upon and treated as a resistance if its dimensions are those of resistance. It would otherwise be quite impossible to add R^2 and $(\omega L)^2$. Now, the dimension of the coefficient of self-induction is simply a length, and that of angular velocity is the reciprocal of time, so that

the dimensions of reactance are $L.T^{-1}$.

These are, however, the dimensions of velocity, which were seen in Section 40 to be also those of resistance. Moreover, since a henry is equal to 10^9 absolute units, and an ohm is also equal to 10^9 absolute units, the product ωL must give the inductive resistance directly in ohms.

With the help of equation (121) we are now in a position to calculate the current for any given values of $I.R$ and ωL. We are no longer compelled to follow the reverse course to find the terminal pressure for an assumed value of the current. We shall assume the same data as already used on page 250, viz.

$$V = 118, \quad R = 10, \quad \omega L = 6{\cdot}28.$$

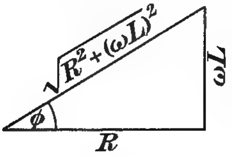

Fig. 245

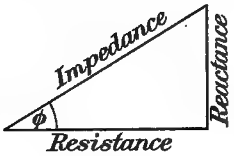

Fig. 246

From equation (121) we have

$$I = \frac{118}{\sqrt{10^2 + 6{\cdot}28^2}} = 10 \text{ amperes.}$$

For the phase-lag of the current behind the terminal pressure we have

$$\tan\phi = \frac{\omega L}{R} = \frac{6{\cdot}28}{10} = 0{\cdot}628.$$

This corresponds to an angle of 32° 10′ or $\frac{\pi}{6}$. This is in evident agreement with the length CD in Fig. 243, which was drawn for the same conditions.

77. Resistance and Inductance in Series.

We shall now consider the case in which two parts of a circuit are connected in series, each part containing both ohmic and inductive resistance, as represented in Fig. 247. The total pressure V is, in this case, the hypotenuse of a right-angled triangle, one side of which is equal to the sum of the pressure drops $I.R_1$ and $I.R_2$ due to the ohmic resistances, while the other side is equal to the sum of the inductive pressure drops E_{s1} and E_{s2}. At the same time, however, the total pressure V is the geometrical sum of the terminal pressures V_1 and V_2 of each half of the circuit. The triangle of pressures can be drawn for each half of the circuit (Fig. 248). The terminal pressure V_1 of the left-hand half forms the hypotenuse to the sides $I.R_1$ and E_{s1}, and makes an angle ϕ_1 with the current vector or side $I.R_1$, the tangent of which is given by the equation

$$\tan \phi_1 = \frac{E_{s1}}{I.R_1} = \frac{\omega L_1}{R_1}.$$

In a similar manner the terminal pressure V_2 of the right-hand half of the circuit forms the hypotenuse to the two sides $I.R_2$ and E_{s2}, and makes an angle ϕ_2 with the current vector, the tangent of which is given by the equation

$$\tan \phi_2 = \frac{E_{s2}}{I.R_2} = \frac{\omega L_2}{R_2}.$$

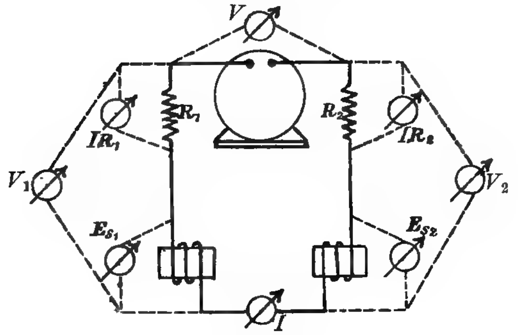

Fig. 247

The phase-lag of the current behind the terminal pressure V of the whole circuit will lie somewhere between ϕ_1 and ϕ_2. Hence, although the current in all parts of a series unbranched circuit must necessarily have the same phase, the pressure differences between various parts of the circuit generally differ in phase (Fig. 248).

The circuit made up of an alternating current generator working on an external circuit containing both resistance and inductance is similar to that just considered. As an example we shall assume the following data:

Terminal pressure of generator ...	$V = 2000$ volts,
Current	$I = 50$ amperes,
Armature resistance	$R_a = 1$ ohm,
Self-induction of armature ...	$L = 0{\cdot}04$ henry,
Frequency	$f = 50$,
Phase-lag in external circuit ...	$\cos \phi = 0{\cdot}8$.

We wish to determine the pressure drop in the generator and the electromotive force induced in the generator, both as regards its magnitude and its phase. For the construction of the diagram shown in Fig. 249, we have the following values:

$$I.R = 50.1 = 50 \text{ volts},$$

$$E_{s\,\text{internal}} = \omega L.I = 0{\cdot}04.2.3{\cdot}14.50.50 = 628 \text{ volts}.$$

From these two values we find the internal drop of pressure OF as the hypotenuse of a right-angled triangle, one side of which is equal to the ohmic resistance drop, while the other side is equal to the E.M.F. due to the armature self-induction. We thus get

$$OF = \sqrt{50^2 + 628^2} = 630 \text{ volts}.$$

We now find the ohmic resistance drop in the external circuit, thus:

$$I.R = V.\cos\phi = 2000.0{\cdot}8 = 1600 \text{ volts}.$$

If $\cos\phi = 0{\cdot}8$ we know that $\sin\phi = 0{\cdot}6$, so that the inductive drop in the external circuit can be found in the following manner:

$$E_{s\,\text{ext.}} = V.\sin\phi = 2000.0{\cdot}6 = 1200 \text{ volts}.$$

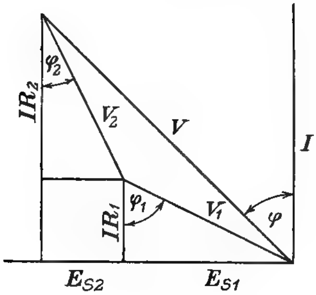

Fig. 248

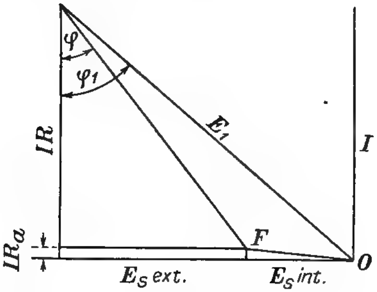

Fig. 249

The total ohmic resistance drop in the whole circuit is equal to

$$I.R_a + I.R = 50 + 1600 = 1650 \text{ volts},$$

and the total inductive drop is equal to

$$E_{s\,\text{int.}} + E_{s\,\text{ext.}} = 628 + 1200 = 1828 \text{ volts}.$$

From these two values we find the E.M.F. of the machine E_1 as the hypotenuse of a right-angled triangle, one side of which is equal to the sum of the resistance pressure drops, while the other is equal to the sum of the inductive pressure drops. We thus have

$$E_1 = \sqrt{1650^2 + 1828^2} = 2460 \text{ volts}.$$

The angle ϕ_1 by which the current lags behind the E.M.F. of the generator can be found from the equation

$$\cos\phi_1 = \frac{\Sigma\,(I.R)}{E_1} = \frac{1650}{2460} = 0{\cdot}67.$$

Neglecting secondary effects, we can say that the terminal P.D. on open circuit will be equal to the electromotive force E_1. On loading the generator the pressure drops from 2460 to 2000 volts, i.e. 460 volts, whereas the pressure required to drive the current through the armature was seen to be $OF = 630$ volts. This apparent contradiction is due to the fact that the pressure drop OF in Fig. 249 is subtracted geometrically and not algebraically from the E.M.F., to obtain the terminal pressure V.

The drop of 460 volts which we have found between no-load and full load represents 18·6 per cent. of the no-load pressure. This is approximately the usual figure found in practice, and is, as we have seen, almost entirely due to the internal self-induction of the machine. This brings out very clearly the detrimental action of the self-induction in increasing the apparent internal resistance of an alternating current generator.

78. Resistance and Inductance in Parallel.

In the case illustrated in Fig. 250 two branches are connected in parallel, each containing both ohmic and inductive resistance. The terminal pressure V is exactly the same both in magnitude and phase for each branch. This is true even if one or both of the branches contain either ohmic or inductive resistance alone, instead of each branch being partly non-inductive and partly inductive, as shown in Fig. 250. We have now to determine the magnitude and phase of the total current, and the current in each branch, from a knowledge of V, R_1, ωL_1, R_2 and ωL_2. The line OG in Fig. 251 is drawn to represent the terminal pressure V, and a semicircle is described upon it. OA is drawn so as to make an angle ϕ_1 with OG. This angle is given by the formula

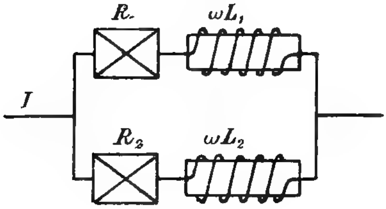

Fig. 250

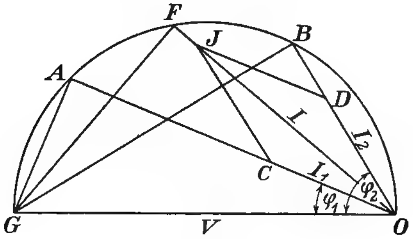

Fig. 251

$$\tan \phi_1 = \frac{\omega L_1}{R_1}.$$

The line OA represents the ohmic pressure drop in the upper branch of Fig. 250. If we divide this by the resistance R_1 we get the current I_1, which may be represented by the length OC. In a similar manner the line OB is drawn at an angle ϕ_2 with the diameter, where

$$\tan \phi_2 = \frac{\omega L_2}{R_2}.$$

The line OB represents the ohmic pressure drop in the lower branch, and by dividing it by the resistance R_2 we get the current I_2, which may be represented by the length OD. The resultant OJ of I_1 and I_2 gives us the total current I both in magnitude and phase. If the vector I be produced to meet the semicircle at F, the length OF represents the ohmic drop, and the length FG the inductive drop of a single coil or piece of apparatus which could replace the branched circuit of Fig. 250. The total current lags behind the terminal pressure by the angle FOG.

79. Effect of Phase Difference on A.C. Power.

If the current is not in phase with the pressure, the instantaneous value of the power is positive whenever the pressure and current are momentarily in the same direction (Fig. 252). On the other hand, the power is momentarily negative whenever the current and pressure are in opposite directions (Fig. 253). By the direction of the pressure we mean the direction from the positive terminal or terminal of high potential to the negative or low potential terminal. We shall not consider the internal electromotive forces, but only external potential differences. When we speak above of power being positive, we mean that the apparatus or part of the circuit under consideration is absorbing electrical energy and transforming it into some other form such as thermal or mechanical energy. When an apparatus has a negative power it is generating electrical energy at the expense of energy in some other form. In a dynamo, for example, the current flows from the negative terminal to the positive, that is, against the terminal pressure, and the power is therefore negative, or, in other words, electrical energy is being generated. In the motor the current flows from positive to negative and the power is positive, that is, the motor is absorbing electrical energy. We are thus led to the conclusion that a simple alternating current generator in which the current is out of phase with the E.M.F. does not act continuously as a generator, but, twice in each period, takes electrical energy from the circuit and is driven for a moment as a motor.

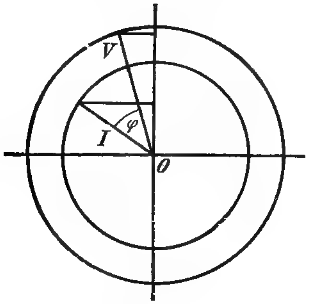

Fig. 252

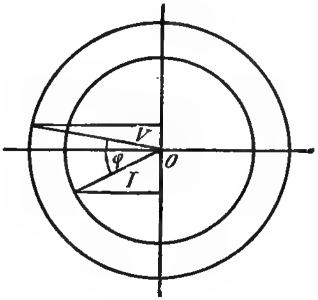

Fig. 253

The same rule can be applied to any apparatus to which electrical energy is supplied. If, at any moment, the current flows in the direction of the terminal pressure, i.e. from + to − , electrical energy is being absorbed, but if the current is flowing in the other direction, i.e. from − to + , the apparatus is acting momentarily as a generator and returning electrical energy to the circuit.

Fig. 254 refers to a coil or other apparatus in which the current I lags by an angle ϕ behind the terminal pressure. This lag is due to the self-induction of the coil. The time is plotted as abscissae and the current and P.D. as ordinates. The angle ϕ by which the current lags behind the P.D. corresponds to the time AB.

A curve is now plotted giving the momentary values of the power found by multiplying together the corresponding values of current and P.D. These values of the power are positive during the intervals OA and BC, but negative during the shorter intervals AB and CD, as shown by the dotted curve P, which represents the rapidly changing power to a certain scale. The area enclosed between this curve and the horizontal axis represents the energy supplied to the apparatus. This area lies partly above the axis and partly below. The mean power is found by subtracting the area below the line from the area above the line, and dividing the difference by the time. It is evident that the power supplied to the apparatus is less than we should expect from multiplying together the pressure and the current as read on suitable instruments.

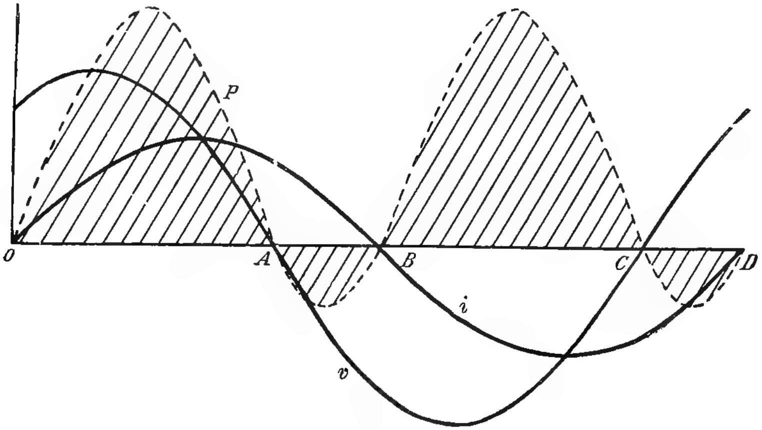

Fig. 254

In order to calculate the mean power, we shall imagine the curves to be plotted to a base representing the angular movement of the vector, instead of the time. If the strength of the current at a given moment be $\hat{i}.\sin\alpha$, the terminal pressure at the same moment will be $\hat{v}.\sin(\alpha+\phi)$, and the momentary value of the power will be equal to their product, thus

$$P = \hat{v}.\hat{i}.\sin\alpha.\sin(\alpha+\phi).$$

We may imagine this power to remain constant over the infinitely small angle $d\alpha$, so that the area of an infinitely narrow vertical strip will be

$$P.d\alpha = \hat{v}.\hat{i}.\sin\alpha.\sin(\alpha+\phi).d\alpha.$$

The mean value of the power is obtained by integrating this expression between the limits 0 and π and dividing the resulting area by the base π. In this way we get

$$P = \frac{1}{\pi}\int_0^\pi \hat{v}\,.\,\hat{i}\,.\sin\alpha\,.\sin(\alpha+\phi)\,.\,d\alpha.$$

Now, $\sin(\alpha+\phi) = \sin\alpha\,.\cos\phi + \cos\alpha\,.\sin\phi.$

Substituting this value for $\sin(\alpha+\phi)$, we have

$$P = \frac{1}{\pi}\int_0^\pi \hat{v}\,.\,\hat{i}\,.\sin^2\alpha\,.\cos\phi\,.\,d\alpha + \frac{1}{\pi}\int_0^\pi \hat{v}\,.\,\hat{i}\,.\sin\alpha\,.\cos\alpha\,.\sin\phi\,.\,d\alpha.$$

Since $\int_0^\pi \sin\alpha\,.\cos\alpha = 0$, the second integral vanishes, and we have

$$P = \frac{1}{\pi}\,.\,\hat{v}\,.\,\hat{i}\,.\cos\phi\int_0^\pi \sin^2\alpha\,.\,d\alpha.$$

We saw on page 237 that $\int_0^\pi \sin^2\alpha\,.\,d\alpha = \frac{\pi}{2}$, wherefore

$$P = \frac{1}{\pi}\,.\,\hat{v}\,.\,\hat{i}\,.\cos\phi\,.\,\frac{\pi}{2}.$$

This may be written in the following form:

$$P = \frac{\hat{v}}{\sqrt{2}}\,.\,\frac{\hat{i}}{\sqrt{2}}\,.\cos\phi.$$

Since the maximum values divided by $\sqrt{2}$ are equal to the R.M.S. or effective values, we have

$$P = V\,.\,I\,.\cos\phi \quad\text{..............................(123).}$$

The law that the power is equal to the product of current and pressure is therefore only applicable to instantaneous values, and not to effective values. In any case where there is difference of phase, the true power is found by multiplying the apparent power by the cosine of the angle of phase difference. For this reason $\cos\phi$ is generally called the power-factor. This designation is the more suitable since the curves of current and pressure depart in practice more or less from the sine form and rob $\cos\phi$ of the exact meaning which we have attached to it in the case of pure sine-curves. The power-factor is then generally defined as the ratio of the actual power, as indicated on a wattmeter, to the apparent power, as calculated from the current and pressure. Thus

$$\cos\phi = \frac{P}{V\,.\,I} \quad\text{..............................(124).}$$

In the example considered in Section 75, for example, the terminal pressure V of the generator was 2000 volts, while the current I was 50 amperes. The apparent power in this case is 2000.50 = 100,000 watts. Since, however, the power-factor was given as $\cos\phi = 0{\cdot}8$, the true power is only 80 per cent. of this, or

$$P = V\,.\,I\,.\cos\phi = 100{,}000\,.\,0{\cdot}8 = 80{,}000 \text{ watts.}$$

The power transmitted to the generator from the steam engine corresponds to the load of 80,000 watts only, so that no direct loss is entailed by this reduction of output from 100,000 to 80,000 watts. On the other hand, however, both the operation of the generator and the efficiency of the whole system are adversely affected by the phase displacement or lag of the current. The output of 80,000 watts is only maintained by a disproportionately great E.M.F., which has to overcome the back E.M.F. of self-induction, in addition to the resistance. The generator must therefore be capable of working at a higher pressure, a part only of which is applicable to the external load.

This is perhaps more evident if we imagine the whole load to consist of resistance and self-induction. We have then

$$P = I^2.R,$$

which may be written in the form

$$P = I.R.I.$$

It is evident from Fig. 244 that $I.R$ may be replaced by $V.\cos\phi$, giving us

$$P = V.\cos\phi.I.$$

This shows us clearly that, in calculating the actual power or output, we must only consider the component $V.\cos\phi$ of the total pressure V, that is, the component in phase with the current.

There is still another point of view from which we may consider this equation. We may look upon the power $P = V.I.\cos\phi$ as the product of the pressure V and the component $I.\cos\phi$ of the total current. This component is equal to the projection OC (Fig. 255) of the current vector I upon the pressure vector V. Hence the true power is found by multiplying the pressure by the component of the current in phase with it.

This component $OC = I.\cos\phi$ is called the watt-component of the current, or the energy-component, while the component OD at right angles to it is called the wattless current. Since a generating station works at a constant terminal pressure, this last method of imagining the current to be divided into two components is the more convenient, and is the one generally adopted, although it is sometimes more in accordance with the real nature of the case to resolve the pressure.

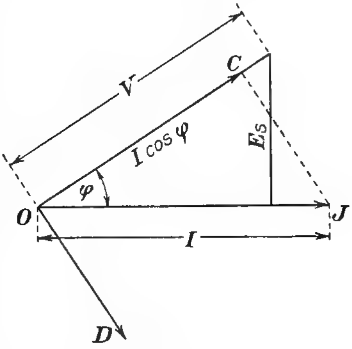

Fig. 255

We saw above that the lag of the current behind the pressure had the effect of unduly increasing the necessary E.M.F. for a given load and current. We might now express this in another way and say that the current must be very large for a given load and pressure, since only a component of the current enters into the calculation of the output. This increased current necessitates a greater cross-section of copper in generators, cables and motors, or else increased losses due to heating. Every effort is made therefore to reduce the self-induction of generators and motors to a minimum.

The effect of phase difference on the power or output of a generator is clearly illustrated by the following interesting experiment (Fig. 256). A variable non-inductive resistance is connected in series with a highly inductive coil across the terminals of an A.C. generator. The terminal P.D. of the generator is maintained constant. The output of the generator is measured by means of a wattmeter. When the rheostat handle is over to the right and the resistance consequently large, the current and therefore also the power will be small. As the rheostat handle is turned to the left, the resistance is decreased and the power increases as one would expect. Beyond a certain point, however, the wattmeter reading decreases in spite of the increased current caused by cutting out the resistance. The reason for this is evident, if we turn to Fig. 244 on page 251, in which the area of the triangle is equal to half the product of the base and height. The area is therefore equal to $\frac{1}{2}V.\cos\phi.\omega L.I$. Since, in our case, ωL is constant, the area is proportional to $V.I.\cos\phi$, and is therefore a measure of the output of the generator. With a constant hypotenuse V, the triangle has a maxi-

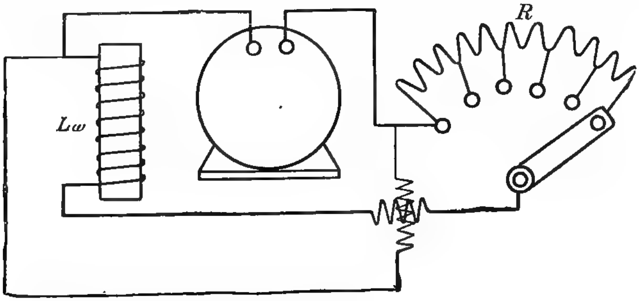

Fig. 256

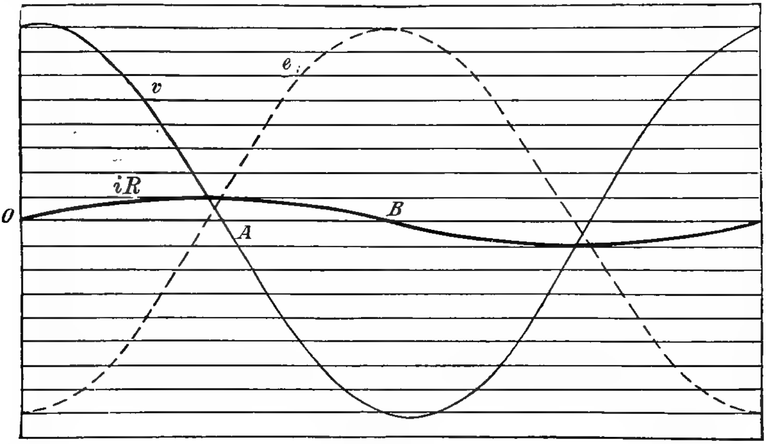

Fig. 257

mum area when the other two sides are equal, that is, when the resistance R is equal to the reactance ωL, and the angle of lag is 45°. The output is then a maximum for the given terminal pressure and inductance.

If the rheostat arm be put right over to the left, the current naturally increases still further, while the wattmeter reading falls almost to zero. This is explained by the fact that the resistance of the circuit is almost entirely inductive, so that the angle of lag, as found from the equation

$$\tan\phi = \frac{\omega L}{R},$$

is almost 90°. The power-factor $\cos\phi$ is therefore nearly 0, and the output consequently very small in spite of the large current. The latter is almost entirely wattless.

In order to show this in a graphical manner, we have drawn in Fig. 257 the sine-curves for a circuit containing much self-induction, but little ohmic resistance. The curve of ohmic pressure drop is so flat that it could almost be neglected in finding the curve of terminal pressure. The latter has little to do beyond counterbalancing the back E.M.F. of self-induction. The curve v is subsequently nearly 180° out of phase with the curve e_s of self-induction, and therefore almost 90° ahead of the current. By forming the products of the momentary values of pressure and current, we find that the power is positive from O to A, and negative from A to B. Since these two intervals are almost equal, the resultant mean power is very small. It would vanish entirely if the winding of the coil had no resistance and the iron were free from hysteresis.

It follows from these considerations that a choking coil can be used to cause a large drop of pressure without absorbing very much power. They have been largely employed as steadying resistances in series with A.C. arc lamps. Their employment is limited, however, owing to the disadvantage, pointed out above, of loading the station and mains with wattless current. If the pressure has to be reduced considerably, it is far preferable to use a small transformer instead of a choking coil.

A further use to which choking coils have been put is in connection with the series running of a number of glow lamps on high pressure mains. Each lamp has a small choking coil connected in parallel with it, through which the current passes when the lamp breaks, thus enabling the other lamps in series with it to continue burning. If the broken lamp were short-circuited the current through the whole circuit would be increased and the other lamps endangered. This does not happen if choking coils are used instead of short-circuiting devices. When the lamp is burning, the current divides between the lamp and the coil but on the lamp breaking, the whole current passes through the coil, with the result that the total current is slightly reduced. This reduction of current is hardly appreciable when the number of lamps in series is very large, and it is further limited by making the choking coils so as to have a variable coefficient of self-induction. When the lamp breaks, the whole current is forced to pass through the coil, and the flux is thereby increased. This increase in the magnetic flux causes an increase in the back E.M.F., and consequently a decrease in the current. If, however, the iron core of the choking coil is already highly saturated, a large increase in the current has but a small effect on the flux and therefore also on the back E.M.F.

Choking coils are sometimes used in connection with the parallel running of alternators. We saw in Section 35 that self-induction acts as a sort of electrical inertia, opposing sudden variations of the current. There is often a danger of heavy currents surging backwards and forwards between two or more alternators when connected in parallel (see Sections 104 and 106). These currents can be largely prevented by connecting choking coils between the generators and the common bus-bars at the switch-board. The advantage gained in this way will more than compensate for the small pressure drop in the coils. The power lost in such a coil is very small, as the current is nearly 90° out of phase with the terminal pressure of the coil.

Swinburne has suggested an arrangement whereby the pressure drop in the choking coils can be prevented. The iron core of the choking coil has two windings, one in series with each of the two generators. These are so connected that the ampere-turns are normally in opposition, and if the currents are equal no flux is produced. If, however, current passes from one generator to the other, the ampere-turns act in conjunction, the iron core is magnetised and the circulating current is limited by the inductive action.

80. Effect of Capacity.

A condenser consists essentially of two metal plates separated from each other by a thin sheet of insulating material. If the two plates are connected to the terminals of a battery or dynamo, the one plate becomes charged positively and the other negatively. Although this way of looking at the matter agrees with the general ideas of frictional electricity, it is hardly in accordance with the supposition of Maxwell that the electric current is due to the motion of positive electricity only. The two ideas can be brought into agreement, however, by assuming that the negative plate is charged by the flowing away of positive electricity instead of the flowing into it of negative electricity. We thus have a current round the circuit at the moment of charging; the current flows from the positive terminal of the generator to the condenser, and from the negative terminal of the condenser back to the generator. This current flows until the condenser is fully charged, that is, until the back E.M.F. of the condenser is equal to the P.D. applied to it. A certain quantity of electricity will have to flow into the positive plate (and out from the negative plate) in order to charge the condenser to the given back electromotive force E_c.

The effect of capacity in A.C. circuits is of great practical interest on account of electric cables acting as condensers. To simplify the problem as much as possible, we shall assume that the cable has neither ohmic resistance nor self-induction. For the instantaneous value of the terminal pressure of the machine we have

$$v = \hat{v} . \sin \alpha = \hat{v} . \sin (\omega t).$$

If this pressure increases by an amount dv in the succeeding interval dt, we have

$$dv = \hat{v} . \cos (\omega t) . \omega . dt.$$

The quantity of electricity dq which passes into the positive plate of the condenser during this interval is proportional to the increase of pressure dv and to the capacity of the condenser (see page 105). The capacity C must be expressed in units corresponding to the ampere and volt, i.e. in farads. We have then

$$dq = C.dv = C.\hat{v}.\cos(\omega t).\omega.dt.$$

Now, $\frac{dq}{dt}$ is the rate at which electricity flows into the condenser in coulombs per second, i.e. in amperes, during the short interval dt. It is the instantaneous value i of the condenser current, and can be expressed as follows:

$$i = \frac{dq}{dt} = C.\omega.\hat{v}.\cos\alpha.$$

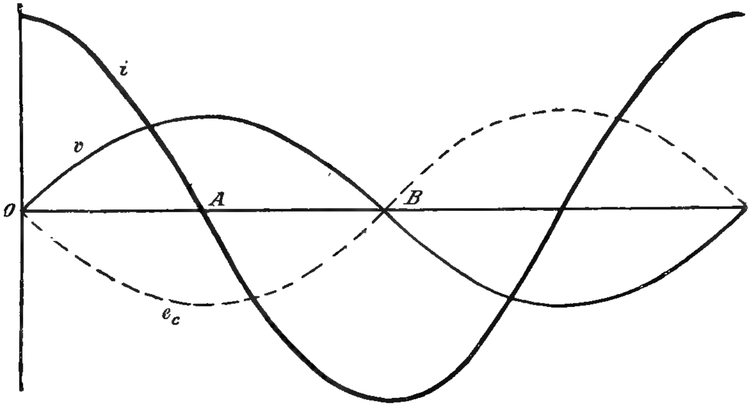

Fig. 258

Hence, the current is a cosine function of the angle α which the terminal pressure vector makes with its zero position. When $\alpha = 0$, $\cos\alpha = 1$, and the current has its maximum value

$$\hat{i} = C.\omega.\hat{v}.$$

By dividing both sides by $\sqrt{2}$ we obtain the effective value of the current, which is therefore given by the equation

$$I = C.\omega.V \quad \text{.............................(125).}$$

Since the terminal pressure of the machine and the back E.M.F. of the condenser are exactly equal and opposite to each other at every moment, they must be numerically equal, so that we may write the above equation in the following form:

$$E_c = \frac{I}{C.\omega} = \frac{I}{2\pi.f.C} \quad \text{.......................(126).}$$

We see, then, that there is another pressure in the circuit in addition to the generator pressure, viz. the back E.M.F. of the condenser, which has to be overcome by the generator. We must now determine the phase of this new electromotive force.

The thin full-line curve in Fig. 258 represents the terminal pressure of the generator. The abscissae represent the angular displacement of the vector v from its initial position. On the assumption that the circuit contains neither resistance nor self-induction, we saw above that the current is proportional to the cosine of the angle α. It will therefore have a positive maximum when $\alpha = 0$, and will vanish when $\alpha = 90°$. In this way we obtain the heavy line curve i for the current. Since we have seen that the back E.M.F. of the condenser is directly opposed to the terminal pressure, the dotted curve e_c must represent the former. The current curve i lags 90° behind the back-pressure curve e_c, but is 90° ahead of the terminal pressure curve v.

It is necessary, however, to check this mathematical result by considering the relation between current and pressure at several characteristic moments during the period. The curve v cuts the axis at the point O, and is steepest at this point. This is the moment at which the terminal pressure is changing most rapidly, and at which the quantity of electricity passing into the condenser per second has its maximum value. The current has therefore its maximum value. So long as the terminal pressure continues to increase, that is, during the interval OA, the current will have the same direction as the pressure, and the ordinates of both are positive.

When the terminal pressure reaches its maximum value at the moment A it is constant for a moment, and the current into the condenser ceases. When, however, the terminal pressure decreases, it is overcome by the back E.M.F. of the fully charged condenser, and a current is driven back through the generator in opposition to its terminal pressure. This continues during the quarter period AB, in which the ordinates of current and terminal pressure are of opposite sign. Hence, the relative positions of the curves in Fig. 258 are quite correct, and we see clearly why the current curve is 90° ahead of the curve of terminal pressure, or why the back E.M.F. of the condenser is 90° ahead of the current.

81. Resistance and Capacity in Series.

We assumed in the last section that no ohmic resistance was interposed between the generator and the condenser, so that the terminal pressure of the condenser was identical with that of the generator. If, however, a resistance R lies between the generator and the condenser, the terminal pressure v of the generator has to cover the drop of pressure $i.R$, in addition to overcoming the back E.M.F. of the condenser. These two components are shown in Fig. 259, in which the thick black line represents the pressure drop $i.R$, while the dotted line represents the back E.M.F. of the condenser leading 90° ahead of $i.R$.

At the moment O there is no back E.M.F., and the terminal pressure v has only to supply the drop in the resistance. Its momentary value will therefore be OF.

At the moment A there is still a considerable ohmic drop and, in addition thereto, a back E.M.F. to overcome. The momentary value of the terminal pressure must therefore be GA.

At the moment B the current is passing through zero, and there is consequently no ohmic drop; the terminal pressure HB has only to counterbalance the equal and opposite back E.M.F. of the condenser.

Finally, at the moment *C* the ohmic pressure drop is exactly covered by the back E.M.F. of the condenser, which is now acting with the current. The terminal pressure of the generator is therefore equal to 0.

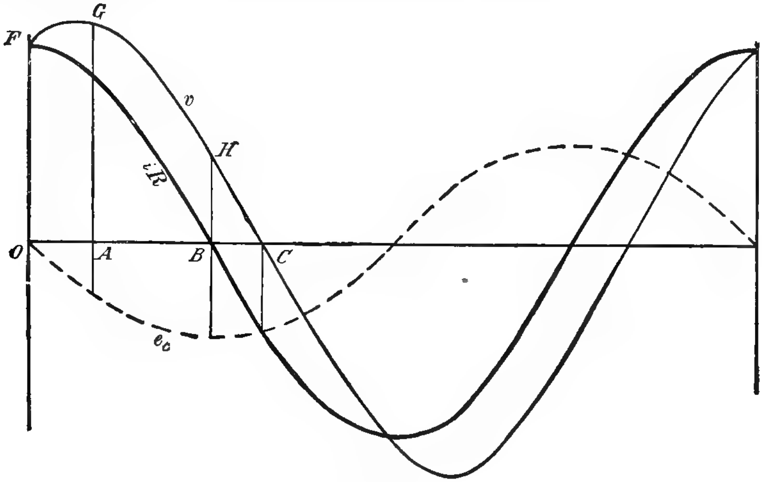

Fig. 259

When the curve *v* has been drawn in this manner, we find that it lags behind the current curve. In Fig. 259 the current is ahead of the terminal pressure by an angle represented by *BC*. The effect of capacity is thus the opposite of that of self-induction, in that it causes the current to lead ahead of the pressure, whereas self-induction causes the current to lag.

82. Circuit containing Resistance, Inductance and Capacity.

We must now consider the case in which ohmic resistance, self-induction and capacity are all connected in series. At the moment represented in Fig. 260 the current *I* is passing through its zero value, and the vector *I.R* is therefore horizontal. The vector E_s of the back E.M.F. due to self-induction lags 90° behind the current vector, and is therefore vertically downwards. This vector must be counterbalanced by an equal and opposite component $-E_s$ of the terminal pressure of the generator. From *I.R* and $-E_s$ we get the resultant *OA*. There is another E.M.F. acting in the circuit, however, viz. the back E.M.F. E_c of the condenser, which is 90° ahead of the current, and is therefore represented by a vector vertically upwards from *O*. This must be counterbalanced by an equal and opposite component of the terminal pressure. The resultant *OC* of the vectors *OA* and $-E_c$ is the required vector

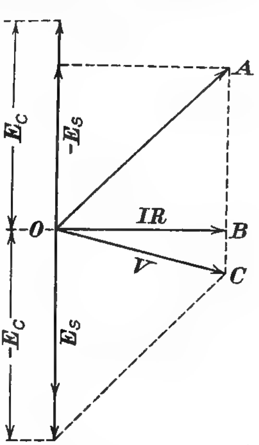

Fig. 260

of the terminal pressure V. Under certain conditions the terminal pressure V may be much smaller than either the pressure E_s due to the self-induction, or the pressure E_c across the condenser.

By means of the diagram in Fig. 260 we are now able to calculate the current for any given values of the terminal pressure, ohmic resistance, self-induction and capacity. From the figure, we have

$$BC = E_c - E_s,$$

so that, for the right-angled triangle OBC we have

$$V^2 = (I.R)^2 + (E_c - E_s)^2.$$

From equation (126) on page 263, we have

$$E_c = \frac{I}{\omega C},$$

and from page 245

$$E_s = \omega L.I.$$

By substituting these values in the above equation, we get

$$V^2 = (I.R)^2 + \left(\frac{I}{\omega C} - \omega L.I\right)^2,$$

which is equivalent to

$$I = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \quad \text{......................(127).}$$

This may be called the general expression of Ohm's law for an alternating current circuit, since the denominator is the apparent resistance or impedance of the circuit. It is represented by the hypotenuse of a right-angled triangle, one side of which is equal to the ohmic resistance, while the other is equal to the difference $\frac{1}{\omega C} - \omega L$. We must not forget that C is to be expressed in farads, so that the value of the capacity in microfarads must be multiplied by 10^{-6} before it is introduced into the above equation.

It is a point of great importance in the above equation that the angular velocity ω occurs in the numerator of one of the two terms within the brackets, and in the denominator of the other. This indicates that there is a value of the frequency for which the two terms will be equal and their difference consequently vanish. To find this frequency we simply put

$$\frac{1}{\omega C} - \omega L = 0,$$

or

$$\omega = \frac{1}{\sqrt{C.L}} \quad \text{..............................(128).}$$

The frequency at which the difference $\frac{1}{\omega C} - \omega L$ vanishes is therefore

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{C.L}} \quad \text{........................(129).}$$

For this value of the frequency, equation (127), which we have called the general expression of Ohm's law for an A.C. circuit, becomes simplified to the following form:

$$I = \frac{V}{R}.$$

The current reaches its maximum value at this frequency, and is as great as if both self-induction and capacity were absent, and the pressure V were applied directly to the resistance R.

Let us consider as an example an alternating current generator with an E.M.F. of 10,000 volts and a coefficient of self-induction L of 0·4 henry. We shall assume that it is connected to a length of cable the far end of which is open, and which has a total resistance of 5 ohms for the double length. The two cores of the cable act as the plates of a condenser, and have a capacity of 2·5 microfarads. A condenser with this capacity may be assumed to replace approximately the distributed capacity of the cable, if connected between the cores halfway between the generator and the free end of the cable. The resistance in series with this condenser will be half that of the whole cable, viz. 2·5 ohms. On this assumption the capacity current at a frequency of 50 can be found from equation (127):

$$I = \frac{10{,}000}{\sqrt{2{\cdot}5^2 + \left(\frac{10^6}{6{\cdot}28 \,.\, 50 \,.\, 2{\cdot}5} - 6{\cdot}28 \,.\, 50 \,.\, 0{\cdot}4\right)^2}} = 8{\cdot}7.$$

This current flows in the cable in spite of the fact that it is open at the further end. The current leads nearly 90° ahead of the pressure, so that $\cos\phi$ is practically equal to 0 and the power is negligibly small.

It is interesting to determine the frequency at which the current in this example would reach its maximum value. From equation (129) we find the critical frequency, as follows:

$$f = \frac{1}{2\pi\sqrt{C \,.\, L}} = \frac{1}{2\pi\sqrt{2{\cdot}5 \,.\, 10^{-6} \,.\, 0{\cdot}4}} = 160.$$

At this frequency the denominator in equation (127) becomes equal to R, and, although the further end of the cable is open, the current reaches an enormous value, viz.

$$I = \frac{V}{R} = \frac{10{,}000}{2{\cdot}5} = 4000 \text{ amperes.}$$

The P.D. across the condenser is then equal to that across the self-induction, and for both we find the following incredible value:

$$\frac{I}{\omega C} = \omega L \,.\, I = 2\pi \,.\, 160 \,.\, 0{\cdot}4 \,.\, 4000 = 1{\cdot}62 \,.\, 10^6 \text{ volts.}$$

The cross-section of the conductor is naturally unable to stand the large current, and the insulation would be broken down by the enormous pressure. This dangerous condition which arises at the critical frequency is known as **resonance.** The above example is exaggerated on account of the value chosen for the self-induction of the generator being much larger than the actual value for a practical alternating current generator. For smaller values of L the critical frequency is higher than 160. The two characteristic peculiarities of resonance are large currents and enormous pressures, in spite of open circuits and normally excited machines.

83. Self-induction and Capacity in Parallel.

In Fig. 261 is shown a circuit containing two parallel branches, one of which consists of a choking coil in series with ohmic resistance, while the other contains a condenser, also in series with ohmic resistance. The terminal pressure V is common to both branches. Its vector must consequently be the hypotenuse of two right-angled triangles, one of which has sides equal to $I_1 . R_1$ and $\omega L . I_1$, while the sides of the other are equal to $I_2 . R_2$ and $\frac{I_2}{\omega C}$ (Fig. 262). The current I_1 must lag behind the pressure V by an angle ϕ_1, which is given by the equation

$$\tan \phi_1 = \frac{\omega L_1}{R_1}.$$

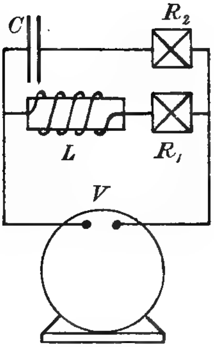

Fig. 261

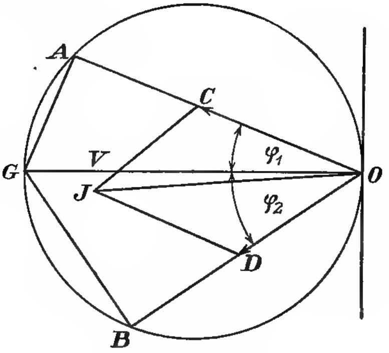

Fig. 262

The current I_2, on the other hand, leads ahead of the terminal pressure by the angle ϕ_2, where

$$\tan \phi_2 = \frac{E_c}{I_2 . R_2} = \frac{1}{\omega C . R_2}.$$

A circle is drawn on $OG = V$ as diameter, and a chord OA drawn so as to make an angle ϕ_1 with the diameter. OA represents the drop in the lower branch due to ohmic resistance, while AG is equal to the inductive drop. The chord OB is drawn so as to make an angle ϕ_2 with OG. OB represents the ohmic pressure drop in the upper branch, and BG represents the pressure lost in counterbalancing the back E.M.F. of the condenser. The angles ϕ_1 and ϕ_2 must naturally lie on opposite sides of the diameter. By dividing the chord OA by the resistance R_1, we get

$$I_1 = \frac{OA}{R_1} = OC.$$

The current I_2 is obtained in a similar manner by dividing the chord OB by the resistance R_2, thus

$$I_2 = \frac{OB}{R_2} = OD.$$

The resultant OJ is found from the components $I_1 = OC$ and $I_2 = OD$ by the parallelogram of forces. The vector OJ represents the resultant current both in magnitude and phase. It is noteworthy that its displacement from the terminal pressure vector is very small. It is evident that by a suitable choice of capacity the displacement between resultant current and terminal pressure could be altogether prevented. Condensers have been employed to a small extent for this very desirable purpose. In Section 110, however, it will be shown that an over-excited synchronous motor, the back E.M.F. of which exceeds the terminal pressure, resembles a condenser, in that it takes a leading current from the mains. Such motors have therefore been used to reduce the displacement between the current and pressure vectors caused by self-induction.

CHAPTER XI

84. The electromotive forces induced in a transformer.—85. The magnetising current.—86. The hysteresis current.—87. Transformer on non-inductive load.—88. Transformer on inductive load.—89. The effect of magnetic leakage.

84. The Electromotive Forces induced in a Transformer.

The principle of the transformer has already been explained in Section 34. The magnetic flux produced by the alternating current alternately grows and dies away, first in the one direction and then in the other. In so doing, it cuts through the turns of two separate windings on the core, and induces in them electromotive forces proportional to the number of turns in each. We shall consider, in the first place, the conditions existing in the primary winding when the transformer is on open circuit, that is, when no current is taken from the secondary winding. The primary winding is then nothing more than a choking coil, in which the flux produced by the alternating current induces a back E.M.F. of self-induction. We shall not, however, make use of the idea of self-induction, but shall speak of the induced or back electromotive force. If Φ be the highest value reached by the flux linking the winding, f the frequency, and T_1 the number of turns in the primary winding, then the primary E.M.F. can be found from equation (119) on p. 246, as follows:

$$E_1 = 4{\cdot}44 . \Phi . f . T_1 . 10^{-8} \quad \ldots\ldots\ldots\ldots\ldots\ldots (130).$$

The external P.D. applied to the primary terminals has, as we have already seen in the fundamental diagram on page 251, a double function to perform, viz. to overcome this back E.M.F. and to drive the current through the ohmic resistance of the winding. The resistance of transformer windings is generally so small that the latter component of the terminal pressure can be neglected. The induced electromotive force E_1 is therefore almost exactly equal and opposite to the terminal pressure V_1, and only allows that current to flow through the winding, which is necessary to produce the magnetic flux. Since the magnetic circuit of a modern transformer consists entirely of iron and has a low reluctance, a very small current suffices to produce the necessary flux, so that the no-load current is always very small.

Whether the secondary circuit be open or closed, the secondary winding is cut by the alternating flux, which induces in it an E.M.F. E_2 in phase with the primary E.M.F. If the number of turns in the secondary winding be T_2, we have, from equation (119) on page 246,

$$E_2 = 4{\cdot}44 . \Phi . f . T_2 . 10^{-8} \quad \ldots\ldots\ldots\ldots\ldots\ldots (131).$$

The electromotive forces are thus proportional to the number of turns in the windings. At no-load there is no secondary current, and the electromotive force E_2 is equal to the secondary terminal pressure V_2. Neglecting the primary copper drop, we have

$$\frac{V_2}{V_1}=\frac{T_2}{T_1} \quad \text{..............................(132).}$$

Hence, at no-load the terminal pressures are proportional to the number of turns in the windings.

It is assumed here that the two windings retain the same relative positions. If, however, one winding be wound on an iron ring as in the stator of an induction motor, whilst the other is wound on the iron drum which constitutes the rotor of the motor, the ratio of the primary to the secondary voltage can be varied by turning the drum. When the axes of the coils coincide the secondary voltage is a maximum, and when they are at 90°, the secondary voltage is zero. Such induction regulators, as they are called, have been used for starting and regulating single phase railway motors.

85. The Magnetising Current.

Since the unloaded transformer is equivalent to a choking coil, the magnetising current is given by equation (120) on page 246, viz.

$$I_m=\frac{\mathbf{B}\,.\,l}{1{\cdot}78\,.\,T_1\,.\,\mu}\,.$$

B is the highest value reached by the number of lines per sq. cm, l is the length of the magnetic path, and μ is the permeability of the iron. The value of μ is not quite constant during the period, but since, in order to reduce hysteresis loss and magnetising current, transformers are always worked at low flux-densities, i.e. on the first part of the magnetisation curve, the value of μ varies very little.

Instead of taking the value of μ from magnetisation curves found by the ballistic method, we can determine it by actually measuring the magnetising current taken by a transformer or annular ring built up of the same iron. The maximum flux Φ can be found by equation (130) on p. 270, and the flux-density **B** is obtained by dividing Φ by the cross-section of the iron. The permeability can then be found from the following formula:

$$\mu=\frac{\mathbf{B}\,.\,l}{1{\cdot}78\,.\,I_m\,.\,T_1} \quad \text{.........................(133).}$$

The value of μ found in this way is, of course, the correct one to employ in the transformer calculation. It is not, however, exactly the same as the value of μ corresponding to **B** in the magnetisation curve, but is a sort of mean value which can hardly be defined, except by means of equation (133).

The curve μ in Fig. 263 has been determined in this manner. It is taken from Kapp's *Dynamo Construction*, and gives what we may call the mean permeability for the values of **B** shown on the abscissa axis. As one would expect from the shape of the magnetisation curve in Fig. 56 on p. 63, the permeability increases up to a certain value of **B**, beyond which it decreases. The highest permeability has hardly been reached in the figure, which only goes up to a flux-density of 7000. It is easy to see from Fig. 56 that the permeability B/H would reach its maximum at the point where the tangent to the curve would pass through the origin.

We shall now determine the phase of the various pressures and no-load current by means of a vector diagram (Fig. 264). The induced electromotive forces lag 90° behind the magnetising current I_m. If the vector I_m is drawn horizontally to the right, the vectors representing E_1 and E_2 must be drawn vertically downwards. Since the electromotive forces are proportional to the number of turns, we have

$$E_1 = E_2 \cdot \frac{T_1}{T_2}.$$

Hence, the vector of the primary electromotive force represents to a certain scale the secondary electromotive force. This greatly simplifies the diagram. All secondary electromotive forces are therefore multiplied by $\frac{T_1}{T_2}$ to reduce them to equivalent primary forces, which are then drawn on the diagram.

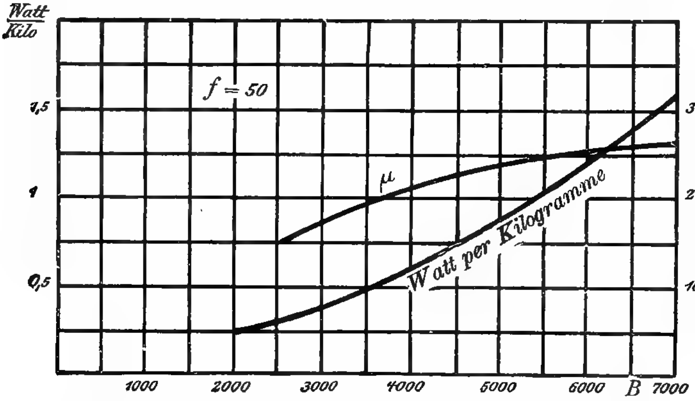

Fig. 263

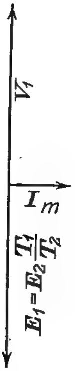

Fig. 264

We have seen that the primary terminal pressure V_1 is almost exactly equal and opposite to the primary electromotive force E_1. The V_1 vector will therefore be vertically upwards. The angle between the terminal pressure V_1 and the primary current I_m is 90°, so that no power is taken by the open-circuited transformer. We have, however, not only neglected the power lost in heating the copper, which is very small, but we have also neglected the power absorbed in continually reversing the magnetic field in the iron. This latter loss is by no means negligible, and we must now investigate this effect of the so-called hysteresis of the iron.

86. The Hysteresis Current.

If the power supplied to a transformer on open circuit be measured by means of a wattmeter, it is found to be by no means negligible. This power is principally absorbed by the hysteresis of the iron. If P_h is the power measured on the wattmeter, I_0 the primary current, and V_1 the primary terminal pressure, we have

$$\cos\phi = \frac{P_h}{V_1 . I_0}.$$

The primary no-load current lags therefore behind the terminal pressure by an angle ϕ, while the primary back E.M.F. is, as before, exactly equal and opposite to V_1. Now, the back E.M.F. must lag 90° behind the magnetic flux, by the variation of which it is produced. When, for example, the E.M.F. is a maximum, as shown in the figure, the magnetic flux must be passing through its zero value, since this is the moment when the steepness of the sine curve or rate of change of the flux is at its maximum. The vector of the flux-density B must therefore be drawn horizontally to the right. Hence, we are led to the conclusion that the magnetic flux-density B lags behind the no-load primary current I_0 by an angle β.

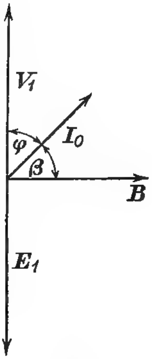

Fig. 265

In order to make this clearer, a hysteresis loop is drawn in Fig. 266, in which the instantaneous values of the no-load current are plotted as abscissae and the values of the flux-density B as ordinates. If the applied terminal pressure V_1 is a sine function of the time, the induced E.M.F. must also be a sine function. The curve representing the flux-density is drawn on the right, and since it is merely displaced 90° from the induced E.M.F. it is also a sine curve. Its maximum value is the same as that of the hysteresis loop on the left of the figure.

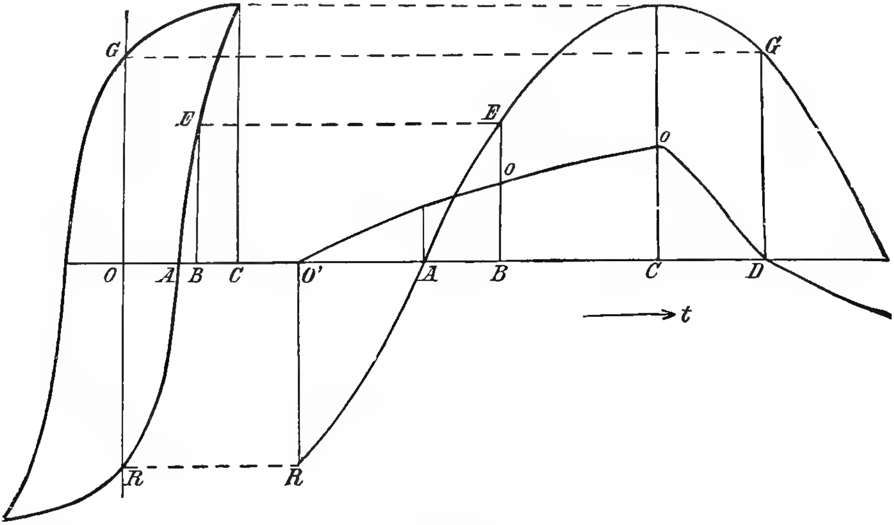

Fig. 266

A point is found in the sine curve having an ordinate equal to the remanent magnetism OR. This point corresponds to the moment O at which the current is consequently zero.

At the moment A, on the other hand, the flux-density has fallen to zero, and we must measure off the corresponding current OA from the loop and set it up as the ordinate at the point A.

At the moment B, the flux-density is seen from the right-hand figure to be EB, which corresponds on the loop to a current OB. The latter is therefore set up as the ordinate at the point B.

Both the flux-density and the current reach their maximum values at the moment C. The point D at which the curve cuts the base line is determined by finding the point for which the ordinate DG is equal to the remanent flux-density OG in the hysteresis loop.

It is evident from Fig. 266 that the current is out of phase with the magnetic flux. The current curve, moreover, is distorted, so that, although both curves reach their maximum values at the same moment, they pass through their zero values at different times. We are met with the difficulty that our rotating vectors represent true sine waves, so that we cannot exactly represent the no-load current by a vector. For practical purposes, however, the distorted current wave can be replaced by an equivalent sine wave, i.e. one with the same effective value I_0. The flux wave will lag behind this sine wave of current by an angle ϕ, which, however, will be less than the angle represented by $O'A$ in Fig. 266.

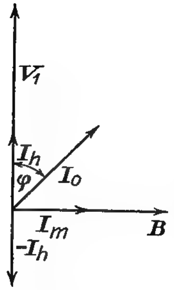

Fig. 267

In Fig. 267 the current I_0 is resolved into two components, one, I_h, in the direction of the terminal pressure V_1, the other, I_m, in the direction of the induction. The former is the hysteresis current, while the latter is the wattless magnetising current.

We see from the figure that

$$I_h = I_0 . \cos \phi.$$

Now, the power absorbed at no-load by hysteresis, viz. P_h, is equal to $V_1 . I_0 . \cos \phi$, or, if the copper drop is not neglected, to $E_1 . I_0 . \cos \phi$. We have then

$$P_h = E_1 . I_0 . \cos \phi,$$

or

$$I_h = \frac{P_h}{E_1} \qquad \text{.................................(134).}$$

We then find I_m from the equation

$$I_m = \sqrt{I_0^2 - I_h^2} \qquad \text{..........................(135).}$$

The physical meaning of this is made plainer, perhaps, by drawing the vector $- I_h$ in Fig. 267 equal and opposite to I_h. Then the theoretical magnetising current I_m is the resultant of the total current I_0 and the current $- I_h$, which may be called the demagnetising effect of hysteresis. Hence, we may consider the current I_0 to be made up of two components, viz. the true magnetising current I_m and the component I_h, which counteracts the demagnetising effect due to hysteresis. If hysteresis could be eliminated the no-load current would be identical with the wattless magnetising current I_m, although the curve would still be distorted from the sine form owing to varying permeability.

We shall now apply the above results to an actual example. The following data apply to a 36-kilowatt transformer, described by Kapp:

Cross-section of iron	$A = 900$ sq. cm,
Length of path in iron ...	$l = 100$ cm,
Number of primary turns ...	$T_1 = 315$,
Frequency	$f = 50$,
Primary terminal pressure ...	$V_1 = 2000$ volts,
Current at no-load	$I_0 = 0{\cdot}36$ ampere,
Power taken at no-load ...	$P_h = 400$ watts,
Weight of iron	$= 765$ kilogrammes.

We are required to determine the permeability μ of the iron. From equation (134) we have

$$I_h = \frac{P_h}{V_1} = \frac{400}{2000} = 0{\cdot}2 \text{ ampere,}$$

and from equation (135), for the wattless magnetising current,

$$I_m = \sqrt{I_0^2 - I_h^2} = \sqrt{0{\cdot}36^2 - 0{\cdot}2^2} = 0{\cdot}3 \text{ ampere.}$$

Before we can determine the permeability we must calculate the maximum flux-density. This can be obtained by means of equation (130) on page 270, thus

$$\mathbf{B} = \frac{\Phi}{A} = \frac{V_1 \,.\, 10^8}{4{\cdot}44 \,.\, f \,.\, T_1 \,.\, A} = 3180.$$

The permeability can now be found from equation (133) on page 271,

$$\mu = \frac{\mathbf{B} \,.\, l}{1{\cdot}78 \,.\, I_m \,.\, T_1} = \frac{3180 \,.\, 100}{1{\cdot}78 \,.\, 0{\cdot}3 \,.\, 315} = 1900.$$

This is approximately the value given by Fig. 263 for a flux-density of 3180. The power absorbed per kilog. at a frequency of 50 works out to $\frac{400}{765} = 0{\cdot}52$ watt. The curve of iron loss per kilogramme in Fig. 263 was experimentally determined in this manner. This curve gives a somewhat lower value than that just found, viz. 0·42 watt per kilogramme at 50 cycles per second. When accurate curves are available, giving the iron loss and permeability for the material to be used, the losses and no-load current of a transformer can be accurately calculated. This is done by simply working backwards through the calculation by which the curves were obtained from the experiment. It should be borne in mind that the experimentally determined iron losses are not entirely due to hysteresis, but contain the eddy current losses in the iron.

87. Transformer on Non-inductive Load.

To any one who has clear ideas on self-induction and of the action of a choking coil, the behaviour of a transformer on open circuit, as dealt with in Section 84, will have presented no difficulties. A clear understanding of the action of a loaded transformer is, however, much more difficult to obtain. It is not easy to see that any current we choose can be taken from the secondary winding, and that the primary current will automatically adjust itself to a corresponding value. Whereas on open circuit the back E.M.F. prevented all but a very small current from flowing through the primary winding, this current grows immediately on current being taken from the secondary winding. We naturally ask the reason for this sudden increase of primary current, or apparent decrease of self-induction of the transformer. The difficulty is due, to some extent, to the lack of visible connection between the two windings which are so completely insulated from each other.

The relation between the primary and secondary currents can be easily determined from a consideration of the conservation of energy. The supply of energy to the primary must be equal to the sum of the energy given out from the secondary and the losses within the transformer. If R_1 be the resistance of the primary winding and R_2 that of the secondary, we can write down the following equation for non-inductive loads, neglecting the small displacement caused by the wattless magnetising current I_m:

$$V_1.I_1 = V_2.I_2 + I_1^2.R_1 + I_2^2.R_2 + P_h.$$

Now, the losses are very small, even when compared with a fraction of the full load. We shall therefore neglect them, and write

$$V_1.I_1 = V_2.I_2,$$

or

$$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{T_2}{T_1} \quad \text{..............................(136).}$$

Hence, the currents are inversely proportional to the pressures or inversely proportional to the number of turns in the respective windings.

Now, although we have obtained this result from a consideration of the principle of the conservation of energy, which we know from experience can be applied with perfect confidence, yet the difficulty mentioned above will not have been removed. It will remain with us until we study the magnetic effect of each winding and draw a diagram showing the magnetomotive forces.

With a non-inductive load, the secondary current is in phase with the induced E.M.F. The primary current, on the other hand, is practically in phase with the primary terminal pressure, which is almost exactly opposed to the induced E.M.F. It is therefore obvious that the primary and secondary currents are nearly 180° out of phase, and consequently neutralise each other in their magnetic effects. The apparent decrease in the self-induction as the load is increased can therefore be explained by the neutralisation of the larger part of the primary ampere-turns by the opposing ampere-turns of the secondary.

We have already seen that, if we neglect the ohmic drop in the primary winding, the electromotive force E_1 must be numerically equal to the constant primary terminal pressure. For the maximum value of the total flux we have, then,

$$\Phi = \frac{V_1 \cdot 10^8}{4{\cdot}44 \cdot f \cdot T_1}.$$

Since everything on the right-hand side of the equation is constant, the maximum value of the flux must also be constant, quite irrespective of the load on the transformer. If an unloaded transformer be suddenly loaded, the sudden rush of secondary current exerts a powerful demagnetising effect. The magnetic flux and back E.M.F. are momentarily decreased, but the primary current instantly rises to such a value that the demagnetising effect is neutralised, and the magnetic flux maintained at its no-load value. In other words, the combined magnetic effect of the primary ampere-turns I_1T_1 and the secondary ampere-turns I_2T_2 is equal to the effect of the no-load ampere-turns I_0T_1. The vector I_0T_1 in the diagram is the resultant of the vectors I_1T_1 and I_2T_2.

If the no-load ampere-turns I_0T_1 are very small compared with the ampere-turns when loaded, I_1T_1 is approximately equal to I_2T_2.

This is also true in the case of the series transformer (Fig. 268) in which the primary winding is inserted in one of the leads connecting the alternator with the external circuit. Its secondary winding can be closed through an adjustable resistance R_2. When the secondary circuit is open, the primary winding acts as a choking coil, but if the number of primary turns is small, the potential drop will be small compared with the terminal P.D. of the alternator; it will, however, have its maximum value under this condition. On closing the secondary circuit through a large resistance as shown in the figure, current flows in this circuit, which will be unaltered on reducing R_2. Assuming a constant primary current, the secondary current is also constant, being dependent namely on the transformation ratio. As R_2 is decreased, the secondary terminal P.D., which is equal to I_2R_2, decreases, and consequently also the primary P.D. Hence the primary acts as a choking coil of adjustable inductance; the P.D. across it will be zero when the secondary winding is short-circuited.

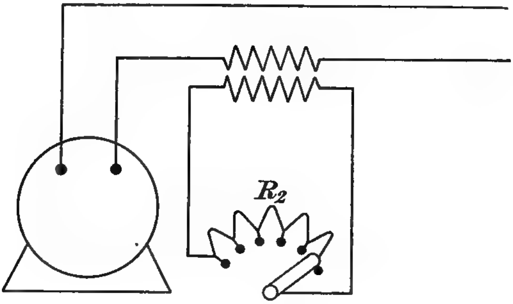

Fig. 268

We shall now consider the vector diagram of a transformer on non-inductive load, taking into account the no-load current, but neglecting in the first diagram the resistance of the windings.

Before making the diagram, however, we divide all the ampere-turns by the number of primary turns T_1. The vectors then represent currents reduced to the primary winding. The diagram of currents in Fig. 269 is obtained in this way, and the vector I_0 is the resultant of the vectors I_1 and $\frac{I_2 T_2}{T_1}$.

To construct the diagram we must first draw the triangle OAB of no-load current from the three vectors I_0, I_h and I_m. If the vector I_m is horizontal, the vector of electromotive force $E_1 = E_2 \cdot \frac{T_1}{T_2}$ will be vertically

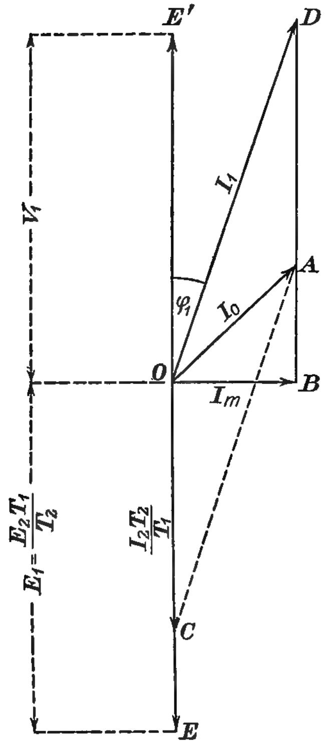

Fig. 269

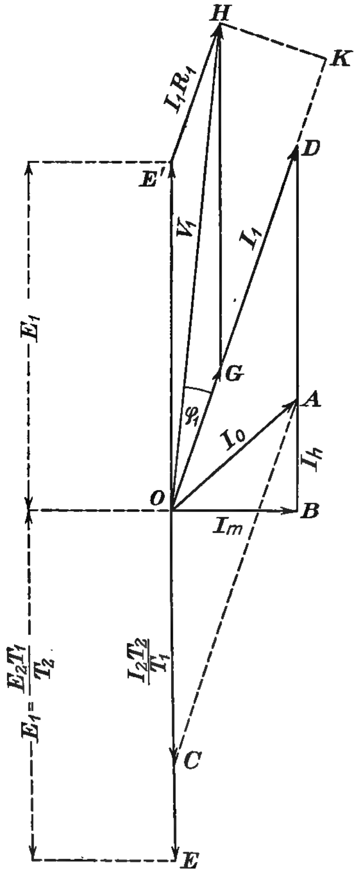

Fig. 270

downwards, since the induced E.M.F. lags 90° behind the magnetic flux. In Fig. 269

$$OE = E_1 = E_2 \cdot \frac{T_1}{T_2}.$$

Since the load is non-inductive, the secondary vector must be in the same direction as the induced E.M.F. As the diagram is derived from the ampere-turns, the actual secondary current must be reduced to an equivalent current in the primary winding, so that

$$OC = \frac{I_2 . T_2}{T_1}.$$

Since the no-load current I_0 is the resultant of the reduced secondary current and the primary current, the latter is easily determined by completing the parallelogram $OCAD$. We find then that

$$OD = I_1.$$

This primary current vector lags behind the primary terminal pressure by an angle ϕ_1, which is exaggerated in Fig. 269 for the sake of clearness, but which, in reality, is negligibly small. A transformer on a non-inductive load is therefore equivalent to a non-inductive load in the primary circuit.

We must now investigate the effect of the primary pressure drop I_1R_1, which we have so far neglected. The vector OG is drawn along the primary current vector, and made equal to $I_1 . R_1$ (Fig. 270). OE' is made equal and opposite to OE. The resultant OH of the vectors OG and OE' represents the terminal pressure V_1 both in magnitude and phase.

The correctness of the diagram can be checked by means of the principle of the conservation of energy. Putting $\phi_1 = \delta - \alpha$ where $\delta = DOE'$ and $\alpha = HOE'$ (Fig. 270), the power supplied to the primary may be expressed as follows:

$$P_1 = V_1 I_1 \cos\phi_1 = V_1 I_1 (\cos\delta . \cos\alpha + \sin\delta . \sin\alpha) \quad \ldots\ldots(a).$$

By projecting the vectors V_1, I_1R_1 and I_1 on to the vertical, it can be seen that

$$I_1 \cos\delta = I_2 \frac{T_2}{T_1} + I_h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b),$$

$$V_1 \cos\alpha - I_1 R_1 \cos\delta = E_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c),$$

$$V_1 \sin\alpha = I_1 R_1 \sin\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(d).$$

By multiplying together equations (*b*) and (*c*) and substituting the result in (*a*) together with the value of $V_1 \sin\alpha$ obtained from (*d*), the following result is obtained:

$$P_1 = E_2 I_2 + E_1 I_h + I_1^2 R_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(137).$$

The product $E_2 . I_2$ represents the total secondary power, both internal and external. The product $I_1^2 . R_1$ is the power lost as heat in the primary winding, while $E_1 . I_h$ represents the iron loss. The total supply of power is thus equal to the sum of the secondary output and the transformer losses.

It is important to notice that the induced back E.M.F. is affected by the ohmic pressure drop in the primary winding. If the applied primary pressure V_1 is maintained constant, the magnetic flux decreases as the load comes on. The no-load test should therefore be made at a pressure equal to the induced E.M.F. when loaded. The flux will then be the same both on open circuit and on load, and the no-load triangle OAB will be exactly correct at that load, but not at any other load. The difference is so small in actual practice, however, that this refinement may be safely neglected.

We turn now to the efficiency of the transformer on non-inductive load. If the secondary winding have a resistance of R_2, the secondary terminal pressure V_2 will be less than the induced E.M.F. E_2 by an amount $I_2.R_2$. The secondary copper loss will be $I_2^2.R_2$, and the real output $V_2.I_2$. The efficiency on non-inductive load is therefore,

$$\eta = \frac{V_2 . I_2}{V_1 . I_1 . \cos\phi_1} = \frac{V_2 . I_2}{V_2 . I_2 + I_1^2 . R_1 + I_2^2 . R_2 + P_h} \quad \ldots\ldots(138).$$

As this equation shows, the efficiency is very bad on small loads, on account of the iron loss being relatively so large. With increasing load the iron loss becomes of less relative importance, and as the copper losses are small compared with the load when the latter is not too large, the efficiency rises rapidly with the load. It reaches a maximum when the variable copper losses are together equal to the constant iron loss. If the load be increased beyond this point, the copper losses, which increase as the square of the current, become so important that the efficiency decreases.

We pass now to a consideration of the most efficient distribution of losses between the windings for a given winding space.

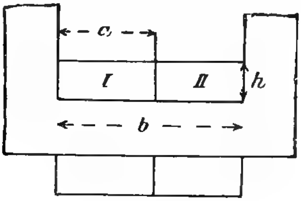

Fig. 271

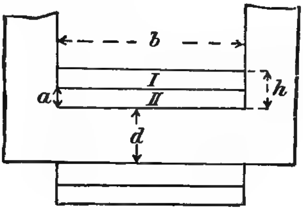

Fig. 272

Since $I_2 = I_1 \frac{T_1}{T_2}$, the total loss in the two windings is given by the following formula:

$$P_l = I_1^2 . R_1 + I_2^2 . R_2 = I_1^2 \left(R_1 + R_2 . \frac{T_1^2}{T_2^2}\right).$$

If l_1 and l_2 be the mean lengths of a turn in the two windings, we have

$$R_1 = \rho . \frac{l_1 . T_1}{A_1} \text{ and } R_2 = \rho . \frac{l_2 . T_2}{A_2}.$$

By substituting these values in the above formula, we get

$$P_l = I_1^2 . T_1^2 . \rho . \left(\frac{l_1}{T_1 . A_1} + \frac{l_2}{T_2 . A_2}\right) \ldots\ldots\ldots\ldots\ldots\ldots(a),$$

where A_1 and A_2 are the cross-sections of the wire with which the transformer is wound.

With the flat coil winding (Fig. 271) the turns of each winding have the same mean length, so that

$$l_1 = l_2 = l.$$

If we assume that the percentage space lost by insulation is the same in both coils, we may, if we omit a constant factor, say that

$$T_1.A_1 = a.h, \quad T_2.A_2 = (b - a).h.$$

The combined copper losses are therefore given by the formula

$$P_l = \frac{I_1^2 . T_1^2 . \rho . l}{h} . \left(\frac{1}{a} + \frac{1}{b - a}\right) \ldots\ldots\ldots\ldots\ldots\ldots(b).$$

To find the minimum value of this expression, we must differentiate it with regard to a and equate the result to 0. This gives us

$$-\frac{1}{a^2}+\frac{1}{(b-a)^2}=0,$$

or

$$a=\frac{b}{2}.$$

That is to say, the most efficient distribution of space is an equal distribution. The cross-section of the wire will then be inversely proportional to the number of turns, and directly proportional to the current. The current density will be the same in each winding. If we put $a=\frac{b}{2}$ in equation (b), we see that the losses are the same in each winding.

If the winding consists of cylindrical coils, as shown in Fig. 272, we have

$$l_2 = (d+a)\,\pi,$$

$$l_1 = [d+2h-(h-a)].\pi = (d+h+a)\,\pi,$$

$$T_1.A_1 = b\,(h-a), \quad T_2.A_2 = b.a.$$

Equation (a) will then take the following form,

$$P_l=\frac{I_1^2.T_1^2\rho\pi}{b}.\left[\frac{d+h+a}{(h-a)}+\frac{d\times a}{a}\right].$$

By differentiating with regard to a and equating the result to 0, we get

$$a^2+a.d-\frac{h.d}{2}=0.$$

This is the same equation as we obtain if we equate the two terms in the bracket in the last equation for P_l. The efficiency for a given winding space is therefore, as before, a maximum when the copper losses are equally distributed. The current density will not be the same in each winding. These calculations, however, are only rough approximations, since the relative loss of space due to insulation will be much greater in the high pressure than in the low pressure winding. A further inaccuracy is caused by the necessary space between the two windings in Fig. 272.

As an example, we shall consider a transformer with a primary terminal pressure of 2000 volts, taking a current of 20 amperes, and having a ratio of 10 to 1, so that the secondary current is 200 amperes and the secondary terminal pressure at no-load 200 volts. If 3 per cent. of the power is lost in heating the conductors, the pressure drop due to resistance will be 3 per cent., and this drop must be equally divided between the primary and secondary windings. We have then

$$I_1.R_1=\frac{1{\cdot}5}{100}.2000=30 \text{ volts},$$

$$I_2.R_2=\frac{1{\cdot}5}{100}.200=3 \text{ volts}.$$

Hence

$$R_1=\frac{30}{20}=1{\cdot}5 \text{ ohms},$$

$$R_2=\frac{3}{200}=0{\cdot}015 \text{ ohm}.$$

The error introduced by subtracting the drop from the terminal pressure algebraically instead of geometrically is negligibly small. We have therefore

$$E_1 = V_1 - I_1 . R_1 = 1970 \text{ volts},$$

$$E_2 = \frac{E_1}{10} = \frac{1970}{10} = 197 \text{ volts},$$

and $$V_2 = E_2 - I_2 . R_2 = 194 \text{ volts}.$$

The drop of pressure between no-load and full load is 200 − 194 = 6 volts, i.e. 3 per cent.

In order to reduce the losses and to economise copper a type of transformer known as an auto-transformer is sometimes employed. This is shown diagrammatically in Fig. 273, in which it may be assumed that the primary voltage V_1 is 220 and that the load to be supplied from the secondary consists of 10 lamps each taking 0·4 ampere at 110 volts. These would be connected between one end and the middle point of the primary winding. The secondary current is 10 × 0·4 = 4 amperes and the secondary output 4 × 110 = 440 watts. The primary current is therefore 440/220 = 2 amperes. This arrangement has the obvious advantage that the separate secondary winding is dispensed with and that the portion of the primary winding, which serves also as the secondary, carries a smaller current than would flow in a separate secondary. Although this device is unsuitable where the transformation ratio is large, it is very advantageous for arc lamp circuits and for starting and regulating single-phase railway motors. If lamps are connected not only across the left-hand half, but also across the right-hand half of the winding, the arrangement constitutes an A.C. three-wire system with the transformer as balancer. It should be pointed out that, even with the conditions shown in Fig. 273, the right-hand half of the winding must not be regarded as a choking coil reducing the pressure from 220 to 110.

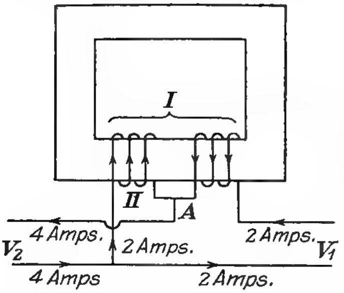

Fig. 273

88. Transformer on Inductive Load.

We pass now to the consideration of the case where the secondary terminals of the transformer are connected to a load consisting of both ohmic and inductive resistance. If R_2' be the resistance and ωL_2 the reactance of the external secondary circuit, the phase difference between the secondary terminal pressure and the secondary current is given by the equation

$$\tan \phi_2 = \frac{\omega L_2}{R_2'} .$$

The difference of phase between the induced E.M.F. E_2 and the current I_2 is given by the equation

$$\tan \phi = \frac{\omega L_2}{R_2' + R_2} .$$

The vector *OE* in Fig. 274 is made equal to $\frac{E_2 . T_1}{T_2}$, which is the secondary induced E.M.F. reduced to its primary equivalent. A right-angled triangle is constructed on the hypotenuse *OE*, so that the side *OM* lags behind *OE* by the above angle ϕ. *OM* is then equal to the secondary pressure used in overcoming ohmic resistance, the part *OL* being the internal drop, and the remainder *LM* the pressure used against the external ohmic resistance. Since the whole diagram is reduced to the primary turns, we have

$$OM = I_2 (R_2' + R_2) . \frac{T_1}{T_2}. \qquad OL = I_2 R_2 . \frac{T_1}{T_2}.$$

Similarly *EM* represents the equivalent inductive drop in the external circuit, or

$$EM = \omega L_2 . I_2 . \frac{T_1}{T_2}.$$

The secondary terminal pressure is found by subtracting geometrically the internal drop *OL* from the induced E.M.F. *OE*. We have therefore

$$RE = V_2 . \frac{T_1}{T_2}.$$

The figure shows that the terminal pressure is the hypotenuse of the right-angled triangle formed by the ohmic and inductive pressure drops in the external circuit.

We must now draw the primary and secondary current vectors. The secondary current is in phase with the ohmic pressure drop produced by it. When reduced to the equivalent primary current, we get

$$OC = \frac{I_2 . T_2}{T_1}.$$

The vector *OD* must now be drawn so that its resultant with *OC* is the no-load current *OA*. Since the no-load ampere-turns are the resultant of the primary and secondary ampere-turns, the vector *OD* must represent the primary current both in magnitude and phase.

The primary ohmic drop *OG* must be equal to $I_1 . R_1$ and must be in phase with the primary current. Equal and opposite to *OE* we draw *OE'*, to represent the pressure required to overcome the back E.M.F. The resultant *OH* of *OG* and *OE'* gives us the applied primary terminal pressure V_1. From an examination of the diagram we learn the following important facts:

1. The primary and secondary current vectors are now almost exactly opposite to one another, and the no-load ampere-turns are almost equal to the algebraic difference between the primary and secondary ampere-turns. Since the no-load current is actually much smaller than shown in the figure, the equation $I_1 = I_2 . \frac{T_2}{T_1}$ is still very nearly correct.

2. Since the ohmic drop in the windings is usually very small, the terminal pressure vectors V_1 and V_2 come almost exactly vertical, that is, they coincide in direction with the induced E.M.F.

3. The angle of lag ϕ_2 in the external secondary circuit is consequently nearly equal to the angle ϕ_1 between the primary terminal pressure and primary current. Hence, the secondary phase displacement reacts directly on the primary circuit.

This last result enables us to look upon the transformer as a single piece of apparatus, and to bridge over, in imagination, the separation of the two windings. Thus, we speak of the pressure drop and copper loss of the transformer as a whole, as if it had but a single winding. The dotted line joining the points G and E in Fig. 274 is parallel with, and equal to, the primary terminal pressure V_1. Moreover, as the ohmic pressure drops, OG and OL are practically in the same straight line, they can be added together to give a total drop, and we can write

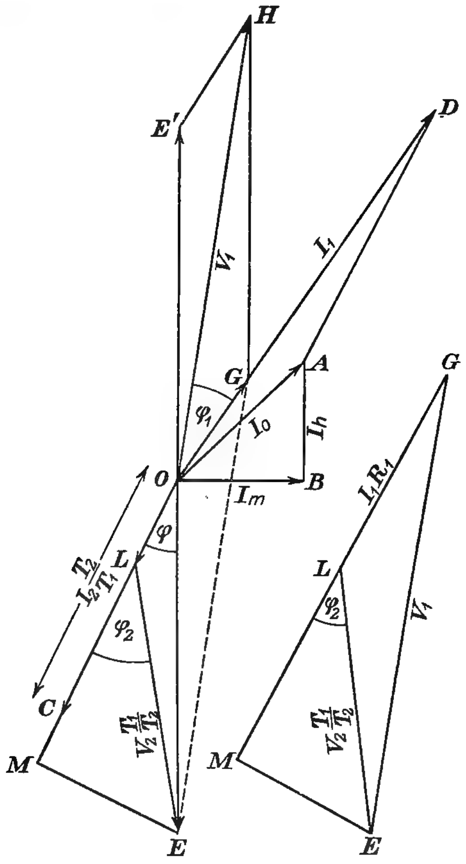

Fig. 274 Fig. 275

$$GL = OG + OL = I_1 . R_1 + I_2 . R_2 . \frac{T_1}{T_2}.$$

Putting I_2 equal to $\frac{I_1 . T_1}{T_2}$, we have

$$GL = I_1 . R_1 + I_1 . R_2 . \frac{T_1^2}{T_2^2} = I_1 \left(R_1 + R_2 . \frac{T_1^2}{T_2^2}\right).$$

It would appear from this equation that the primary current passed through the two resistances R_1 and $R_2 . \frac{T_1^2}{T_2^2}$ in series. The latter is the secondary resistance reduced to its equivalent primary value. After overcoming this total resistance the initial terminal pressure V_1 is reduced to the secondary terminal pressure V_2. The transformer is somewhat similar to a generator with an E.M.F. V_1, an internal resistance $R = R_1 + R_2 . \frac{T_1^2}{T_2^2}$, and a terminal pressure $V_2 . \frac{T_1}{T_2}$. To make this quite clear, the corresponding diagram for the generator is drawn in Fig. 275, in exactly the same position as it has in the transformer diagram. The electromotive force is the hypotenuse, the side GM is the total ohmic drop, both internal and external, while the side EM is the external inductive drop. The generator is assumed to have no internal self-induction, because we have neglected leakage.

If, for example, we assume that the transformer considered in the last section is connected to a secondary circuit with a power-factor of $\cos \phi_2 = 0{\cdot}7$, and that the primary current I_1 is 20 amperes, we can find the primary terminal pressure which is necessary to maintain a secondary terminal pressure of 200 volts. We have

$$R_1 + R_2 . \frac{T_1^2}{T_2^2} = 1{\cdot}5 + 0{\cdot}015 . 100 = 3 \text{ ohms.}$$

The total internal ohmic drop reduced to the primary winding is therefore

$$GL = I_1 \left(R_1 + R_2 . \frac{T_1^2}{T_2^2} \right) = 60 \text{ volts.}$$

The drop of pressure in external ohmic resistance, when reduced to the primary winding, is found as follows:

$$LM = V_2 . \cos \phi_2 . \frac{T_1}{T_2} = 200 . 0{\cdot}7 . 10 = 1400 \text{ volts.}$$

The side GM representing total ohmic drop both internal and external is therefore equal to 1460 volts. Since $\cos \phi_2 = 0{\cdot}7$, $\sin \phi_2$ must equal $0{\cdot}714$, and the inductive drop EM is found as follows:

$$EM = V_2 . \sin \phi_2 . \frac{T_1}{T_2} = 200 . 0{\cdot}714 . 10 = 1428 \text{ volts.}$$

For the hypotenuse EG we have $\sqrt{GM^2 + EM^2}$,

or $$V_1 = \sqrt{1460^2 + 1428^2} = 2040 \text{ volts.}$$

This then is the primary pressure required to give a pressure of 200 volts on the secondary terminals. This, however, is on the assumption that every line of force produced by the primary winding passes through the secondary winding and *vice versa*, so that the two windings have a common field and no other. As a matter of fact, there is a partial leakage of lines, which increases the drop of pressure due to increasing load. In the following section it will be shown that, with these actual relations, we can still imagine the two windings to be combined to form, as it were, a single winding.

89. The Effect of Magnetic Leakage.

In Section 84 we compared the back E.M.F. induced in the primary winding with the E.M.F. of self-induction of a choking coil. Since then, however, we have purposely avoided the use of the term "self-induction," and have always spoken of the induced E.M.F. This was necessary because, in addition to this useful induced E.M.F., there is, in both windings, a useless and harmful self-induction due to magnetic leakage.

In order to make this clear, we shall consider a very bad arrangement, which would never be adopted in practice, viz. a core-type transformer with the primary winding on one limb and the secondary on the other (Fig. 276). The lines of force produced on no-load pass, for the most part, completely round the iron circuit and thus thread the secondary winding, but some of the primary lines pass through the air and constitute the no-load leakage. The number of lines threading the secondary winding is therefore smaller than the total number threading the primary winding. The E.M.F. induced in the secondary winding must consequently be less than that induced in the primary winding. We may consider that a part of the primary terminal pressure is used in overcoming the back E.M.F. due to this leakage self-induction, leaving the remainder to overcome the back E.M.F. E_1 due to the flux which links both windings. The latter E.M.F. will therefore be smaller than the terminal P.D., and consequently the secondary terminal pressure will be correspondingly less. The effect of magnetic leakage is, however, quite negligible at no-load, as the reluctance of the iron path is so small, compared with that of the air, that the leakage flux bears a very small ratio to the useful flux.

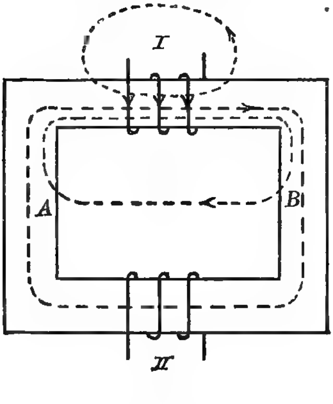

Fig. 276

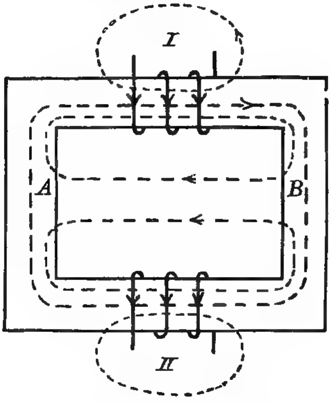

Fig. 277

The effect of leakage increases immensely, however, directly the transformer is loaded. We have seen that the secondary current is almost exactly opposed to the primary current. If, at the moment represented in Fig. 277, the primary current is flowing inward on the side of the coil which is turned towards us, then the secondary current will be flowing outward. By applying the right-handed screw rule, it is easily seen that the two windings are in opposition with regard to the main flux threading the two windings, but are acting together with regard to the leakage flux. The primary and secondary ampere-turns are in opposition for the iron path, but in parallel for the leakage through the air. This causes a rapid increase in the ratio of the leakage to the useful flux.

The increase of leakage with load is explained very clearly if we consider the difference of magnetic potential between the points *A* and *B*. At no-load (Fig. 276) this difference is only sufficient to drive the magnetic flux through the lower iron core. When loaded (Fig. 277) the difference of magnetic potential has also to overcome the counter magnetomotive force of the secondary ampere-turns. This increased difference of magnetic potential naturally drives a larger leakage through the air. It is analogous to forcing water through a leaky pipe; the greater the back-pressure opposing the passage of the water, the greater will be the quantity of water lost through the leaks.

Fig. 277 cannot represent the actual distribution of flux, since, in the lower half of the transformer, the leakage lines are shown passing through the iron in the opposite direction to the useful lines. The actual flux through any part of the magnetic circuit must be the resultant of these opposing fluxes, and the actual distribution will be as shown in Fig. 278 at any moment when the primary current exceeds the secondary current. The actual distribution changes entirely in character as well as in magnitude during a period, and since the primary and secondary currents are not exactly 180° out of phase, there are moments when there is no primary current, but yet a small secondary current which produces a flux distribution the reverse of that shown in Fig. 276. Hence, we see that the leakage at any moment and at any point is the resultant effect of the instantaneous values of the primary and secondary currents.

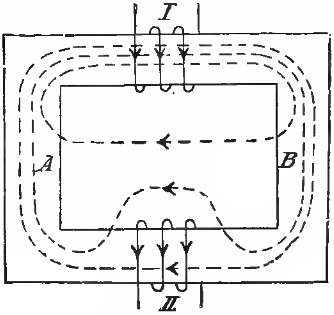

Fig. 278

In order to reduce the magnetic leakage as much as possible, the two windings are put as close together as possible (Figs. 271 and 272), or even subdivided and alternate primary and secondary coils sandwiched together. There is, however, always a small drop due to magnetic leakage.

Fig. 279 shows the paths of the main and the leakage fluxes with cylindrical coils. It is assumed that the primary is outside the secondary winding. Fig. 280 shows the fluxes when flat coils are employed, two primary and three secondary coils being sandwiched alternately together..

In some circumstances it is impossible to arrange for the primary and secondary windings to be as close together as one would desire. In an electric furnace of the induction type for melting steel, the secondary winding is represented by an annular channel containing metal maintained in a molten condition by the heat developed in it by the induced current (see Fig. 281). A considerable distance must obviously separate the molten metal and the primary winding. In order to keep the voltage drop due to the leakage flux within reasonable limits, the useful flux is made very large by the employment of an iron core of very large section.

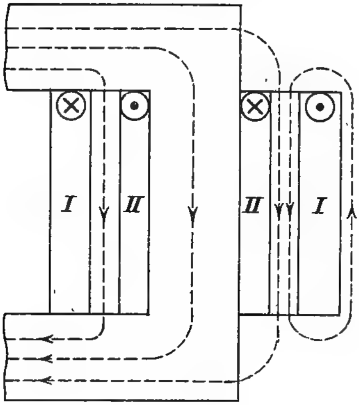

Fig. 279

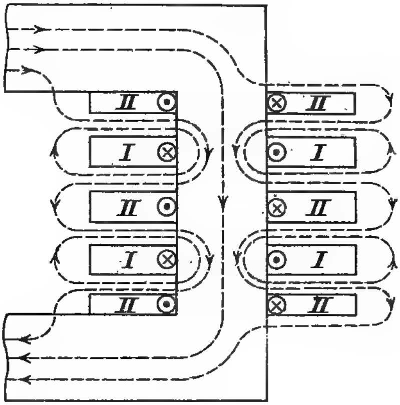

Fig. 280

To show the effect of leakage on the vector diagram, we shall neglect the small no-load current, and assume that the primary and secondary currents are exactly opposed to each other*. The angle by which the secondary current I_2 lags behind the induced electromotive force $OE\left(=E_2 . \frac{T_1}{T_2}\right)$ is chosen arbitrarily in Fig. 282. It depends, as a matter of fact, upon the relation between the total secondary self-induction and the total secondary ohmic resistance. The pressure drops due to the ohmic resistance of the windings are drawn in the direction of the currents, viz.

$$OG = I_1 . R_1,$$

$$OL = I_2 . R_2 . \frac{T_1}{T_2}.$$

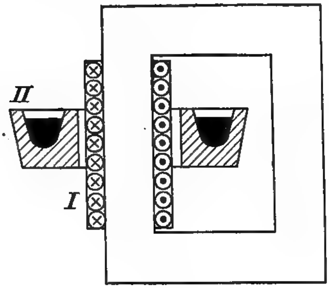

Fig. 281

We know from page 247 that the induced E.M.F. due to the self-induction of leakage lags 90° behind the current. To overcome that caused by the secondary current, an equal pressure ON, 90° ahead of the current I_2, will be necessary. The total pressure drop in the secondary winding will be equal to the resultant OR. The line RE will represent the secondary terminal pressure both in magnitude and phase, so that

$$RE = V_2 . \frac{T_1}{T_2}.$$

---

* The exact diagram will be given when considering the induction motor in Section 123.

The effects of resistance and self-induction are greatly exaggerated in Fig. 282. In reality they are much smaller, so that the terminal pressure differs very little from the induced E.M.F.

A component OP will be required in the primary circuit to counterbalance the back E.M.F. due to the leakage of the primary winding. OP will be 90° ahead of I_1. The resultant OQ of the primary ohmic and inductive drops must represent the total primary pressure drop. By compounding OQ with the pressure OE', which counterbalances the back E.M.F. OE, we get the applied primary terminal pressure $OH = V_1$.

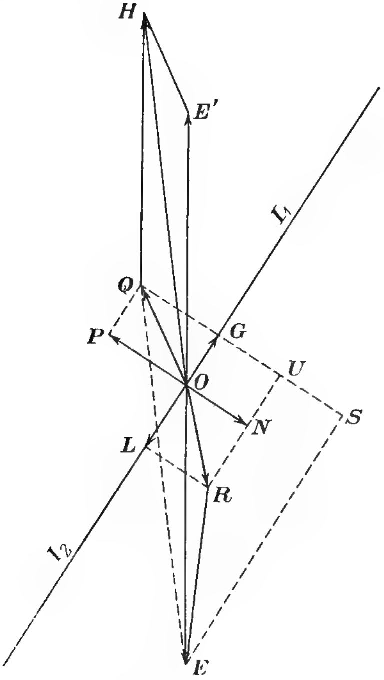

Fig. 282

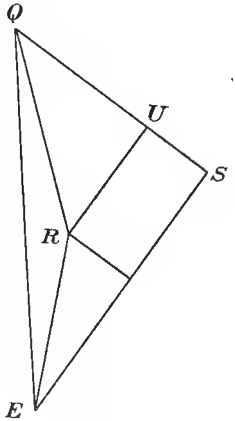

Fig. 283

We have thus determined the magnitude and phase of all the vectors, and we can now determine the equivalent resistance and leakage self-induction of a single apparatus to replace, in imagination, the two separate windings. For this purpose we join QG and QE, and drop perpendiculars from R and E on to QG produced. The figure thus obtained is given in Fig. 283, and shows the various pressures with which we are concerned. The hypotenuse QE is equal to OH, that is, to the primary terminal pressure V_1. QU represents the sum of the pressure losses due to magnetic leakage, and RU the sum of the ohmic pressure losses. The total drop of pressure in the transformer is given by QR, leaving RE for the secondary terminal pressure. We need hardly mention that the internal pressure drop is in reality much smaller than indicated in Figs. 282, 283.

It is of great importance that the triangle QRU can be found experimentally. By short-circuiting the secondary terminals we make $RE = 0$, so that the point E falls on the point R. The primary pressure must naturally be chosen so small that the normal current flows in the short-circuited transformer. This primary pressure is then directly equal to QR. The resistances are easily measured and the ohmic pressure drop RU calculated from them.

The triangle QRU can then be drawn and the effect of magnetic leakage determined.

We can now construct Kapp's transformer diagram for constant current but variable external power-factor (Fig. 284). The triangle QRU, which we have just determined for the normal full load current, is drawn in the position shown. With a radius equal to V_1 two circles are drawn, one with the centre Q, and the other with the centre R. For any given power-factor ϕ_2 in the external secondary circuit, RE gives the secondary terminal pressure, or rather, its primary equivalent, while QE or RV is equal to the constant V_1. The difference between them is EV, which gives the fall of secondary pressure between open circuit and full load. For any other value of the current, QR will be different, so that a new diagram must be drawn.

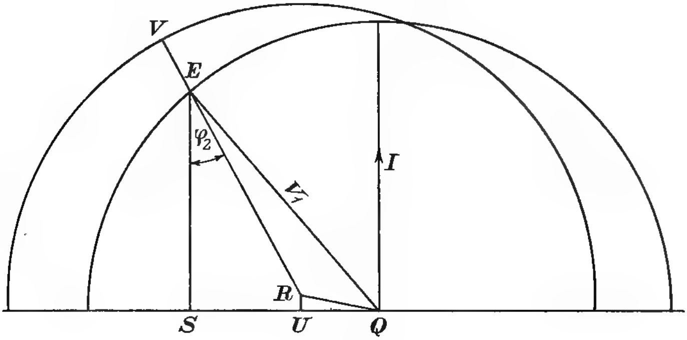

Fig. 284

The diagram shows that the fall of pressure between open circuit and full load increases as the point E moves to the right, that is, as the power-factor decreases.

If the current leads ahead of the terminal pressure, so that the point E moves over to the left, the drop of pressure may not only disappear, but may be turned into a rise, so that the terminal pressure increases with the load. This may occur if the secondary circuit contains condensers or over-excited synchronous motors. (See the corresponding diagram in Section 96, also Sections 81 and 109.)

CHAPTER XII

90. Types of A.C. generators.—91. The mean E.M.F. of a generator.—92. The effective E.M.F. with sinusoidal field.—93. The E.M.F. with 1, 2, or 3 slots per pole per phase.—94. The E.M.F. of a distributed winding.—95. The alternating E.M.F. of a closed D.C. winding.

90. Types of Alternating Current Generators.

Many very different types of alternating current generators have been proposed and constructed from time to time, but, as in so many things, there has been a gradual elimination and standardization, until, to-day, nearly every A.C. generator is constructed with an external stationary armature and an internal revolving field magnet system. It is important, however, to study the various types of machines, and this we shall do, not so much in their historical order as in the order of their simplicity. We shall deal in a general way with both single-phase and polyphase machines, but shall reserve the consideration of the points peculiar to polyphase generators until we come to the theory of polyphase currents. The single-phase machine has a single winding on its armature. The side of each coil may be placed in a single hole or slot, as shown in Fig. 285, or distributed between several slots, as shown in Fig. 286. In both of these cases there is a single coil-side under each pole. In polyphase machines, on the other hand, the armature carries two or three distinct windings, which may or may not be connected, but which are displaced from each other around the armature. For the present, we may consider these phases or windings to be entirely disconnected and quite distinct, so that a two-phase generator is nothing more than a machine with two single-phase windings. In Fig. 287, for example, we consider only the winding 1 and neglect the winding 2. In this way we shall be able to consider single-phase and polyphase machines together.

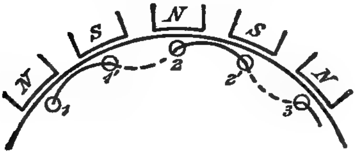

Fig. 285

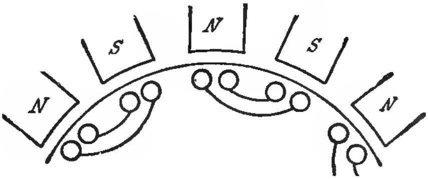

Fig. 286

External pole machines were formerly made with ring and with drum armatures. In the ring armature shown in Fig. 288 all the coils are wound in the same direction but are connected up alternately in opposite directions. By drawing arrows on the ends of the coils to show the direction of the induced E.M.F., and tracing the circuit from coil to coil, we find that all the induced electromotive forces are added together.

Machines with external poles and drum armatures (Fig. 285) are still built for small powers and high speeds. There is, as a rule, one coil per pair of poles, and this coil, as in the D.C. machine, spans an arc equal to the pole-pitch. In Fig. 285, for example, the wire passes up slot 1 from back to front, and then down slot 1′ from front to back, and so on. When this coil is completely wound, we pass to the coil 2, 2′ by means of the dotted connection at the back end of the armature.

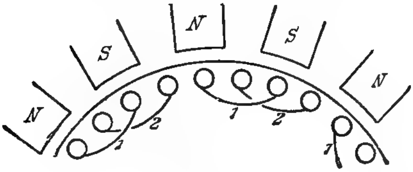

Fig. 287

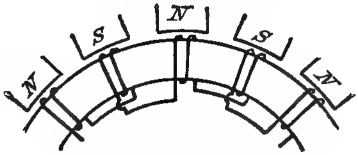

Fig. 288

The internal pole machine (Fig. 289) is strikingly simple both in principle and construction, and is the type almost universally adopted. The external and stationary armature is built up of sheet-iron stampings, secured in a cast-iron frame. The winding is carried in slots or tunnels on the inner surface of the annular ring. These slots are lined with insulating material, micanite in machines for high pressures, and leatheroid, or some other such material, in low-pressure machines. The winding is similar to the ordinary drum windings, or like that shown on the rotating armature in Fig. 285, but here the ends are brought out to fixed terminals, so that the collection of current at high pressures by means of slip rings is avoided. The poles, which are alternately north and south, are fixed on the periphery

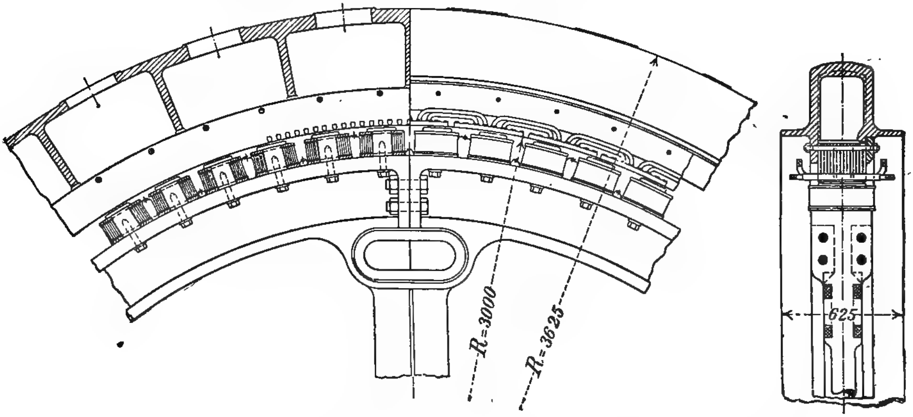

Fig. 289

Three-phase generator of the Société Électricité et Hydraulique, Charleroi.

of a flywheel, which rotates within the armature. The magnetising winding on the poles consists either of insulated wire or of bare copper strip wound on edge, with a thin layer of insulation between successive turns. The exciting current is supplied by a direct-current dynamo or an accumulator battery, and is led into the field coils through brushes and slip rings.

We have, so far, considered machines with a single ring of poles. In machines with disc armatures, however, there are two crowns of poles, so arranged that unlike pole faces are opposite to one another. We shall consider, in the first place, those machines in which the poles of each ring or crown are alternately north and south. Such is the case in the old Ferranti type of machine (Fig. 290). The armature consists of a wave-shaped copper band, which rotates in the narrow gap between the two crowns of poles. The poles shown in the figure lie behind the paper, while a second set of poles is arranged in front of the paper so that unlike poles are exactly opposite. This machine shows very clearly how the wire passes down, in front of a north pole, and up, in front of the succeeding south pole. The winding has quite lost its coil-like character and has become a pure wave winding.

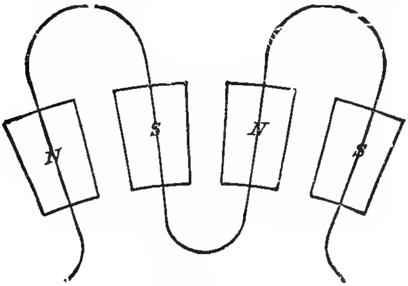

Fig. 290

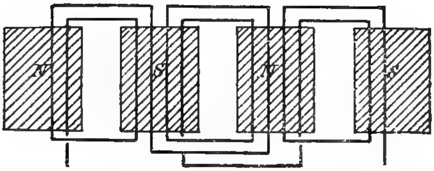

Fig. 291

Later Ferranti machines had the same field system, but the armature was built up of thin disc-like coils. The same construction was adopted in the earliest Siemens and Halske alternators. To show the essential similarity of disc and drum windings, a winding is shown in Fig. 291 which is equally applicable to either. For a Ferranti machine the poles have simply to be bent round to form an arc of a circle, like Fig. 290, and another set of poles arranged in front of them. The figure may equally well represent the development of a drum winding, in which case the opposite crown of poles is produced by induction in the armature iron.

In order that the electromotive forces may be added together, the coils must be connected up alternately in opposite directions, as is shown in Fig. 291. We notice that we have here the sides of two adjacent coils under one pole. The electromotive forces induced in all the wires under one pole are in the same direction, and the wires virtually form the side of a coil of twice the number of turns.

In the so-called *inductor alternators* there is no reversal of flux, but simply an introduction and withdrawal of lines through the armature coils. The characteristic feature of these machines is that neither the armature winding nor the field winding rotates, the only moving part consisting of masses of iron. These masses of iron conduct the magnetic flux across the gap, first at one point and the next moment at another. The flux through any coil on the armature is continually changing and thus inducing alternating electromotive forces in the coil. Machines of this type are specially suitable for running at high peripheral speeds. A portion of an inductor generator by the A.E.G. of Berlin is shown in Fig. 292. There are two armatures, *A* and *B*, built up of sheet iron, and secured by the frame *C* which acts as the magnetic yoke. At the bottom of the annular space thus formed, the magnetising coil *F* is wound. On the left is shown the flywheel which carries a number of iron blocks, *G*. These blocks offer a path of low reluctance to the lines of force, which consequently pass, for the most part, through those parts of the armature between which the blocks are momentarily situated. The magnetic flux practically rotates with the iron blocks, which are therefore equivalent to rotating electromagnets. The armature coils must naturally span a distance equal to half the pitch of the blocks.

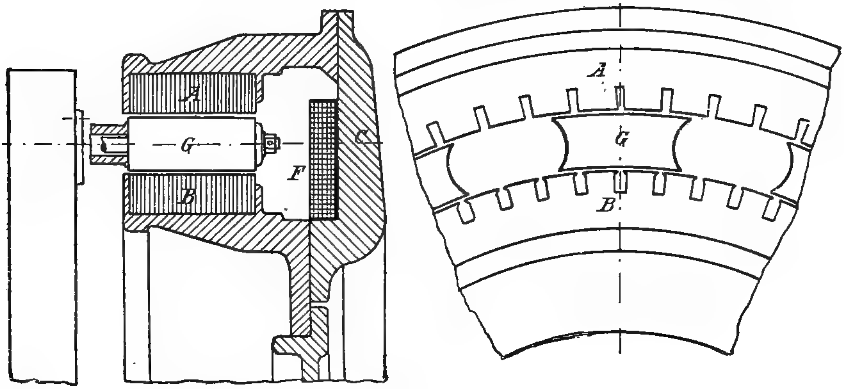

Fig. 292

Were magnetic leakage entirely absent, the armature wires in the neutral zone, i.e. between the blocks, would be entirely inactive. As the result of leakage, however, each coil has opposing electromotive forces induced in its two sides. If the flux passing across a block is Φ_p while the leakage flux, that is, the flux which crosses between two blocks, is Φ_l, then the E.M.F. is proportional to $\Phi_p - \Phi_l$. When we take into account that the flux does not reverse, but simply varies between zero and maximum, it is evident that, for Φ in equation (143) on page 296, we must put

$$\Phi = \frac{\Phi_p - \Phi_l}{2} \quad \text{..............................} (139).$$

The great leakage in this type of machine caused it to be given up in favour of the alternate pole type. The chief disadvantage is not so much the additional ampere-turns required to drive the larger flux through the yoke as the heavy drop of pressure caused by the increased leakage when the machine is loaded (see Section 101).

91. The Mean E.M.F. of a Generator.

As a general rule the armature windings of D.C. and A.C. generators differ from one another in that the wires of the D.C. winding are equally distributed over the whole armature, while the wires of a coil-side of an A.C. winding are either contained in a single slot or distributed between two or three slots in close proximity. The width of a coil-side is therefore less than the width of the pole, and, as a rule, even less than the width of the interpolar arc. The instantaneous values of the induced electromotive forces in the various wires are added together at every moment. To obtain the mean value of the E.M.F. induced in the armature, we must multiply the number of wires connected in series by the number of lines cut by each in a complete revolution, divide the product by the time taken for the revolution, and by 10^8 to get the result in volts. If, then,

Φ be the flux per pole,
p the number of pairs of poles, and
z' the number of conductors in series,

then the change of linkages per revolution will be $2p.\Phi.z'$. Since the time taken by the revolution is $\frac{1}{n}$ of a second, we have for the mean or average E.M.F.

$$E_{av} = \frac{2p.\Phi.z'}{1/n}.10^{-8} = 2\Phi.p.n.z'.10^{-8} \quad \text{..........(140).}$$

Putting $$f = p.n,$$
this becomes $$E_{av} = 2\Phi f z'.10^{-8} \text{ volts} \quad \text{..................(141).}$$

It is interesting to compare this result with that obtained for a series-wound D.C. armature. Since there are only two parallel paths through the armature, $a = 1$ and equation (79) on page 124 may be written

$$E = p.\Phi.n.Z.10^{-8}.$$

Z is here the total number of wires on the armature, of which only a half are in series, so that $z' = \frac{Z}{2}$. This gives for the D.C. armature

$$E = p.\Phi.n2z'.10^{-8} = 2\Phi f z'.10^{-8}.$$

This is exactly the same equation as was obtained above for the mean E.M.F. of the alternator.

If an alternator and a D.C. dynamo have the same total number of wires on the armature, and are so connected that the number of parallel paths through the winding is the same in each case, then the average P.D. of the alternator will be the same as the P.D. of the D.C. machine.

92. The Effective E.M.F. with Sinusoidal Field.

On the assumption that the flux-density around the armature varies from point to point in accordance with the sine law, and that the conductors of a coil-side are concentrated in a single slot, the maximum E.M.F. is given by equation (112) on page 234, viz.

$$\hat{e} = \pi.\Phi.f.z'.10^{-8}.$$

As the effective or R.M.S. value is equal to the maximum divided by $\sqrt{2}$, we have

$$E = \frac{\pi}{\sqrt{2}} \,.\, \Phi \,.\, f \,.\, z' \,.\, 10^{-8} = 2{\cdot}22 \,.\, \Phi f z' \,.\, 10^{-8} \ldots\ldots\ldots\ldots (142).$$

The ratio of the effective to the mean value is known as the form-factor. It depends on the shape or form of the field distribution curve and on the type of armature winding. In our case the form-factor is $\frac{2{\cdot}22}{2} = 1{\cdot}11$.

We must now pass to the consideration of the effective E.M.F. for other cases, differing from the ideal one just considered. We therefore write formula (142) in a more general form, viz.

$$E = k.\Phi.f.z'.10^{-8} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (143).$$

By distributing the coil-side between two or more slots the E.M.F. is reduced. If the field is sinusoidal it is a simple matter to calculate the E.M.F. for each slot from equation (142). The electromotive forces for the several slots can then be compounded by the parallelogram or polygon of forces, taking into consideration the phase difference due to the displacement of the slots around the armature.

Suppose, for example, that we have a sinusoidal field distribution and a double slot winding, that is, one in which the coil-side is laid in two slots, and that the distance between these two slots is equal to a third of the pole-pitch. The pole-pitch corresponds to an angle of π radians, so that the electromotive forces induced in the two slots differ by a phase angle of $\pi/3$ or 60°. The whole winding may therefore be considered as made up of two parts, and the E.M.F. in each part is given by equation (142) as

$$E' = 2{\cdot}22 \,.\, \Phi f \frac{z'}{2} \,.\, 10^{-8} \text{ volts.}$$

When we compound two electromotive forces E', having a phase difference of $\pi/3$, we find that the resultant is equal to $2.\cos \pi/6.E' = 1{\cdot}73E'$. The E.M.F. of the whole armature is therefore

$$E = 1{\cdot}73E' = 1{\cdot}92.\Phi f z'.10^{-8} \text{ volts.}$$

A case of special interest is that in which the coil-side is distributed between so many slots that the winding is practically equivalent to a smooth core winding, in which the wires lie side by side upon the surface.

Let the ratio of the width of the coil-side to the pole-pitch be 2γ, the wires being distributed uniformly over this width. The winding will occupy 2γ of the available space on the armature surface. The total number of wires in series is z', and if these wires were all simultaneously under the middle of the poles the R.M.S. electromotive force would be (see equation 142)

$$E = 2{\cdot}22.\Phi f z'.10^{-8}.$$

The E.M.F. of each wire varies according to a sine law, and instead of reaching the position of maximum E.M.F. corresponding to the middle of the pole, at the same moment, the wires are spread out uniformly at that moment over an angle of $2\gamma\pi$. The E.M.F. will thus be reduced from a value corresponding to $\cos 0 = 1$ to a value corresponding to

$$\frac{1}{2\gamma\pi}\int_{-\gamma\pi}^{+\gamma\pi}\cos\phi\,.\,d\phi\,. = \frac{\sin\gamma\pi}{\gamma\pi}\,.$$

The effective E.M.F. will therefore be reduced in this ratio, that is

$$E = 2{\cdot}22\,\frac{\sin\gamma\pi}{\gamma\pi}\,.\,\Phi f z'\,.\,10^{-8}\text{ volts}\,\ldots\ldots\ldots\ldots\ldots\ldots(144).$$

As an example, we may consider a three-phase distributed winding, in which each coil-side has a width equal to a third of the pole-pitch. We have here

$$2\gamma = \tfrac{1}{3},$$

$$\sin\gamma\pi = \sin\frac{\pi}{6} = 0{\cdot}5,$$

$$\frac{\sin\gamma\pi}{\gamma\pi} = \frac{0{\cdot}5}{\pi/6} = \frac{3}{\pi},$$

and from equation (144)

$$E = 2{\cdot}22\,.\,\frac{3}{\pi}\,.\,\Phi f z'\,.\,10^{-8} = 2{\cdot}12\,.\,\Phi f z'\,.\,10^{-8}\ldots\ldots\ldots\ldots(145).$$

For a width of coil-side equal to two-thirds of the pole-pitch, such as we have in some three-phase windings, the relations are as follows:

$$2\gamma = \tfrac{2}{3},$$

$$\sin\gamma\pi = \sin\frac{\pi}{3} = 0{\cdot}866,$$

and from equation (144)

$$E = 2{\cdot}22\,.\,\frac{0{\cdot}866}{\pi/3}\,.\,\Phi f z'\,.\,10^{-8} = 1{\cdot}84\,.\,\Phi f z'\,.\,10^{-8}\,\ldots\ldots\ldots(146).$$

These equations (145) and (146) will prove very useful to us when we come to the consideration of induction motors, but for single and polyphase generators other equations will be required, which we now proceed to establish.

93. The E.M.F. with 1, 2, or 3 Slots per Pole per phase.

Instead of assuming, as we did in the previous section, that the field is distributed according to a sine law, we shall now assume that the distribution is rectangular, having a constant value under the pole, and falling to zero at the pole-tip. The induced E.M.F. will then have a similar wave form. As a matter of fact, such a distribution of the field is impossible, as there is naturally a falling off near the edges of the pole and a fringing from the pole into the interpolar space. This results in the E.M.F. curve approaching more nearly to the sine form, and gives an effective E.M.F. about 10 per cent. smaller than that theoretically calculated.

The theoretical curve of E.M.F. for a single-slot winding is shown in Fig. 293. The E.M.F. attains its maximum value directly the slot comes under the pole, and remains at this value until the slot emerges from under the pole. The time taken by a slot to move through the angle π is $\frac{1}{2f}$ second, and to move through the angle $\beta\pi$ the time taken is $\frac{\beta}{2f}$ second. This is the time taken by each wire to cut through the flux Φ from each pole. The rate of cutting is therefore $\frac{\Phi \, . \, 2f}{\beta}$, and as there are z' wires in series, the maximum E.M.F. will be

$$\hat{e} = \frac{2}{\beta} \, . \, \Phi f z' \, . \, 10^{-8} \text{ volts} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (147).$$

In Fig. 294 we have plotted the square of the E.M.F. from moment to moment and constructed a rectangle on the base 2π with an area equal to the shaded areas. The height of this rectangle is $\beta \, . \, \hat{e}^2$. By taking the square root of this height we find the effective value of the E.M.F., viz.

$$E = \sqrt{\beta} \, . \, \hat{e} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (148).$$

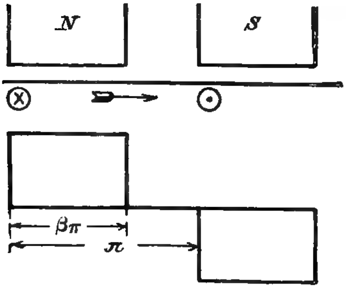

Fig. 293

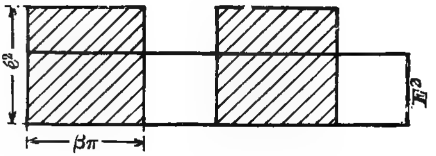

Fig. 294

If, for example, the polar arc is $\frac{2}{3}$ of the pole-pitch, we get from equation (147)

$$\hat{e} = 3\Phi f z' \, . \, 10^{-8},$$

and from equation (148)

$$E = \sqrt{\frac{2}{3}} \, . \, \hat{e} = 2{\cdot}45 \Phi f z' \, . \, 10^{-8} \text{ volts.}$$

If we take into consideration the fact that the actual E.M.F. is roughly 10 per cent. less than this, we get, for the ratio $\beta = \frac{2}{3}$, almost the same value as we obtained for the single-slot winding and sinusoidal field.

In a double-slot winding the width of the coil-side $2\gamma\pi$ is equal to the pitch of the holes (Fig. 295)*. We need only consider the case in which this distance is less than the interpolar arc, since this condition is always satisfied in actual machines. If the pole-pitch is π and the width of pole $\beta\pi$, the interpolar arc is $\pi - \beta\pi$ and we assume that $2\gamma\pi < \pi - \pi\beta$, i.e. that $2\gamma < 1 - \beta$.

The E.M.F. curve is made up of two rectangles, displaced relatively to each other by an angle $2\gamma\pi$. During the time that both slots are simultaneously under the pole, that is, over the angles $(\beta - 2\gamma)\pi$, the E.M.F. is a maximum, and, since it is quite immaterial whether the wires are in one or two slots during this interval, the maximum E.M.F. is given by equation (147)

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* In Figs. 295 to 304 the abscissae represent the various widths in terms of the pole-pitch; when multiplied by π they represent electrical angles.

$$\hat{e} = 2\frac{1}{\beta}\cdot \Phi f z' \cdot 10^{-8}.$$

When only one of the two slots is under the pole, i.e. over the angle $2\gamma\pi$, the E.M.F. has half this value.

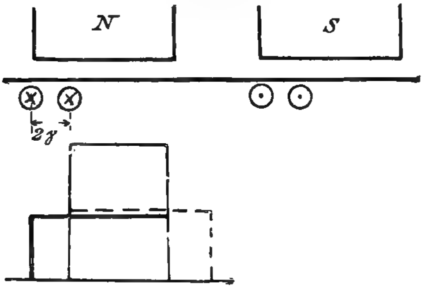

Fig. 295

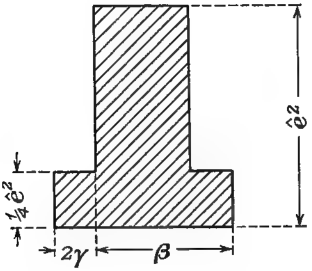

Fig. 296

We set up, as before, the squares of the E.M.F. from moment to moment (Fig. 296), and calculate the shaded area thus obtained. This is equal to

$$2\cdot 2\gamma\cdot\left(\frac{\hat{e}}{2}\right)^2 + (\beta - 2\gamma)\cdot\hat{e}^2 = (\beta - \gamma)\cdot\hat{e}^2.$$

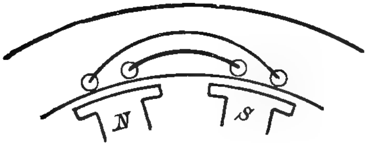

Fig. 297

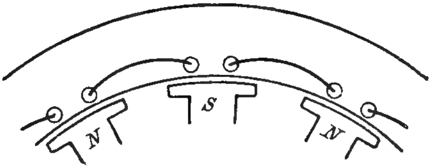

Fig. 298

By dividing this area by the base unity, since β and γ are in terms of the pole-pitch, and taking the root of the quotient, we get the effective value of the E.M.F., viz.

$$E = \sqrt{\beta - \gamma}\cdot\hat{e}\ldots\ldots\ldots\ldots(149).$$

This equation is equally applicable to the windings shown in Figs. 297 and 298. The portion of winding shown in Fig. 297 is equivalent to a single coil distributed between two slots per coil-side. The whole armature will contain p such coils, whereas the winding shown in Fig. 298 will consist of $2p$ single coils with a separate slot for each coil-side. For the purpose of calculation, however, Fig. 298 is to be treated as a double-slot winding with

p coils on the armature. The winding must naturally be so carried out that the current flows in the same direction through two adjacent slots.

We now come to the case of three slots per pole and phase, and we assume, as before, that the breadth of the coil-side is less than the interpolar gap, so that

$$2\gamma < (1 - \beta).$$

The curve of E.M.F. is obtained by adding the ordinates of three rectangles, displaced from each other by the pitch of the slots or half the breadth of the coil-side. For the sake of clearness, the rectangles are shown in Fig. 299 as having unequal heights. At first, when only one slot is under the pole, the E.M.F. corresponds to a third of the armature wires. This lasts for a time corresponding to γ, where 2γ is the breadth of the coil-side. Over an equal arc γ the E.M.F. corresponds to $\frac{2}{3}$ of the total wires, and over an arc $\beta - 2\gamma$ it has a constant value corresponding to the whole number of armature wires. As the coil-side leaves the pole, the E.M.F. drops in the same way as it rose when the coil-side came under the pole. The ordinates in Fig. 300 represent the squares of the momentary values of the E.M.F., and the shaded area is found to be equal to

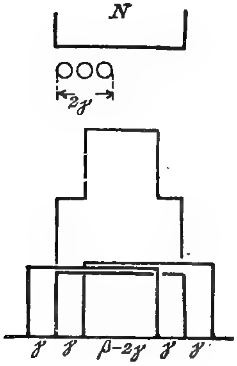

Fig. 299

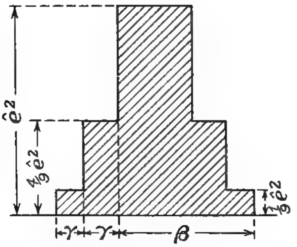

Fig. 300

$$2\gamma \,.\, (\tfrac{1}{3} \,.\, \hat{e})^2 + 2\gamma\, (\tfrac{2}{3} \,.\, \hat{e})^2 + (\beta - 2\gamma) \,.\, \hat{e}^2$$

If we divide this area by the base unity and extract the square of the resultant height, we get, for the effective value of the E.M.F.,

$$E = \sqrt{\beta - \tfrac{8}{9} \,.\, \gamma} \,.\, \hat{e} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (150).$$

For $\hat{e}$ we have equation (147) on page 298, viz.

$$\hat{e} = \frac{2}{\beta} \,.\, \Phi f z' \,.\, 10^{-8}.$$

We may consider as an example a three-phase generator with treble-slot winding, or, as it is more often expressed, with 3 slots per pole and phase. Since there are $3 \,.\, 3 = 9$ slots per pole, the pitch of the slots must be $\pi/9$.

Hence $\gamma = \frac{1}{9}$, and, assuming that the ratio β of polar arc to pole-pitch is $\frac{1}{2}$, we have

$$\hat{e} = \frac{2\,.\,2}{1}\,.\,\Phi f z'\,.\,10^{-8} = 4\,.\,\Phi f z'\,.\,10^{-8}.$$

Then, from equation (150), it follows that

$$E = \sqrt{\frac{1}{2} - \frac{8}{9}\,.\,\frac{1}{9}}\,.\,\hat{e} = 2{\cdot}53\,.\,\Phi f z'\,.\,10^{-8}.$$

94. The E.M.F. of a Distributed Winding.

In the foregoing considerations we have always assumed that the breadth of the coil-side was less than the polar arc $\beta\pi$. We shall now, however, consider a case in which the coil-side is broader than either the polar arc or the interpolar gap, so that $2\gamma > \beta$.

At the moment of maximum E.M.F. (Fig. 303) the coil-side will project beyond the pole, so that the value of $\hat{e}$ depends on β and not on 2γ. We shall consider, in the first place, the moment when the coil-side is symmetrical about the neutral axis (Fig. 301). The electromotive forces induced under each pole will exactly neutralise each other. If the coil moves forward through an arc having the ratio α to the pole-pitch, the E.M.F. induced by the south pole will increase by an amount proportional to α. The opposing E.M.F. induced under the north pole will decrease simultaneously by the same amount. The total E.M.F. will therefore correspond to an arc 2α. Since the maximum value of the E.M.F. depends on the polar arc ratio β, we have, for the E.M.F. at any moment,

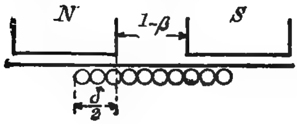

Fig. 301

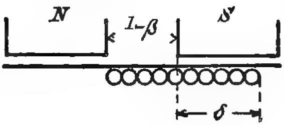

Fig. 302

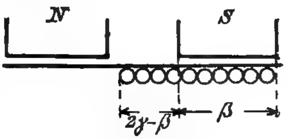

Fig. 303

$$e = \hat{e}\,\frac{2\alpha}{\beta}.$$

This applies to the interval between the positions shown in Fig. 301 and Fig. 302, i.e. until the coil-side is beyond the influence of the north pole.

If the breadth of the coil-side exceeds the interpolar arc by an amount δ, it is evident from Fig. 302 that

$$\delta = 2\gamma - (1 - \beta) = 2\gamma + \beta - 1 \quad \text{..............(151).}$$

The distance through which the coil moved during the interval considered above is seen from Fig. 301 to have been $\delta/2$. We draw ordinates equal to the squares of the E.M.F. from moment to moment during this interval, and calculate the area F_1 thus produced (Fig. 304), viz.

$$F_1 = \int_0^{\delta/2} \hat{e}^2\,.\,\left(\frac{2\alpha}{\beta}\right)^2\,.\,d\alpha = \frac{4\,.\,\hat{e}^2}{\beta^2}\,.\left[\frac{\alpha^3}{3}\right]_0^{\delta/2}.$$

By substituting the limits, we get

$$F_1 = \frac{\hat{e}^2 . \delta^3}{6 . \beta^2}.$$

The second interval to be considered is that between Fig. 302 and Fig. 303, when the coil-side is partly under a single pole. If α represents now the amount of the coil which is under the pole at any moment, then the instantaneous value of the E.M.F. will be given by the equation

$$e = \hat{e} . \frac{\alpha}{\beta}.$$

At the beginning of the interval (Fig. 302) α is equal to δ, while at the end (Fig. 303) α is equal to β. We must square the values of the E.M.F., and integrate between the limits $\alpha = \delta$ and $\alpha = \beta$. In this way we get for the area F_2 in Fig. 304

$$F_2 = \int_\delta^\beta \hat{e}^2 . \frac{\alpha^2}{\beta^2} . d\alpha = \frac{\hat{e}^2 . (\beta^3 - \delta^3)}{3\beta^2}.$$

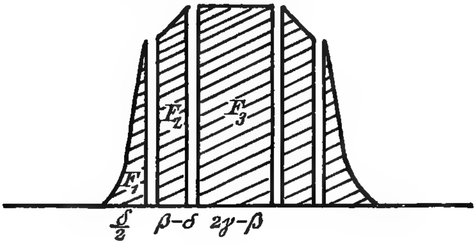

Fig. 304

During the third interval the whole pole face is covered with the coil-side, and the E.M.F. has a constant maximum value. This interval corresponds to an arc $2\gamma - \beta$ (Fig. 303). By squaring the E.M.F. we find the area F_3 in Fig. 304, viz.

$$F_3 = \hat{e}^2 . (2\gamma - \beta).$$

Although the various areas are shown separated in Fig. 304, they are, of course, parts of a single area, divided into five parts by four vertical lines. For the total area we have

$$2F_1 + 2F_2 + F_3 = \hat{e}^2 . \left(2\gamma - \frac{\beta}{3} - \frac{\delta^3}{3\beta^2}\right).$$

If this be divided by the base unity and the root extracted, the effective E.M.F. is obtained; thus

$$E = \hat{e}\sqrt{2\gamma - \frac{\beta}{3} - \frac{\delta^3}{3\beta^2}} \qquad (2\gamma > \beta) \text{ ....(152).}$$

This equation is based on the assumption that the coil-side is broader than the pole face. The value $\hat{e}$ will be smaller than the value given by equation (147) on page 298 in the ratio $\frac{\beta}{2\gamma}$, since only this fraction of the total number of wires on the armature can be under the poles at the same moment. For the value of $\hat{e}$ to be put in equation (152), we have therefore

$$\hat{e} = \frac{1}{\gamma} \cdot \Phi f z' \cdot 10^{-8} \qquad (2\gamma > \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (153).$$

If, however, the coil-side is narrower than the pole face, $\hat{e}$ must be found from equation (147). In calculating the effective E.M.F., moreover, we must notice that the maximum E.M.F. depends on the coil-side 2γ, and not on the polar arc β. The limits of integration for the second interval will be δ and 2γ, and the third interval will spread over the arc $\beta - 2\gamma$. Proceeding in a similar manner to the above, we find that

$$E = \hat{e}\sqrt{\beta - \frac{2\gamma}{3} - \frac{\delta^3}{12\gamma^2}} \qquad (2\gamma < \beta) \quad \ldots\ldots\ldots\ldots (154).$$

e is, in this case, given by equation (147).

If the coil-side 2γ is even narrower than the interpolar arc, equation (151) will give a negative value for δ. It must then be put equal to 0, that is, the last term in equation (154) is to be omitted. The proof of this follows from the consideration that there is no area F_1, and that the limits between which the area F_2 is to be calculated are 0 and 2γ. Both the conditions are fulfilled by putting $\delta = 0$.

95. The Alternating E.M.F. of a Closed D.C. Winding.

A very important example of distributed winding is found in the rotary converter. This is practically an ordinary D.C. machine with taps taken from two diametrically opposite points on the armature winding and connected to two slip rings. If the machine is multipolar, and has a parallel wound armature, the connections to the slip rings are joined to every equipotential point on the winding, so that each ring has p connections.

These machines can be used for the simultaneous supply of both D.C. and A.C. current, or for transforming either of these into the other. In the vast majority of cases, however, it is driven as a motor from the A.C. side, and supplies D.C. from the commutator side for the purposes of electric traction. When the coils to which the slip rings are directly connected are in the neutral zone (Fig. 305), the brushes on the slip rings are equivalent to those on the commutator. The P.D. between the slip rings must, at this moment, be equal to the P.D. applied to the D.C. side of the converter. From equation (79) on page 124, we have then

$$\hat{e} = \frac{p}{a} \cdot \Phi n Z \cdot 10^{-8},$$

where Z is the total number of wires on the armature. Now, the number of wires in one branch of the armature winding is $\frac{Z}{2a}$, so that

$$z' = \frac{Z}{2a}.$$

Further, $$f = p \cdot n.$$

The above equation for the maximum E.M.F. on the A.C. side may therefore be written as follows:

$$\hat{e} = 2\Phi f z' . 10^{-8}.$$

We could have found this value directly from equation (153).

When the coil to which the slip ring connection is made passes under a pole (Fig. 306), each branch of the armature becomes the seat of two opposing electromotive forces, proportional respectively to the arcs y and x. The E.M.F. decreases until, when the slip ring connections come under the mid-points of the poles, it is equal to 0.

To find the effective E.M.F. of a single-phase converter, we must put 2γ equal to 1 in equation (152) on page 302. If we assume that $\beta = \frac{2}{3}$, which is about the usual value of this ratio, equation (151) on page 301 becomes

$$\delta = 2\gamma + \beta - 1 = \tfrac{2}{3}.$$

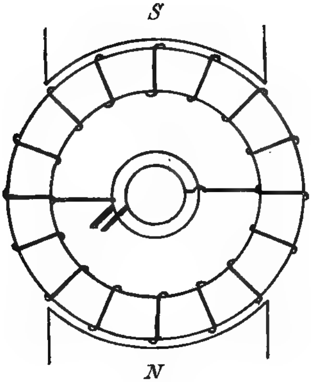

Fig. 305

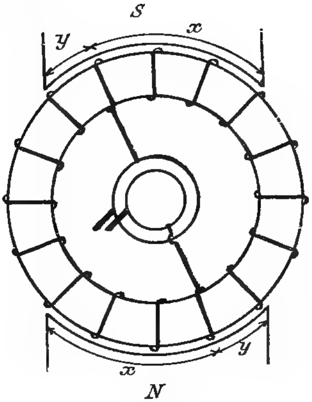

Fig. 306

From equation (152) we get for the effective value

$$e = \hat{e}\,.\sqrt{2\gamma - \frac{\beta}{3} - \frac{\delta^3}{3\beta^2}} = 0{\cdot}745\,\hat{e},$$

and, substituting the value of $\hat{e}$ found above,

$$E = 0{\cdot}745 . 2\Phi f z' . 10^{-8} = 1{\cdot}49 . \Phi f z' . 10^{-8}.$$

The effective alternating E.M.F. is therefore, in our example, 0·745 of the P.D. between the D.C. brushes, or the D.C. pressure is 1·34 times the A.C. pressure. This constant ratio of the pressures is very important. If we wish to convert high-pressure alternating current into low-pressure direct current, we must first transform the high-pressure alternating current to alternating current of a pressure equal to 0·745 of the required D.C. pressure. This is done by means of ordinary static transformers. Commutation difficulties limit the D.C. pressure to about 1000 volts, generally speaking, and high pressures are not suitable for lighting purposes, except in exceptional cases. An alternative to the use of static transformers and rotary converters is the employment of motor generators, in which the high-pressure alternating current is supplied directly to an A.C. motor which is coupled to an ordinary D.C. dynamo. Rotary converters have been used almost exclusively in America, whereas, on the Continent they are quite the exception. In England, both systems have been largely adopted, and their relative merits are continually under discussion.

CHAPTER XIII

96. The E.M.F. diagram for an A.C. generator.—97. The ampere-turns diagram for an A.C. generator.—98. Calculation of armature reaction.—99. Experimental determination of armature reaction and armature leakage.—100. Predetermination of exciting current and pressure regulation.—101. Effect of polar leakage.

96. The E.M.F. Diagram for an A.C. Generator.

If the assumption be made that the coefficient of self-induction of the armature of a machine has a constant value for all conditions of load and excitation, the vector diagram is extremely simple. The electromotive force E_1, induced in the armature by the rotating magnetic flux, forms the hypotenuse of a right-angled triangle, one side of which is equal to the total inductive pressure drop, both internal and external, while the other side is equal to the sum of the internal and external ohmic pressure drops. The direction of the current vector in Fig. 307 is vertically upwards. The self-induction induces an E.M.F. lagging 90° behind the current, which must be counterbalanced by a component OA of the E.M.F. OA consists of two parts, one $E_{s\,\text{int}}$ lost in the armature, and the other $E_{s\,\text{ext}}$ overcoming the back E.M.F. in the external circuit. The component AB consists similarly of a part AD lost in the machine, and a part DB which overcomes the external ohmic resistance. To find the terminal pressure we must complete the rectangle FDA and join OF and FB. The latter represents the terminal pressure V, for it is the hypotenuse of a right-angled triangle with sides equal to the external inductive and ohmic pressure components. The line OF represents the total drop of pressure in the generator. We have already seen in Section

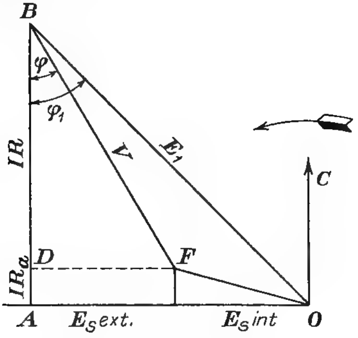

Fig. 307

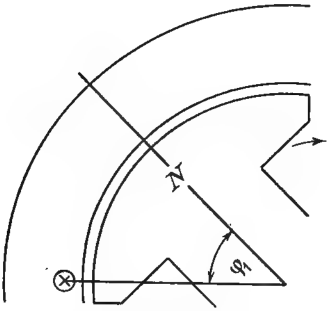

Fig. 308

77 that this drop cannot be subtracted algebraically from the E.M.F., but must be subtracted geometrically, as shown in Fig. 307.

The angle *FBD* is the phase difference ϕ between the terminal pressure and the current; it is the angle of lag in the external circuit, and would be determined by means of an ammeter, voltmeter and wattmeter, suitably connected to the main generator leads. The angle ϕ_1, on the other hand, is the angle by which the current lags behind the induced E.M.F. To make the meaning of this latter angle quite clear, a pole is drawn in Fig. 308 in the position it occupies at the moment of maximum armature current. Were the angle ϕ_1 equal to 0, the current would be a maximum at the moment when the centre of the pole coincided with the armature conductor or coil-side. In the present case the current lags behind the E.M.F. so that the pole has got past the conductor by an angle ϕ_1 before the current in the conductor reaches its maximum value. It is, of course, only in a 2-pole machine that the actual angle ϕ_1 (Fig. 308) is equal to the angle ϕ_1 in the vector diagram (Fig. 307). If a machine has p pairs of poles, the actual angle will only be equal to ϕ_1/p, since one revolution of the generator will correspond to p complete periods or revolutions of the vector diagram.

Before we can construct Fig. 307 we must determine both the open-circuit characteristic and the self-induction of the generator armature. The characteristic can easily be found experimentally by running the generator at normal speed and observing the terminal pressure, while the exciting current is varied, either by means of a rheostat or by altering the P.D. of a separate exciting dynamo. A curve is drawn with abscissae equal to the exciting current or to the ampere-turns per pair of poles, and ordinates equal to the terminal P.D., which, on open circuit, is equal to the induced E.M.F.

The curve obtained in this way is variously known as the magnetisation curve, static characteristic, or open-circuit characteristic. Such a curve is shown in Fig. 332. So long as the magnetic flux is small the curve is practically a straight line, since the air-gap forms the principal part of the reluctance, and the flux, and consequently the E.M.F., is proportional to the exciting current. As the iron becomes saturated, its permeability decreases, and the curve bends over more and more.

Having determined the open-circuit characteristic either experimentally or by calculation, as described in Section 57, we have now to find the self-induction. We shall assume that it is independent of the relative position of poles and armature winding. That this assumption is hardly justified can be seen from Figs. 309 and 310, where the paths of the lines of force are indicated for two positions of the rotating field system. We are here considering the flux produced by the armature currents only, and it is evident that the self-induction of the armature is not constant during a period. Moreover, the phase relation of the current, that is, the angle ϕ_1, must have an effect on the self-induction, as it is evidently of some importance whether the current in a coil-side has its maximum value when it is opposite a pole or when in the neutral zone.

If we neglect these effects, we can find the self-induction of the armature, as we would that of any coil or piece of apparatus, by applying an external alternating P.D. to the terminals of the stationary machine. The P.D. must be of so low a value that the normal full-load current flows through the armature*. If we neglect the small resistance of the armature, the applied P.D. is equal to the back E.M.F. of self-induction, and the quotient obtained by dividing this P.D. by the current is equal to the reactance $2\pi fL$.

By means of the open-circuit characteristic and the self-induction of the armature, we can now predetermine the terminal pressure for a given load and a given exciting current. The power-factor $\cos\phi$ of the load must also be known. From the open-circuit curve we find the value of E_1 for the given exciting current, and, with the centre O, describe an arc with a radius $OB = E_1$ (Fig. 311). The triangle OCF is made with the base $OC = 2\pi fL.I$ and the side $CF = I.R_a$. To construct the diagram for a given power-factor $\cos\phi$ in the external circuit, a vector is drawn from O, making an angle ϕ with the vertical. This indicates the direction of the vector of terminal P.D. and a line is drawn parallel through the point F. This line cuts the circle at the point B. By referring to Fig. 307 we see that FB represents the required terminal pressure V, the letters OFB having the same significance in both diagrams. The algebraic difference between OB and FB represents the fall of terminal pressure between no-load and full load. In order that this difference may be easily seen we have drawn another circle with the same radius E_1, but with the point F as centre. The line FB is produced to meet this second circle at G, and the intercept BG is evidently equal to the pressure drop $E_1 - V$. The figure shows plainly that this drop of pressure increases as the point B moves round to the left, that is, as the angle ϕ increases.

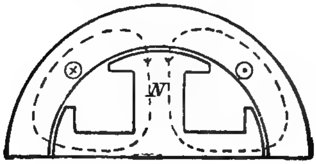

Fig. 309

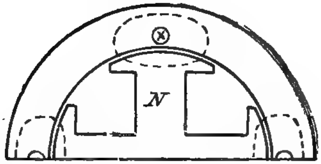

Fig. 310

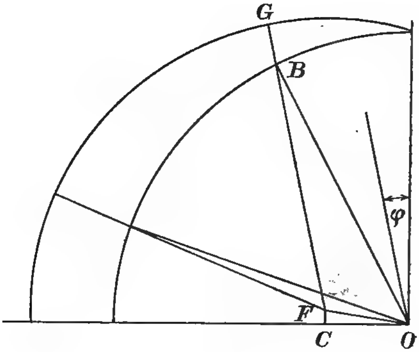

Fig. 311

The pressure drop has a maximum value when the angle ϕ is equal to 90° (Fig. 312). The external circuit will then consist entirely of inductive resistance, and the P.D. will be equal to the back E.M.F. of self-induction in the external circuit. The internal pressure drop *OF* is subtracted almost algebraically from the E.M.F. *OB*, and the terminal pressure *FB* reaches its lowest value.

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* This is often a dangerous experiment to carry out on a large machine, since the alternating flux produced by the armature induces very high electromotive forces in the field windings.

If the load consists entirely of glow-lamps, the power-factor cos ϕ will be equal to 1, and the current will be in phase with the terminal pressure. The diagram for this case is given in Fig. 313, from which it is seen that the fall of terminal pressure between open circuit and full load is very small. In a modern generator the pressure will fall about 5 per cent. in such a case, while on an external circuit with a power-factor of 0·75, the drop will be nearly 20 per cent.

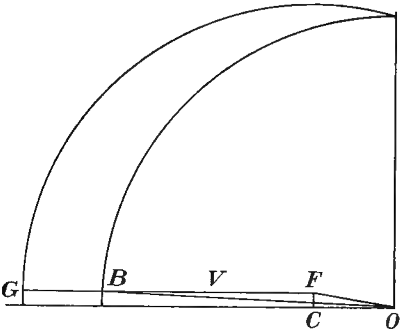

Fig. 312

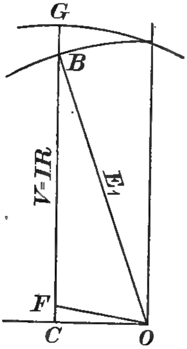

Fig. 313

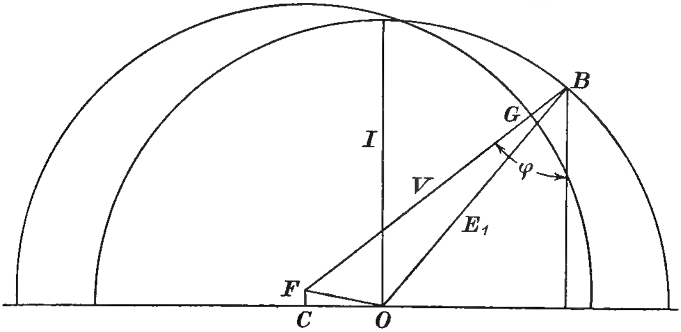

Fig. 314

A case of special interest is that in which the external circuit causes the current vector *I* to lead ahead of the terminal pressure vector *V* by an angle ϕ (Fig. 314). This can occur when the circuit contains condensers or over-excited synchronous motors. In such a case the terminal pressure may exceed the E.M.F. (Fig. 314). This striking result will be more fully dealt with in Sections 105 and 109. We might mention here, however, that the back E.M.F. of an A.C. motor may be greater than the applied terminal P.D., or even than the E.M.F. of the generator. If, for example, the E.M.F. of the generator is 1000 volts, while the back E.M.F. of the motor is 1200 volts, their common terminal pressure will automatically adjust itself to some intermediate value. The state of affairs, so far as the generator is concerned, is shown in Fig. 314.

97. The Ampere-turns Diagram.

The diagram of electromotive forces described in the previous section gives one a very clear insight into the relations determining the terminal pressure under various conditions of load. If it be used, however, to determine the excitation necessary to maintain the P.D. between the terminals at various loads, the calculated results will be found to differ from the results obtained by actual experiment. This is due to the fact that the drop of pressure is largely caused by a decrease in the magnetic flux entering the armature. This decrease in the flux is caused by the demagnetising effect of the armature current, which opposes, to a greater or less degree, the ampere-turns on the field-magnets, and therefore leads to a decrease in the E.M.F. induced in the armature.

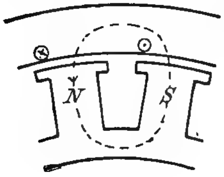

Fig. 315

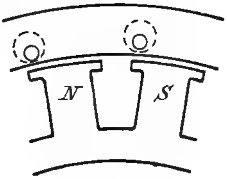

Fig. 316

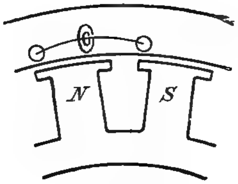

Fig. 317

We must now divide the loss of pressure, which we have attributed to the self-induction of the armature, into two parts. The first part, which we shall call E_2, is caused by the flux which follows the path indicated in Fig. 315. It is assumed in the figure that the field-magnets are rotating clockwise, and that the current lags behind the induced E.M.F. by such an amount that the former has just reached its maximum value in the figure. It is evident that the magneto-motive force of the armature winding is opposed to that of the poles.

The second part of the loss of pressure we shall call E_l and refer to as the armature leakage drop. This is due to the flux which is produced round the armature conductors, but which does not link the field winding. A part of this flux follows the path shown in Fig. 316, that is, it encircles the conductor which is embedded in the slot, some of it even crossing the air-gap and running through the pole face. The remainder of this leakage flux encircles the end connections, that is, those parts of the armature winding which project beyond the iron core, as shown in Fig. 317.

By keeping these two sources of pressure loss separate, we obtain Fig. 318 in the place of Fig. 307. E_1 is the E.M.F. which would be induced by the given field current were there no armature reaction. The effect of the armature reaction in weakening the flux is represented by the drop E_2, so that the E.M.F. corresponding to the flux actually entering the armature is represented by E. From this we have to take the leakage drop E_l and the $I.R_a$ drop, leaving the pressure V across the terminals.

It is evident that the electromotive forces E_1 and E_2 are purely fictitious, and, further, that the very lines of force by which they would be induced do not exist, except as imaginary components of the resultant flux which produces E. As a matter of fact, the E.M.F. E is also fictitious, as the effect of the self-induction or armature slot leakage is to prevent the flux which enters the armature from cutting the conductors by driving a part of it across the slots. Hence, although the flux which crosses the gap is sufficient to produce an E.M.F. E, the flux which actually cuts the conductors is only sufficient to produce the E.M.F. BC in Fig. 318. Since the poles are excited by a steady direct current, and the armature reaction is rapidly varying between 0 and a maximum, we might be led to expect a pulsating flux. This is prevented, however, to a great extent by the self-induction of the field windings and the eddy currents in the poles, so that the flux passing through the poles has a practically constant value Φ as the resultant effect of the constant ampere-turns X_1 on the field and the mean value X_2 of the pulsating armature reaction.

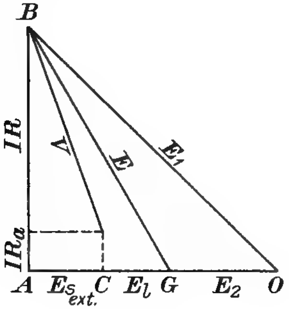

Fig. 318

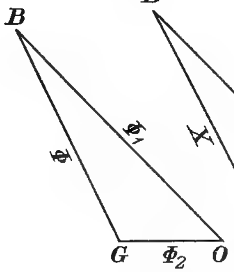

Fig. 319

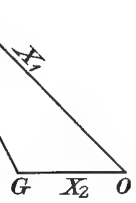

Fig. 320

Hence, neither the electromotive forces E_1 and E_2 in Fig. 318, nor the fluxes Φ_1 and Φ_2 in Fig. 319, have any real existence. The ampere-turns X_1 and X_2, on the other hand, are really present, and act on a common magnetic circuit, in which their resultant X produces a flux Φ. The E.M.F. E corresponding to X ampere-turns is found from the open-circuit characteristic.

If the load be suddenly thrown off the machine, the ampere-turns X_2 of armature reaction entirely disappear, and the points G and O in the figure are brought together. The flux, and with it E, increases to a value corresponding to X_1 ampere-turns. The flux and E.M.F. will not increase, however, in the ratio of X_1 to X, on account of magnetic saturation.

98. Calculation of Armature Reaction.

The calculation of the armature reaction is more difficult in the case of single-phase generators than in the case of two or three-phase machines. If the armature current lags 90° behind the E.M.F., the current reaches its maximum value at the moment that the coil-side is midway between the poles (Fig. 322). The lines of force due to the armature current are shown dotted and are evidently in direct opposition to the field-magnets. The conditions both before and after Fig. 322 are shown in Figs. 321 and 323; the fact that the current is smaller is indicated by lighter crosses and dots on the armature conductors.

If there are z' wires in series on the armature, then the wires per pole or the turns per pair of poles will be $\frac{z'}{2p}$. For the armature reaction at the moment represented in Fig. 322, we have

$$X_{2\,\max} = \frac{z'}{2p} \cdot \hat{i} \text{ amp.-turns per pair of poles.}$$

This is the maximum value, and, assuming a sine-wave current, the mean armature reaction will be

$$X_2 = \frac{z'}{2p} \cdot \frac{2}{\pi} \cdot \hat{i} = \frac{z'}{2p} \cdot \frac{2}{\pi} \cdot \sqrt{2} \cdot I,$$

or

$$X_2 = 0{\cdot}9 \, \frac{I \cdot z'}{2p} \text{ per pair of poles,}$$

where I is the effective or root-mean-square value of the current.

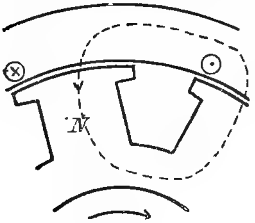

Fig. 321

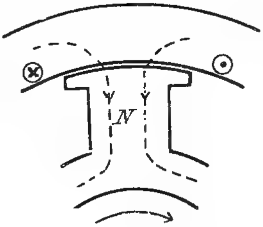

Fig. 322

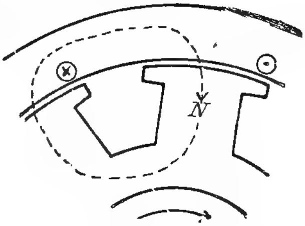

Fig. 323

We have not taken into account the fact that the pole is directly under an armature coil during a part of the period only*. It is more to our purpose, however, to understand the various factors affecting armature reaction than to calculate them with great accuracy. The calculation is, after all, more or less uncertain, as the flux pulsates appreciably, in spite of the self-induction of the field coils and the eddy currents in the poles.

The calculation is much more trustworthy in the case of polyphase machines, in which there are two or three separate windings on the armature. Although we have not yet studied the subject of polyphase currents, we may make use of the simple fact that, when the current in any coil-side of a three-phase winding is a maximum, the current in each of the adjacent coil-sides has half this maximum value. If the current is lagging 90°, the coil-side midway between the poles carries the maximum current, while the coil-side on either side of it carries a current in the same direction but of half the amount (Fig. 324).

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* By taking everything into consideration Kapp has found a coefficient 0·736 instead of 0·9 when $\beta = \frac{2}{3}$, and 0·8 when $\beta = \frac{1}{2}$.

In order to determine the armature reaction per pair of poles, we must first find the ampere-turns in each coil. If there are z' wires per phase, the wires per coil-side will be $\frac{z'}{2p}$, and the ampere-turns of coil 2—5 will be $\frac{z'}{2p} \cdot \hat{i}$. Coils 3—6 and 1—4 will each have $\frac{1}{2} \cdot \frac{z'}{2p} \cdot \hat{i}$ ampere-turns. The flux which passes between holes 3 and 4 links all three of these coils, and has a M.M.F.

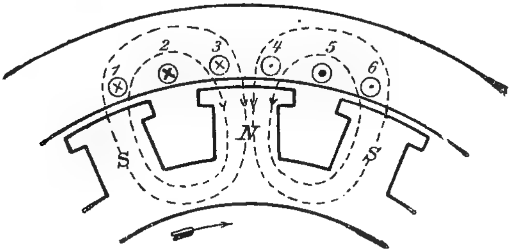

Fig. 324

due to $\frac{2z'}{2p} \cdot \hat{i}$ ampere-turns. The flux which passes to the left of 3 and to the right of 4 links only the coil 2—5, as shown by the dotted lines in the figure. It is produced by $\frac{z'}{2p} \cdot \hat{i}$ ampere-turns. If the polar arc is two-thirds of the pole-pitch, the space between holes 3 and 4 covers half the pole face, so that

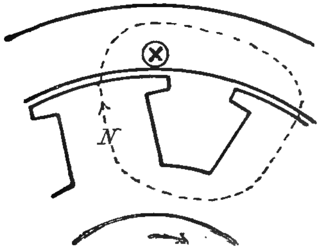

Fig. 325

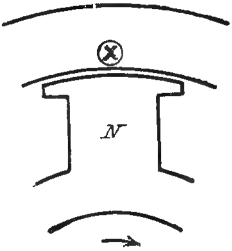

Fig. 326

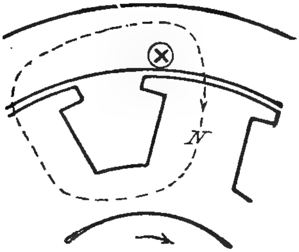

Fig. 327

half the pole face is crossed by flux produced by $2\frac{z'}{2p} \cdot \hat{i}$ ampere-turns, and the other half by flux produced by $\frac{z'}{2p} \cdot \hat{i}$ ampere-turns. Taking the mean value for the whole pole, the armature reaction of a three-phase generator is

$$X_2 = 1{\cdot}5\,\frac{z'}{2p}\,\hat{i} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (155).$$

It might be objected that we have only considered the conditions existing at the moment represented in the figure. As the pole wheel rotates, however, the current in coil 2—5 decreases and that in coil 3—6 increases, so that the armature reaction is hardly altered. When the wires 3 and 6 are midway between the poles the current in these wires will be at its maximum value, so that the conditions represented in the figure are continually recurring. As already mentioned, these results make no claim to mathematical exactness, but merely serve to make the effect of armature reaction as clear as possible.

We have assumed, so far, that the current lagged 90° behind the E.M.F., so that the armature reaction was directly opposed to the field-magnets. We must now consider the effect of power-factor on the armature reaction. If the current is in phase with the E.M.F., the current in any coil-side is a maximum when that coil-side is approximately under the middle of the pole (Fig. 326). At this moment the current can produce no flux round the field magnetic circuit. A moment earlier the growing current produced a flux which strengthened the field, while a moment later the decreasing current weakens the field. The net result is that the armature neither strengthens nor weakens the field.

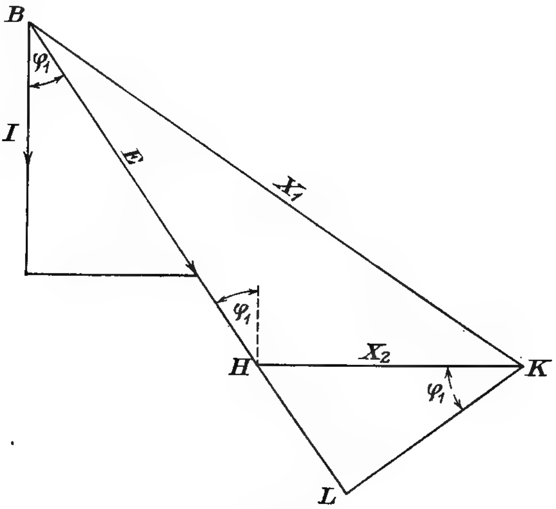

Fig. 328

If the current lags behind the E.M.F. by an angle ϕ_1, the effect of armature reaction is shown in Fig. 328. The vertical line represents the current I lagging behind the electromotive force E. This triangle corresponds to BAG in Fig. 318. BH is the excitation X which would produce E on open circuit, and it must therefore be the resultant of X_1 and X_2. The triangle BKH corresponds to Fig. 320. It is evident that the armature reaction X_2 becomes of increasing importance as ϕ_1 approaches 90°. When $\phi_1 = 90°$, X_1 and X_2 will be co-linear, and X will equal $X_1 - X_2$. This corresponds to Fig. 322.

Another way of looking at it is as follows: Of the total armature reaction only a component $HL = X_2 . \sin\phi_1$ acts in the direction of the resultant excitation. If we represent this direct demagnetising component by X_d, we have

$$X_d = X_2 \, . \, \sin\phi_1 = HL.$$

It can be seen from the figure that $HB + HL$ is very nearly equal to X_1, so that

$$X_1 = X + X_d,$$

or

$$X = X_1 - X_d.$$

For negative values of ϕ_1, that is, for leading currents, the demagnetising effect also becomes negative. This can be shown in Fig. 322 by reversing the direction of the armature current. We can sum this up by saying that lagging currents act demagnetisingly, while leading currents act magnetisingly. This explains the rise of P.D. with increasing load in Fig. 314.

99. Experimental Determination of Armature Reaction and Armature Leakage.

Generally speaking, the results of experiments will be found to agree with the calculations of the previous section. An important experiment in this connection is the short-circuit test of an alternator. The armature is short-circuited through an ammeter and the machine is run at its normal speed with very weak field excitation, as the armature current would otherwise be excessive. The field current is varied, and simultaneous readings are taken of field and armature current. It may be safely assumed that the armature resistance is very small and that the self-induction due to slot-leakage, etc., is comparatively small, especially in a modern machine with open slots, large air-gap and several slots per pole and phase.

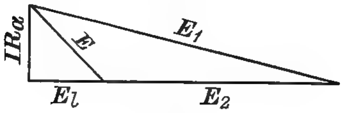

Fig. 329

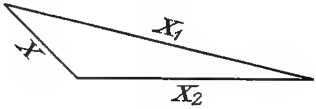

Fig 330

The diagram for short circuit will then be like Fig. 329, in which

E_1 is the E.M.F. which would be given by the field excitation if the permeability were constant,

E_2 is the E.M.F. which would be produced by the flux caused by armature reaction under like conditions,

and E is the actual E.M.F. induced by the resultant flux, which has to drive the current against the resistance and inductance of the armature.

The corresponding diagram of ampere-turns is given in Fig. 330. It is evident that the armature reaction on short circuit is almost equal to the field excitation X_1.

It is not necessary to know the number of turns on either the armature or field, but a curve can be drawn showing the relation between the current in the short-circuited armature and the field current (Fig. 331). This short-circuit characteristic tells us the amount of field current neutralised by the reaction of any given armature current. The curve is almost exactly a straight line. The field current is not exactly neutralised, however, but gives a resultant X as shown in Fig. 330.

In order to calculate the field current required to give a certain terminal P.D. on a given load, it is necessary to know, in addition to the armature reaction X_2, the self-induction L of the armature winding, due to slot-leakage, etc. This is a very difficult quantity to calculate and considerable experience is necessary in applying the various formulae which enter into the calculation.

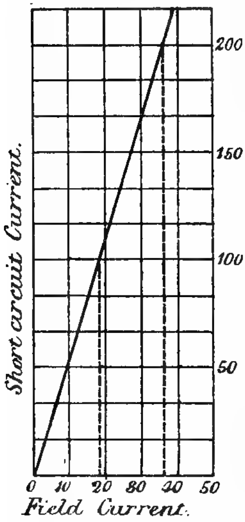

Fig. 331

The method which we shall now describe is due to Potier. To apply the method to a generator, it is necessary to make an experiment which we have not yet mentioned; this consists in loading the machine with a very inductive load, made up, say, of choking coils, so that the armature supplies a heavy lagging current with a power-factor as near as possible to zero. We shall illustrate the method by applying it to an actual example, viz. the three-phase generator shown in Fig. 289. The figures for this machine were given by Heyland in the *E. T. Z.* for 1900. Its short-circuit characteristic is given in Fig. 331 and the open-circuit characteristic in Fig. 332. When loaded with choking coils, the field current was 150 amperes, the armature current 100 amperes and the terminal pressure 2140 volts. This point A is plotted on Fig. 332, as is also the point on the base line representing a field current of 18 amperes, since this is the field current which we see from Fig. 331 would give 100 amperes on short circuit, that is, when the terminal P.D. is equal to 0. The lower curve is drawn through all such points representing the relation between terminal pressure and field current for a wattless current of 100 amperes. The manner in which this curve has been drawn will appear immediately.

A horizontal is drawn through the point A and made equal to the field current on short circuit, viz. $OB = 18$ amperes. Through the point D, thus obtained, a line is drawn parallel to the lower portion of the open-circuit characteristic. A perpendicular is dropped from the point F where this line cuts the characteristic. A right-angled triangle FGA is obtained in this way, which fits in between the two curves over the whole range. The side FG is always vertical, and represents, therefore, a certain difference of pressure; while the side GA is always horizontal and represents a certain field current or excitation.

If, when the machine is working on the inductive load at the point A, we could, in some way, do away with the demagnetising effect of the armature reaction, the field current could be reduced by some such amount as AG, without lowering the terminal pressure. Further, if we could then do away with the self-induction of the armature coils, the drop of pressure due to this cause would disappear, and the terminal pressure would rise by some such amount as FG. Since we have successively removed both armature reaction and self-induction, the machine is now on open circuit and the point F is a point on the open-circuit characteristic. Now, both armature reaction and self-inductive drop are proportional to the armature current, and are constant when the current is constant, as it is for all points on the lower curve in Fig. 332. Further, as the current is wattless for every point on the curve, the armature demagnetising effect must be subtracted algebraically from the field excitation, and the drop must be subtracted algebraically from the induced E.M.F. Hence, there must be some such right-angled triangle which will fit in everywhere, and it is easily seen that FGA is the only possible one. It follows, therefore, that the armature reaction due to a current of 100 amperes is represented by AG, and the self-inductive

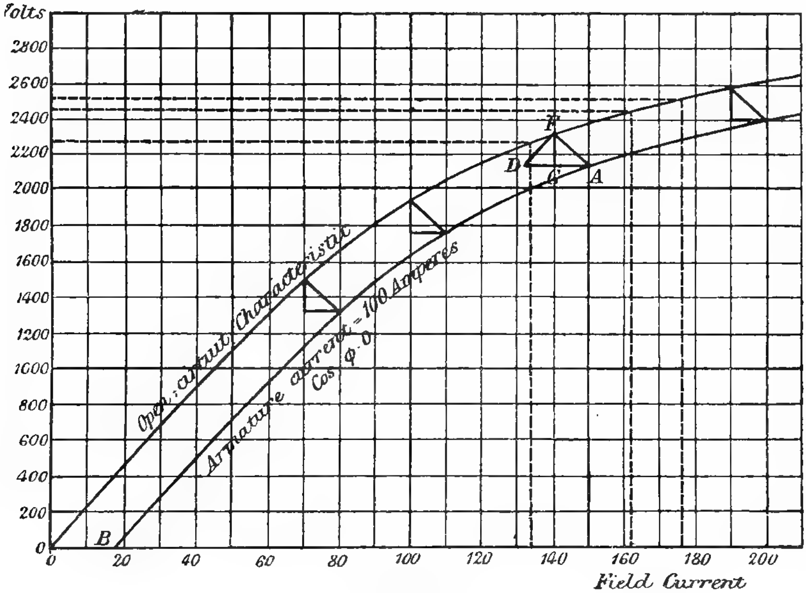

Fig. 332

drop due to slot-leakage, etc., by FG. Since they are both proportional to the current, the corresponding values for other currents can be readily found. In our case

$$X_2 = AG = 10 \text{ amperes},$$

which is a tenth of the armature current, and

$$E_l = FG = 170 \text{ volts},$$

whence

$$\omega L = 2\pi f L = \frac{E_l}{I} = 1{\cdot}7 \text{ ohms}.$$

How these quantities can be used to determine the terminal pressure under any condition of load of any power-factor will be explained in the next section.

100. Predetermination of Exciting Current and Pressure Regulation.

We now proceed to determine the excitation necessary to maintain a terminal pressure of 2200 volts on the machine experimented on in the previous section, under the following conditions:

1. A load of 200 amperes and $\cos\phi = 1$.
2. ,, ,, ,, $\cos\phi = 0{\cdot}8$.
3. ,, ,, ,, $\cos\phi = 0$.

We shall also determine in each case the value reached by the terminal pressure when the load is suddenly switched off, and we shall express the increase of P.D. as a percentage of the normal P.D., i.e. 2200 volts.

The armature resistance R_a is 0·22 ohm.

In the first case $\cos\phi = 1{\cdot}0$, so that the external load is non-inductive, and the whole terminal P.D. is used in overcoming ohmic resistance. The ohmic pressure drop in the armature is

$$I.R_a = 200.0{\cdot}22 = 44 \text{ volts.}$$

The total E.M.F. must have a component in phase with the current, equal to the sum of these ohmic pressure drops. This is represented by AB in Fig. 333, where

$$AB = 2200 + 44 = 2244 \text{ volts.}$$

The pressure required to overcome the back E.M.F. due to the self-induction of the armature is represented by GA, where

$$GA = I.\omega L = 200.1{\cdot}7 = 340 \text{ volts.}$$

This is at right angles to the ohmic drop, and we get for the total E.M.F.

$$GB = \sqrt{2244^2 + 340^2} = 2270 \text{ volts.}$$

This E.M.F. is seen from the open-circuit characteristic to require a resultant excitation of 134 amperes, so that

$$X = BH = 134 \text{ amperes.}$$

We have seen that the field current neutralised by armature reaction is a tenth of the armature current, so that in our case

$$X_2 = 20 \text{ amperes.}$$

Adding X and X_2 vectorially, as in Fig. 328, we get for the field excitation X_1, the length BK in Fig. 333. By scaling this off on the diagram, we find that

$$X_1 = 138 \text{ amperes.}$$

This is the necessary field excitation on non-inductive load. If the load is suddenly switched off so that the machine is on open circuit, it can be seen from Fig. 332 that the terminal pressure will rise to 2310 volts, since this is the P.D. for an excitation of 138 amperes. The rise of terminal pressure on switching off the load is therefore 2310 − 2200 = 110 volts, which is 5 per cent. of the terminal pressure. This is a very good result.

We now proceed to the second case, in which the power-factor is 0·8. We have (see page 253)

$$\begin{aligned} I.R_a &= 200.0{\cdot}22 \quad \ldots \quad \ldots &= 44 \text{ volts,} \\ I.R_{\text{ext}} &= V.\cos\phi = 2200.0{\cdot}8 &= 1760 \quad ,, \\ E_{s\text{ext}} &= V.\sin\phi = 2200.0{\cdot}6 &= 1320 \quad ,, \\ E_{s\text{int}} &= \omega L.I = 1{\cdot}7.200 \ldots &= 340 \quad ,, \end{aligned}$$

The sum of the ohmic pressure is 1804 volts, and the total inductive drop is 1660 volts. The total induced E.M.F. is represented by BG in Fig. 334, where

$$BG = E = \sqrt{1804^2 + 1660^2} = 2455 \text{ volts.}$$

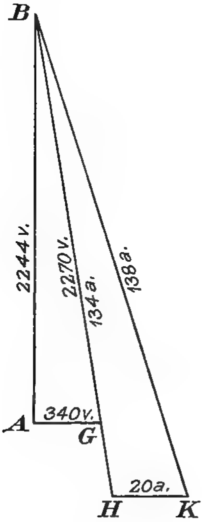

Fig. 333

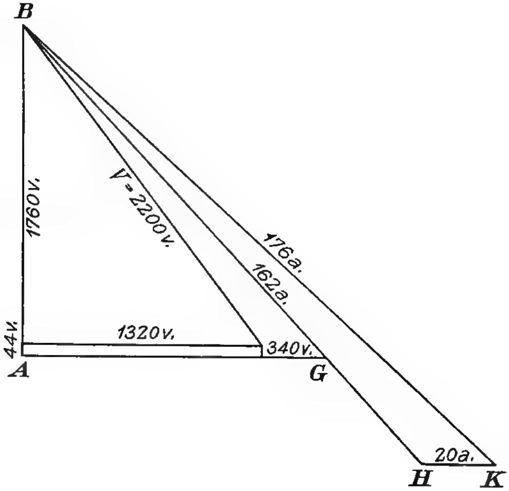

Fig. 334

The resultant excitation corresponding to this E.M.F. is seen from Fig. 332 to be 162 amperes, or

$$X = 162 \text{ amperes.}$$

The armature demagnetising effect $KH = 20$ amperes is added geometrically giving a field excitation BK, where

$$BK = X_1 = 176 \text{ amperes.}$$

Hence, to maintain the terminal P.D. at 2200 volts with the lagging armature current, it is necessary to increase the field current to 176 amperes. On throwing off the load the P.D. will naturally rise to a higher value than before, and Fig. 332 shows this value to be 2520 volts, corresponding to a field current of 176 amperes. The rise of the terminal pressure is therefore 2520 − 2200 = 320 volts, which is 14·6 per cent. of the normal pressure.

The last case which we shall consider is one which is hardly likely to occur under normal working conditions, viz.

$$\cos\phi = 0.$$

The whole 2200 volts will be used in neutralising the back E.M.F. of the external purely inductive load. By adding the internal inductive drop $I.\omega L$ we get the total inductive pressure to be $2200 + 340 = 2540$ volts. The only ohmic pressure component is the 44 volts internal loss, which is practically negligible, so that the E.M.F. is equal to 2540 volts. The no-load characteristic gives a resultant excitation of 180 amperes, or

$$X = 180 \text{ amperes.}$$

The armature demagnetising effect of 20 amperes can be added algebraically, since the armature and field are in direct opposition. For the field excitation we have then

$$X_1 = 180 + 20 = 200 \text{ amperes.}$$

On switching off the load the terminal pressure will now rise to 2620 volts, an increase of 420 volts or 19·1 per cent. of the normal pressure.

The results thus obtained will be modified, however, if the machine has considerable magnetic leakage from pole to pole. The effect of this leakage is considered in detail in the next section. The machine considered will have little polar leakage, since the poles are short and separated from each other by a considerable distance.

101. Effect of Polar Leakage.

It might appear at first sight as if the effect of magnetic leakage between the poles had been already taken into consideration in the open-circuit characteristic. If, however, a machine be carefully designed for a given load, and the diagrams be constructed as already explained, it will be found that the actual relations when loaded differ considerably from those calculated from the diagrams.

This difference is due to the fact that a greater resultant excitation is necessary to produce an E.M.F. when the machine is loaded than to produce the same E.M.F. on open circuit. To understand this, we must consider the machine under the two conditions of open circuit and full load, with the same flux entering the armature in each case, and consequently the same E.M.F. being induced. The number of ampere-turns necessary to drive this flux through the air-gap teeth and armature is equal to the sum $X_g + X_t + X_a$. This same number of ampere-turns acts across the interpolar space on open circuit, and drives the leakage flux from pole-shoe to pole-shoe.

If the component of the armature reaction which acts demagnetisingly when the machine is on full load be X_d, the above ampere-turns must be increased to $X_g + X_t + X_a + X_d$, in order to maintain the same armature flux. Since this increased difference of magnetic potential acts between the pole-shoes, the leakage flux will be increased in the same proportion. The total flux passing through the pole will be increased by this increased leakage, and the ampere-turns required to drive the flux through the pole will be increased. Hence, to maintain the same E.M.F., the field excitation has not only to be increased by the amount X_d when the machine is loaded, but also by an amount equal to the increase in X_m. We see, therefore, that the resultant excitation taken from the open-circuit characteristic for a given E.M.F. is too small if the machine is loaded.

Another way of looking at the matter is to consider the leakage as increasing the armature reaction and thus causing a further drop in the E.M.F. for a constant field current.

It is evident that the polar leakage will have little effect if the pole-cores are unsaturated, and X_m is consequently small. In modern machines, however, the poles are purposely worked at a very high flux-density, in order to minimise the fluctuation of terminal pressure with changing load. In such machines the polar leakage has a marked effect, and the constancy of terminal pressure will depend largely on the reduction of this leakage to a minimum.

The exact calculation of the effect of the polar leakage is complicated by the varying value of the permeability of the iron. We shall first put the question in the following form: How many ampere-turns would be necessary on open circuit to produce the same flux in the poles as actually passes through them on full load?

We have therefore to consider the machine under two conditions, viz. on full load and on no-load, with the assumption that the flux in the poles is the same in each case. We make the further assumptions that the whole leakage occurs between the pole-tips, and that the reluctance of the armature core and teeth is negligible compared with that of the air-gap.

If R be the reluctance of the air-gap,
and R_l ,, ,, leakage path,

then we can make out the following table:

	Ampere-turns used in overcoming reluctance		Air-gap flux	Leakage flux	Total ampere-turns on pole
	of air-gap	of leakage path			
At no-load	X_{g_0}	X_{g_0}	$\Phi_0=\dfrac{0{\cdot}4\pi X_{g_0}}{R}$	$\Phi_{l_0}=\dfrac{0{\cdot}4\pi X_{g_0}}{R_l}$	$X_0=X_{g_0}+X_m$
At full load	X_g*	X_g+X_d	$\Phi=\dfrac{0{\cdot}4\pi X_g}{R}$	$\Phi_l=\dfrac{0{\cdot}4\pi\,(X_g+X_d)}{R_l}$	$X_1=X_g+X_d+X_m$

Since the flux in the pole is the same in each case, the decrease in the useful flux must be equal to the increase of leakage flux, so that

$$\Phi_0-\Phi=\Phi_l-\Phi_{l_0} \quad \text{....................}(156).$$

Since the useful flux, that is, the flux crossing the air-gap, is less when the machine is loaded, the ampere-turns required to drive the flux across the air-gap will decrease, viz. from X_{g_0} to X_g. As the flux in the pole itself is kept constant X_m is the same in each case, so that the resultant ampere-turns are reduced from $X_0 = OB$ to DB, where $OD = X_{g_0} - X_g$. The total ampere-turns X_1 on the pole will, of course, have to be increased so as to exceed this resultant DB by an amount equal to the demagnetising ampere-turns X_d. This, however, does not concern us at present, as one would naturally subtract X_d from X_1, and obtain the resultant DB before referring to the characteristic. Turning now to the useful flux, we see that, since the curve near the origin is practically the magnetisation curve for the air-gap only, $O'D$ will represent the decrease of flux crossing the gap owing to the decrease $OD = X_{g_0} - X_g$ in the air-gap ampere-turns. Hence the useful flux is reduced from AB to AF, *if we take O' as the origin, the curve represents the relation between the resultant ampere-turns and the useful flux.*

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* The total ampere-turns acting across air-gap and armature are X_g+X_d. The part X_d is neutralised, however, by the armature ampere-turns, leaving X_g to overcome the reluctance. It is analogous to charging an accumulator.

We must now determine the position of the point O'. To do this we substitute in equation (156) the values given in the table, and obtain

$$X_{g_0} - X_g = X_d \cdot \frac{R}{R+R_l} \quad\text{......................(157).}$$

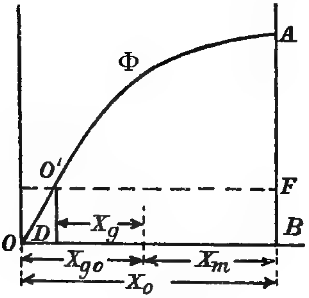

Fig. 335

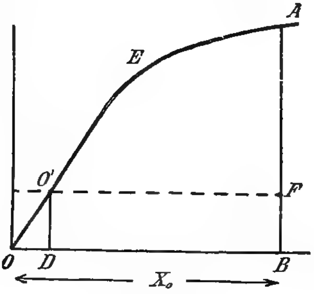

Fig. 336

At no-load the useful and leakage fluxes will be inversely proportional to the reluctances of their respective paths, i.e.

$$\frac{\Phi_0}{\Phi_{l_0}} = \frac{R_l}{R}.$$

This gives for the value of the leakage coefficient at no-load,

$$\nu = \frac{\Phi_0 + \Phi_{l_0}}{\Phi_0} = \frac{R+R_l}{R_l}.$$

From this it follows that

$$\frac{R+R_l}{R} = \frac{\nu}{\nu - 1},$$

and

$$OD = X_{g_0} - X_g = X_d \cdot \frac{\nu - 1}{\nu} \quad\text{....................(158).}$$

Hence, from a knowledge of the open circuit leakage coefficient ν, the value of OD can be calculated for any given armature current and power-factor. It is obviously immaterial whether the above construction be applied to the flux curve (Fig. 335) or to the open-circuit characteristic (Fig. 336).

The effect of leakage on the characteristic obviously depends very much on the point on the curve at which we are working.

CHAPTER XIV

102. Alternator with constant excitation and constant terminal pressure.—103. Synchronising power of armature.—104. The parallel connection of alternators.—105. The effect of field regulation on alternators in parallel.—106. Phase-swinging of alternators.—107. The automatic voltage regulation of alternators.

102. Alternator with Constant Excitation and Constant Terminal Pressure.

If several alternators are connected in parallel to the same bus-bars, or common switchboard terminals, the terminal pressure of each is no longer dependent on itself alone, but is affected by the other machines in parallel with it. If we assume that the other machines are very large compared with the one under consideration, the terminal pressure will not be affected by changes in the latter, so that it may be assumed to have a constant terminal pressure V.

We shall assume, for the sake of simplicity, that the armature demagnetising effect can be included with the self-induction of the armature, due to slot-leakage, etc., by simply increasing the latter. The fundamental diagram in Fig. 307 was drawn on this assumption, and is therefore applicable to the present case. It is drawn again in Fig. 337, in which E_1 represents the induced E.M.F., assumed to be the same on load as on open circuit. So long as the excitation remains unchanged, E_1 is a constant for all loads.

For our purpose we must construct Fig. 337 so that it represents the given machine under certain conditions of load. With O as centre, an arc is struck with radius OB equal to the constant E.M.F. E_1. The line OM is drawn at an angle α to the horizontal, so that

$$\tan \alpha = \frac{I \, . \, R_a}{E_s} = \frac{R_a}{\omega L}.$$

Both E_s and L are used here in the same sense as in Section 96, that is, so as to include the effect of armature reaction. Since E_s is proportional to the current, we can make OC to suit any given current I. A perpendicular erected at C meets OM at F, and this point F is taken as the centre about which to describe an arc with a radius equal to the constant terminal pressure V. This arc cuts the former arc in the point B, which is joined to O and to F, and from which a perpendicular is dropped on to OA.

The line BD should represent the ohmic resistance of the external circuit, multiplied by the current I. Since, however, the alternator is one of several connected in parallel we can no longer speak of its external circuit. We can say, however, that the machine is running under conditions equivalent to an external load with an ohmic drop BD and an inductive drop FD.

Fig. 338 is a similar figure except that it is drawn for a larger current, represented by a larger value of E_s, viz. OC' instead of OC. The values of E_1, V and α are the same in both figures. There are two important points in which the figures differ; the angle γ between the E.M.F. and the terminal pressure has changed, and the total power given out by the machine has changed. Both the angle γ and the power have been increased by the increase of current.

We will not, at this point, inquire into the method by which this increase of load can be effected; whether by some alteration on the generator, or by a direct increase in the output of the engine. The point we must first consider is the relation between the load on the generator and the angle γ. The total electric power generated is

$$P_1 = E_1 . I . \cos(\phi + \gamma).$$

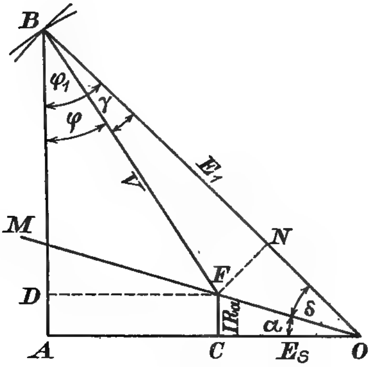

Fig. 337

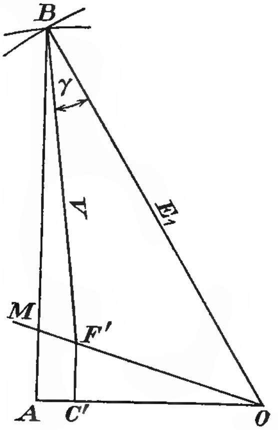

Fig. 338

If we assume that the ohmic resistance of the armature R_a is negligibly small compared with its self-induction, the angle α vanishes and $\cos(\phi + \gamma)$ becomes equal to $\sin\delta$. At the same time OF becomes equal to $I . \omega L$, and we have

$$\sin\delta = \frac{V}{I . \omega L} . \sin\gamma,$$

or

$$P_1 = \frac{E_1 . V}{\omega L} . \sin\gamma.$$

Now, $\dfrac{E_1}{\omega L}$ is the short-circuit current I_0, so that $P_1 = V . I_0 . \sin\gamma$.

The angle γ is the angle by which the terminal pressure vector lags behind the vector of induced E.M.F. We consider the terminal pressure as being nearly in phase with the E.M.F., and not nearly 180° out of phase with it. In a certain sense, however, they are in opposition.

The power generated is therefore proportional to the sine of the angle γ. If the vector V be set up vertically from the point O (Fig. 339), and a horizontal line equal to $V.I_0$ be taken as the diameter on which a semicircle is drawn, then the chord OR, making an angle γ with V, will represent the power P_1.

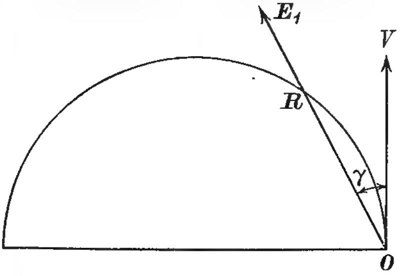

Fig. 339

If the resistance of the armature be taken into account, the relations are as follows:

From Fig. 337

$$P_1 = E_1 I.\cos(\phi+\gamma) = E_1 I.\sin(\alpha+\delta)$$

$$= E_1 I.(\sin\alpha.\cos\delta + \cos\alpha.\sin\delta).$$

Now,

$$\cos\delta = \frac{ON}{OF} = \frac{E_1 - V.\cos\gamma}{I\sqrt{R_a^2+(\omega L)^2}},$$

and

$$\sin\delta = \frac{FN}{OF} = \frac{V.\sin\gamma}{I\sqrt{R_a^2+(\omega L)^2}}.$$

Hence,

$$P_1 = E_1 \frac{\sin\alpha(E_1 - V.\cos\gamma) + V.\sin\gamma.\cos\alpha}{\sqrt{R_a^2+(\omega L)^2}}.$$

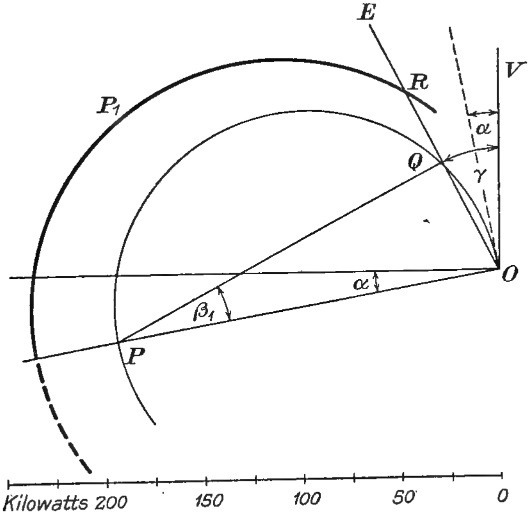

Fig. 340

Now, the short-circuit current is given by the equation

$$I_0 = \frac{E_1}{\sqrt{R_a^2+(\omega L)^2}},$$

and we have

$$V.\sin\gamma.\cos\alpha - V.\cos\gamma\sin\alpha = V\sin(\gamma-\alpha),$$

therefore

$$P_1 = V.I_0.\sin(\gamma-\alpha) + E_1.I_0.\sin\alpha \;\ldots\ldots\ldots\ldots (159).$$

In Fig. 340 the vector V is drawn vertically, as before, and the line OP is made equal to VI_0, but is drawn so as to make an angle α with the horizontal, where

$$\tan\alpha = R_a.$$

A semicircle is drawn on OP, and a line OE is set out at an angle γ with the vector V. If this line cuts the semicircle in a point Q we have

$$OQ = OP.\sin\beta_1 = V.I_0.\sin(\gamma - \alpha),$$

because the angle $\gamma - \alpha$ between chord and tangent is equal to the internal angle β_1.

If we add to OQ the constant amount

$$QR = E_1.I_0.\sin\alpha,$$

we have, from equation (159),

$$P_1 = OQ + QR.$$

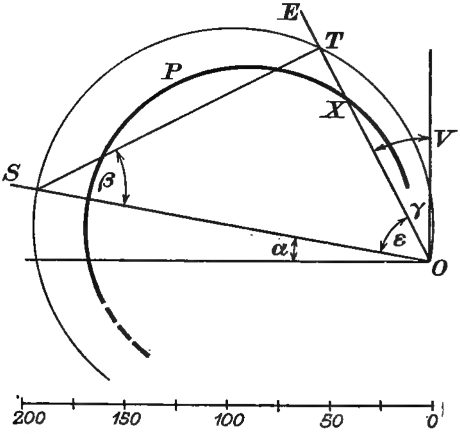

Fig. 341

The locus of all the points R obtained in this manner is represented by the heavy black curve in Fig. 340. Since the angle α is always very small in actual practice, the length QR which is equal to $E_1.I_0.\sin\alpha$, is negligibly small, and the heavy curve is practically a circle. Fig. 340 is drawn for the following values:

$$R_a = 1, \quad \omega L = 6, \quad V = 1000, \quad E_1 = 1200.$$

We have, in the foregoing, considered only the total power generated by the machine, and we now turn to the power given out by the machine to the external circuit. For this we have

$$P = V.I.\cos\phi.$$

From Fig. 337 we see that

$$\cos\phi = \sin(\gamma + \alpha + \delta) = \sin(\gamma + \alpha).\cos\delta + \cos(\gamma + \alpha).\sin\delta.$$

By substituting the values found above for $\cos\delta$ and $\sin\delta$, we get

$$P = V.\frac{(E_1 - V.\cos\gamma).\sin(\gamma + \alpha) + V.\sin\gamma.\cos(\gamma + \alpha)}{\sqrt{R_a{}^2 + (\omega L)^2}}.$$

If we substitute I_0 for $\dfrac{E_1}{\sqrt{R_a{}^2 + (\omega L)^2}}$ and put

$$-V.\cos\gamma.\sin(\gamma+\alpha) + V.\sin\gamma.\cos(\gamma+\alpha) = -V.\sin\alpha$$

we get $$P = V.I_0.\sin(\gamma+\alpha) - \frac{V^2}{E_1}.I_0.\sin\alpha \quad \text{..............(160).}$$

In order to obtain this output graphically, the line OS in Fig. 341 is drawn so as to make an angle α with the horizontal, and is made equal to $V.I_0$. On OS a semicircle is described. This semicircle has the same diameter as that in Fig. 340, but the angle α is now above the horizontal. For the length of any chord, making an angle γ with the vector V, we have

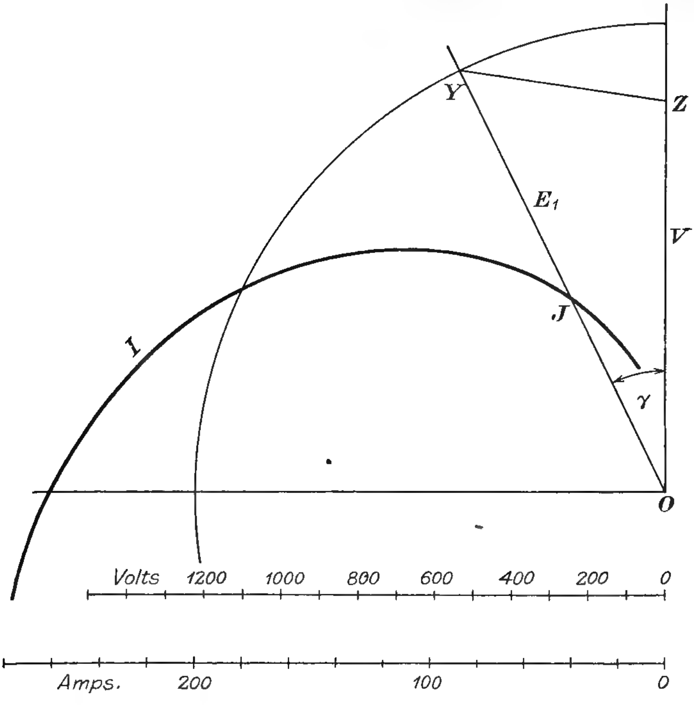

Fig. 342

$$OT = OS\sin\beta = V.I_0.\sin(\alpha+\gamma).$$

From OT we subtract a constant quantity XT, where

$$XT = \frac{V^2.I_0}{E_1}.\sin\alpha,$$

and the remainder $OX = OT - XT = P$.

The length OX is therefore equal to the output P. The locus of all such points as X is the heavy black curve, which, as before, differs but little from a circle. The constants for Fig. 341 are as follows: $R_a = 1$, $\omega L = 6$, $V = 1000$, and $E_1 = 1200$; a scale of kilowatts is added.

We proceed now to determine the current corresponding to any value of the angle between E.M.F. and terminal pressure. The vertical OZ in Fig. 342 represents V, and the vector OY the E.M.F. E_1. The line ZY corresponds to the pressure OF in Fig. 337. Hence,

$$ZY = I \cdot \sqrt{R_a^2 + (\omega L)^2} = I \cdot \frac{E_1}{I_0},$$

or

$$I = \frac{I_0}{E_1} \cdot ZY \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Instead of measuring ZY in volts and multiplying it by I_0/E_1, we can make a new scale, as shown in the figure, on which the length ZY gives the current directly. A point J is marked on the vector OY, so that OJ is equal to ZY. By repeating the construction for different values of γ, and drawing a curve through the points thus obtained, we get the locus I, which shows at once the current corresponding to any value of γ.

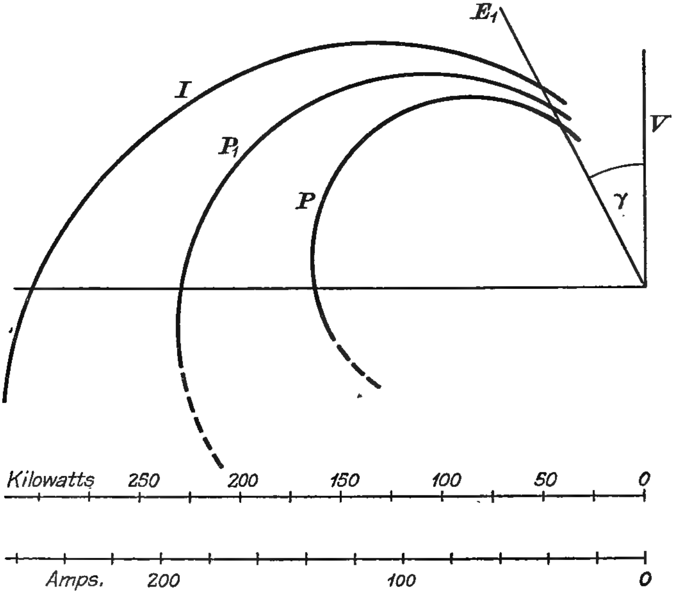

Fig. 343

The three curves P_1, P and I, from Figs. 340, 341 and 342, are shown side by side in Fig. 343. We have already given the constants of the machine, but repeat them here, together with other data necessary for the construction of the diagrams.

$$R_a = 1, \quad \omega L = 6, \quad V = 1000, \quad E_1 = 1200,$$

$$\tan\alpha = \frac{R_a}{\omega L} = \frac{1}{6} = 0\cdot166, \quad \sin\alpha = \frac{R_a}{\sqrt{R_a^2 + (\omega L)^2}} = 0\cdot164,$$

$$I_0 = \frac{E_1}{\sqrt{R_a^2 + (\omega L)^2}} = 197 \text{ amperes.}$$

For the diameters OP and OS, we have

$$OP = OS = V \, . \, I_0 = 197{,}000 \text{ watts},$$

$$QR = E_1 \, . \, I_0 \, . \sin\alpha = 38{,}600 \text{ watts},$$

$$XT = \frac{V^2}{E_1} \, . \, I_0 \, . \sin\alpha = 97{,}000 \text{ watts}.$$

The scales of pressure and power have been chosen arbitrarily. The current scale is found by multiplying each figure on the pressure scale by $I_0/E_1 = 0{\cdot}164$. Hence, the same length will represent either 200 amperes or 1220 volts.

It is evident that each value of the displacement between E.M.F. and terminal pressure corresponds to a definite current, total power and output. The part of the vector E_1 intercepted between the curves P and P_1 represents the power lost in heating the armature winding. It increases with the angle γ, while the output increases slowly beyond a certain point and soon reaches a maximum. To obtain a good efficiency it would be necessary to keep to the right-hand part of the diagram.

It is interesting to notice that the power taken from the prime mover and converted into electrical power increases as the angle γ increases, until $\gamma = 90° + \alpha$, when the power P_1 reaches its maximum value. This maximum is seen from Fig. 340 to be equal to $OP + QR$, or

$$P_{1\max} = V \, . \, I_0 + E_1 I_0 \, . \sin\alpha.$$

The increase of power with increasing displacement between E.M.F. and terminal pressure is of the utmost importance for the parallel working of two or more alternators, as we shall see in the following section.

103. Synchronising Power of Armature.

If several exactly similar alternators are working in parallel, and their exciting currents are exactly equal, and, moreover, the power given to each by its prime mover is the same, then Fig. 343 tells us that the displacement between induced E.M.F. and terminal pressure must be exactly the same in each machine. Now, as the terminal pressure is common to all the machines, it follows that their electromotive forces must coincide in phase. This means that an armature wire on one machine occupies, at any moment, the same position relatively to the poles as the corresponding wire on any other machine. The machines are then said to be running synchronously, by which we mean that they are not only running at exactly the same speed, but that they are also in phase at every moment, the E.M.F. curve of each machine reaching its maximum at the same instant.

We are not concerned, for the present, with the way in which the machines were brought into synchronism, but will first consider how this agreement of speed and phase, when once obtained, is continually maintained. The importance of this question is obvious. As in the parallel working of direct-current dynamos, so here, the electromotive forces must be so connected that the terminals of like sign are connected together. If we consider the individual electromotive forces, we see that each one is in opposition to all the others. From this we see that the electromotive forces of all the machines in parallel must have the same value and direction at every moment, and must also reverse simultaneously. This can only be attained by absolute equality of speed, or, if the machines have unequal numbers of poles, by absolute equality of frequency.

Such equality of speed could never be obtained by external means. It is fortunate, therefore, that the generators themselves exert a mutual influence on one another, tending to maintain synchronism. The effect of this can be seen from the fact that the connecting rods of two engines driving alternators in parallel will remain perfectly in step hour after hour. The cause of this can be seen by assuming that one of the generators, as a result of some inequality in the turning moment of the engine, gets a little ahead. This is equivalent to an increase in the angle γ by which the E.M.F. vector of that machine leads ahead of the vector of terminal pressure. The latter is dependent on the other machines and will not be materially affected. This increase of the angle γ causes, as Fig. 343 shows, an increase of the output or load on the machine. Since this load exceeds the mechanical power supplied by the engine, the speed falls and brings the machine back to its normal position, that is, the angle γ is reduced to its normal value.

If, on the other hand, one of the generators lags slightly behind the other machines, its load is decreased in accordance with Fig. 343. The mechanical power supplied by the engine will then exceed the electrical load, and the excess of power will accelerate the machine until its poles occupy the same position, relatively to the armature, as the poles of the other machines. In this way the alternators keep the engines in synchronism with mathematical accuracy.

We have assumed that the machines are working on the right-hand part of Fig. 343, where an increase of the angle γ causes an increase of the electrical power P_1. If, however, the machines are working in the neighbourhood of their maximum power, an increase of γ causes a very slight increase in the load, so that the tendency to maintain synchronism is small. If we go still further, an increase of γ causes a decrease in the load, as is shown by the dotted part of the curve P_1. As a consequence, the mechanical power supplied to the alternator exceeds the electrical load by an ever-increasing amount, which accelerates the machine more and more until it is 180° out of phase, when it is evident that enormous currents will circulate between the parallel machines and cause the whole station to shut down. It is necessary, therefore, to keep to the right-hand side of the diagram, so that the inevitable tendencies of the engines to get ahead or lag behind are counterbalanced by the powerful synchronising forces of the alternators. The importance of this in alternators with a large number of poles is evident when we consider that a small mechanical displacement of the pole-wheel is equivalent to a large displacement of phase.

In the foregoing we have looked upon the synchronising force as being due to the increase or decrease of electrical output or load caused by the leading or lagging of the pole-wheel. In practice it is more usual to speak of the synchronising current, which circulates between the two machines, adding itself to the normal current of the leading machine and subtracting itself from that of the lagging machine. This current constitutes a load on the leading machine, which is thereby retarded, while to the lagging machine the current is a motoring current and causes it to accelerate.

It is evidently important that a small displacement should cause a large change in the value of P_1, since this is the condition for a powerful synchronising force. We might define the synchronising power of the machine as the value of the differential quotient $dP_1/d\gamma$. Now, in equation (159) on page 325, we found that

$$P_1 = V.I_0.\sin(\gamma - \alpha) + E_1.I_0.\sin\alpha.$$

Hence,
$$\frac{dP_1}{d\gamma} = V.I_0\cos(\gamma - \alpha).$$

By means of this equation we can study the effect of armature self-induction on the synchronising power. We shall consider the moment when the machine is first switched into parallel, when, as we shall see in the next section, the E.M.F. and terminal pressure are exactly in phase, so that $\gamma = 0$. At this moment

$$I_0 = \frac{E_1}{\sqrt{R_a^2 + (\omega L)^2}} = \frac{E_1}{R_a} \cdot \frac{R_a}{\sqrt{R_a^2 + (\omega L)^2}} = \frac{E_1}{R_a} . \sin\alpha,$$

so that, for $\gamma = 0$, we have

$$\frac{dP_1}{d\gamma} = \frac{V.E_1}{R_a} . \sin\alpha . \cos(-\alpha) = E_1 . \frac{V.\sin 2\alpha}{2 . R_a} .$$

For given values of E_1, V and R_a this expression becomes a maximum when $\alpha = 45°$, i.e. when the reactance and resistance are equal. Although it is not practicable to make the self-induction of the armature so small, it is obvious that it should be made as small as possible. If, on the other hand, $\omega L = 0$ then $\alpha = 90°$ and $\sin 2\alpha = 0$. The synchronising power of the armature disappears, therefore, with its self-induction, so that, although the latter should be small, its presence is absolutely essential to the parallel running of the machine.

104. The Parallel Connection of Alternators.

The connection of an alternator to bus-bars on which other alternators are already working is, in many respects, like the similar operation with a direct-current generator. If a shunt dynamo has to be put in parallel with other machines which are already running, it is run up to speed on open circuit, and its field current adjusted until its E.M.F. is equal to the P.D. between the bus-bars. The switch is then closed and the terminals of the dynamo connected to the main terminals or bus-bars of the same sign.

With alternators the machine is started on open circuit and run up to speed. The speed must, however, be carefully adjusted so that the frequency is that of the bus-bars. The field current is then adjusted until the E.M.F. is equal to the P.D. between the bus-bars. As in the case of direct current, the switch must not be closed unless we are certain that the machine terminal which is positive at the moment is switched on to the bus-bar which is positive at that moment. The momentary value of the E.M.F. must also be equal to the momentary value of the P.D. between the bus-bars. If these conditions are fulfilled, the bus-bar P.D. will be exactly counterbalanced by the equal and opposite E.M.F. of the machine, and no rush of current will occur on closing the switch.

The principal difficulty in connecting an alternator in parallel with others is to obtain this exact opposition of phase. It is almost impossible to maintain it for very long, and some indication is required as to the correct moment for closing the switch. Glow-lamps are generally used for this purpose, as shown in Figs. 344 and 345. In the first figure the lamps are connected across the terminals which will be connected by the switch. When the speed of the machine is almost correct, the lamps will slowly alternate between darkness and brightness. The switch should be closed during the period of darkness, as the P.D. of the bus-bars and the E.M.F. of the machine are then in opposition, as shown by the signs in Fig. 344. These signs refer, of course, to one moment only.

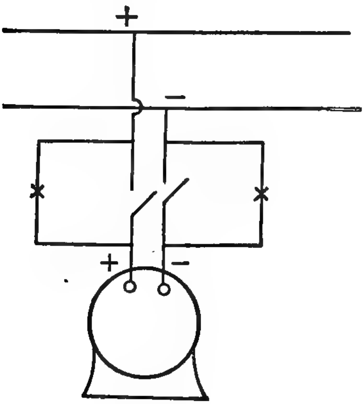

Fig. 344

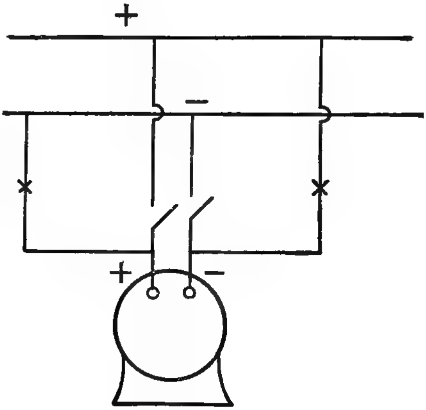

Fig. 345

In the arrangement shown in Fig. 345 the correct moment is indicated by the lamps burning with their maximum brilliancy. The plus and minus signs indicate the state of affairs at the correct moment for closing the switch, and it is evident that, so far as the two lamps are concerned, the bus-bars and the generator are connected in series, thus causing the lamps to burn brightly.

Some experience is required to be able to close the switch at the correct moment. It is evident that it is a far more difficult operation than switching in a D.C. dynamo, since neither the frequency nor the phase can be determined with mathematical accuracy. Such great accuracy, however, is not necessary, as the synchronising power of the machine comes into play directly the switch is closed, and almost immediately pulls the machine into synchronism.

So long as the machine remains unloaded the conditions are the same as with direct current. The P.D. between the bus-bars and the E.M.F. of the machine are equal in magnitude and in direct opposition, so that both the current and the load on the machine are zero. The fundamental difference between the parallel operation of D.C. and A.C. machines is seen, however, when we consider how the newly-connected machine can be made to take its share of the load on the station.

We may consider a D.C. shunt dynamo running light, i.e. without delivering current to the bus-bars, to which it, in common with several large dynamos, is connected. To make the machine take a share of the load its field current is strengthened, and thereby its E.M.F. increased. The armature current increases in accordance with the equation

$$I_a = \frac{E - V}{R_a}.$$

This increase of current causes an increase in the electrical power generated by the dynamo. The former supply of mechanical power is not sufficient to meet this new demand, and the machine slows down. As the engine slows down, the balls of the governor fall and admit more steam, until the mechanical power supplied by the engine is equal to the electrical power or load $E . I_a$*.

When this new state of equilibrium has been reached the machine runs steadily at a speed somewhat lower than that at no-load. That the speed is lower follows from the fact that the governor balls must occupy a lower position in order to admit the increased steam supply. The fall of speed is generally very small. We have seen, therefore, that the load on a shunt dynamo is varied by means of the excitation. The dynamo reacts on the prime mover and causes a small variation of speed, by means of which the supply of power is varied between wide limits. It might be said that the supply adjusts itself to suit the demand, i.e. the steam engine supplies what the dynamo demands from it.

In an alternating current system, however, the relations are quite different. We shall assume that the machine under consideration is small and that there are several large machines working on the same bus-bars, so that the terminal pressure and speed of the large machines are not appreciably affected by manipulations on the smaller one. We can therefore assume that the terminal P.D. and the frequency are constant. The question is: How can we load the small alternator which we have put in parallel with the other machines? The first suggestion is naturally to increase the excitation, and with it the E.M.F. As we should expect, we obtain an increased current, but, to our surprise, neither the wattmeter deflection nor the supply of steam is increased. Since the power is unchanged in spite of increased current and E.M.F., we conclude that the phase displacement between E_1 and I must have increased.

On further examination, however, we see that it is impossible that the load on the generator could be varied by means of the excitation. The machine either runs at the synchronous speed or it falls out of step, so that the increase of excitation has no effect on the speed of the engine or on the position of the governor balls. The steam supply is the same as before, and the power is consequently unchanged. The mechanical power transmitted from engine to generator is the same as when running light before switching into parallel, and is not affected by varying the excitation. To increase the power we must shift the weight on the governor or adjust it in some way, either by hand or by means of a small motor controlled from the switchboard. By this means the steam supply is increased without any variation from the constant speed corresponding to the frequency of the station. Hence, the load on an alternator is varied by altering the mechanical power supplied to it.

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* As we have assumed the total load on the station to be constant, the load on any machine can only increase by a simultaneous decrease occurring in the load on the other machines. The only way this can occur, however, is by a variation in their terminal pressure. This change in V is very small, and we have neglected it.

The question naturally arises as to the manner in which this automatic adjustment of the electrical output takes place with every change in the supply of mechanical power. The first result of an increased steam supply is an acceleration of the generator. This acceleration lasts only a moment, since a slight forward movement of the E.M.F. vector in Fig. 340 increases the load until it corresponds to the steam supply. By steadily increasing the output of the steam engine, that of the generator can be increased until the machine is working in the neighbourhood of its maximum power (Fig. 340), when there is great danger of falling out of step.

There is a similar difference between the procedure to be followed when disconnecting D.C. and A.C. machines from the bus-bars. Before disconnecting a shunt dynamo its load must be removed by weakening its excitation. The current from an alternator, on the other hand, is reduced to zero by manipulating the governor or gradually reducing the steam supply.

From what we have just said it might appear that there is no need for an automatic governor on the steam engine, since the speed of the engine is fixed by the frequency of the station, that is, by the speed of the other machines. This conclusion is quite right so far as one machine is concerned, since its governor could be clamped without any change in the working of the station. If, however, we desire the load on the station to be equally shared between all the generators, even when the load is very variable, it is evident that the governor on each machine must automatically regulate the steam supply. A large load will then cause a general drop in the frequency, and a greater supply of steam to every machine. The frequency can be brought up to its previous value by adjusting the steam supply to every engine.

Another interesting question deals with the effect of a change in the excitation, since we have seen that it is without effect on the power. It is evident that the terminal pressure of the station would be increased by strengthening the field of every machine. What we wish to consider, however, is the effect on a machine of strengthening its field current without changing the excitation of the other machines, and therefore without materially altering the terminal pressure. This question will be considered in the next section.

105. The Effect of Field Regulation on Alternators in Parallel.

In this section we shall consider the changes which occur in an alternator, working in parallel with a number of other alternators, when its field current is varied. The terminal P.D. will be practically constant, and we shall assume that the mechanical power supplied to the generator by the engine is also constant.

If, in the first place, we neglect the armature resistance, we have from page 324,

$$P_1 = \frac{E_1 \, . \, V}{\omega L} \, . \sin\gamma \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

In Fig. 346, the terminal P.D. V is drawn vertically, lagging behind the vector E_1 by an angle γ. OF is the direction of the current vector, lagging behind V by an angle ϕ.

Since P_1 and V are constant, the triangle OCE must have a constant area, for it is equal to $V.E_1.\sin\gamma$, which from the above equation is equal to $P_1.\omega L$. Hence, as E_1 is varied by strengthening or weakening the field current, the point E must move up and down the vertical line EB.

The line CE represents the drop in the armature $\omega L.I$, 90° out of phase with the current. If the excitation be altered until $E_1 = OD$, the drop $\omega L.I$ will then have its minimum value CD, and as $\sin\gamma$ will then equal $\frac{CD}{E_1}$, we see, from equation (a), that

$$CD = \frac{P_1 \, . \, \omega L}{V}.$$

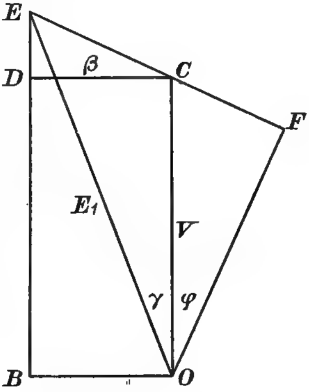

Fig. 346

This is also obvious from the fact that the current is then in phase with the terminal P.D. Hence, if the field current be adjusted until the armature current is a minimum, the machine is then loaded non-inductively. If the excitation be increased, the current will lag, whereas if it be decreased, the machine will deliver a leading current to the bus-bars.

The figure shows, moreover, that the value of E_1 cannot be reduced below OB. If the field be further weakened, the machine breaks out of step with the bus-bars, i.e. with the other machines with which it is in parallel.

If the armature resistance be taken into account, the results obtained are very similar, as we shall now show.

If we take equation (159) on page 325, multiply each side by $\frac{E_1}{I_0 \, . \sin\alpha}$, and then add $\left(\frac{V}{2\sin\alpha}\right)^2$, we get the following equation, in which we have put $-\cos(90° + \gamma - \alpha)$ instead of $\sin(\gamma - \alpha)$:

$$E_1^2 + \left(\frac{V}{2\sin\alpha}\right) - \frac{E_1 \, . \, V}{\sin\alpha} \, . \cos(90° + \gamma - \alpha) = \frac{P_1 \, . \, E_1}{I_0 \, . \sin\alpha} + \left(\frac{V}{2\sin\alpha}\right)^2 \ldots (161).$$

Since $\frac{E_1}{I_0}$ is the internal impedance of the machine, and is therefore constant, we see that the right-hand side of the equation is constant for all excitations, for P_1 and V are both unaffected by the variable excitation. For the sake of shortness we may represent the right-hand side of the equation by R^2. Now, an equation

$$a^2 + b^2 - 2a.b \cos \theta = c^2$$

represents a triangle with sides a, b and c, of which the sides a and b enclose an angle θ. Equation (161) is of this form, and can therefore be represented by a triangle with sides E_1, $\frac{V}{2 \sin \alpha}$ and R, in which the two former sides enclose an angle $90° + \gamma - \alpha$. This leads to a very simple diagram, from which the value of the angle γ between the E.M.F. and P.D. vectors can be seen for any given value of E_1. From the point O in Fig. 347 a vertical line OC is drawn equal to the terminal pressure V. Another line OA is drawn at an angle α with the horizontal, equal to $\frac{V}{2 \sin \alpha}$. The point A is half as high as the point C. With A as centre, an arc is drawn with a radius equal to R, i.e. equal to the root of the right-hand side of equation (161). A line OE is drawn, making any given angle γ with OC, and meeting the arc at the point E. The triangle OAE must be the above-mentioned triangle, for its two sides OA and OE enclose the angle $90° + \gamma - \alpha$ and its third side is R. The side OE is consequently equal to the E.M.F. E_1, and it is ahead of the P.D. vector OC by the angle γ. The point E moves on the circular arc and gives the corresponding values of E_1 and γ. The diagram is actually drawn for the relations previously assumed, viz.

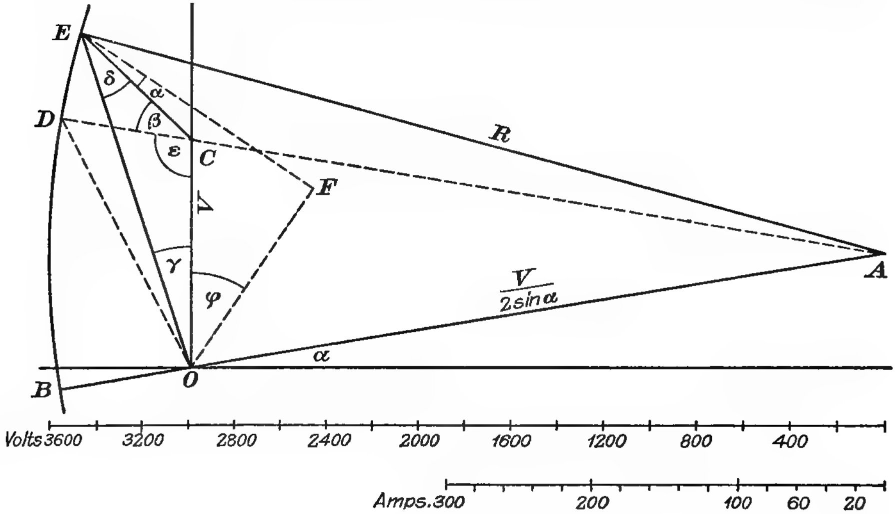

Fig. 347

$$V = 1000, \quad R_a = 1, \quad \omega L = 6,$$

and the constant power P_1, which, it will be remembered, includes the loss in the armature winding, is taken as 100,000 watts. It is evident that, for the same value of E_1, viz. 1200 volts, Fig. 347 must give the same angle γ as is found in Fig. 340 for $P_1 = 100{,}000$ watts.

The effect of varying the excitation can be seen very plainly from the diagram. By gradually decreasing the field current, the length of the vector E_1 is decreased, the point E moves down the arc towards B, and the angle γ grows larger and larger. This is exactly what we should expect from a mechanical point of view. On suddenly decreasing the E.M.F. the electrical power is momentarily reduced, with the result that the supply of mechanical power is, for a moment, greater than that converted by the machine into electrical power. This causes a slight acceleration and an increase of the angle γ, which brings P_1 up to its former value, corresponding to the unaltered supply of mechanical power.

If the weakening of the excitation be carried far enough, the point E arrives at B, so that OB is in line with OA. This length OB represents the smallest E.M.F. with which the machine can take an input of 100,000 watts, or inversely, 100,000 watts is the largest power which the machine can take for an E.M.F. equal to OB. If the excitation is decreased any further, or the input from the engine increased, or even if the angle γ is momentarily increased owing to any slight irregularity, the machine will fall out of step. As in Fig. 340, the angle γ corresponding to this maximum power is equal to $90° + \alpha$. For ordinary practical working it is evident that γ must be small, so that there is no danger of the machine falling out of step.

It is of great interest to know which value of the excitation would give the most efficient working for the given input P_1. If we assume that the I^2R_a loss in the armature is the only variable loss, the efficiency will be a maximum when this loss is a minimum, that is, when the current is a minimum. The drop of pressure in the armature, which is represented in the figure by the line CE, is proportional to the armature current. This will be clearer if we turn to Fig. 337, where the triangle BOF corresponds to the triangle OCE in Fig. 347. As in equation (*a*) on page 328

$$I = \frac{I_0}{E_1} \, . \, CE,$$

in which

$$I_0 = \frac{E_1}{\sqrt{R_a^2 + (\omega L)^2}} \, .$$

By substituting the values $R_a = 1$, $\omega L = 6$, and $E_1 = 1200$ we obtain the scale of amperes in Fig. 347, viz.

$$I = \frac{197}{1200} \, . \, CE = 0{\cdot}164 \, CE.$$

The scale at the foot of the figure expresses the length CE directly in amperes. For highest efficiency the vector CE must be a minimum, that is, it must be in a straight line with AC. CD must therefore represent the minimum current, which is found from the scale to be 91·6 amperes, the corresponding E.M.F. OD being 1220 volts. The loss in the armature winding will be

$$I^2 . R_a = 91{\cdot}6^2 . 1 = 8400 \text{ watts.}$$

This is a very excessive armature copper loss for a machine of 100 kilowatts input, but it has been so assumed for the sake of clearness in the diagrams.

Since the input P_1 is constant, the minimum value of the losses will give the maximum nett output P.

For any other value of the excitation, whether the point E lies above or below D, the current will be larger, and the efficiency consequently lower. It is evident, however, that the variation is very small in the neighbourhood of the maximum, since the excitation can vary considerably on either side of D without any great change in the current.

This latter circumstance is shown very clearly by plotting a curve with the E.M.F. as abscissa and the corresponding current as ordinate. The curves obtained in this way are known as the V-curves, because of their peculiar shape. Two such curves are shown in Fig. 348, in which the abscissae are made equal to the various values of OE in Fig. 347, while the ordinates are equal to the corresponding values of CE. It is evident that there is a certain excitation for each load, at which the current is a minimum. The left-hand branch of the curve is that along which we are running the danger of letting the machine fall out of step by decreasing its excitation beyond the allowable limit.

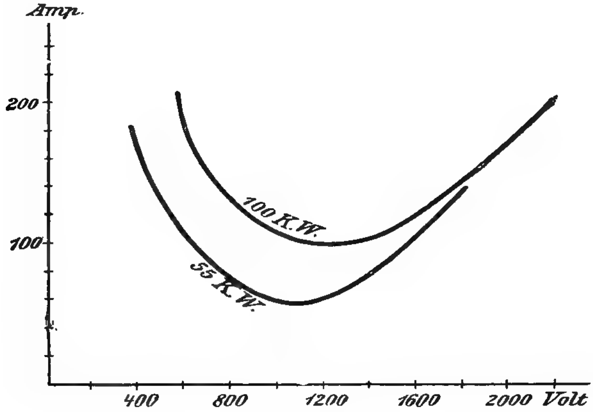

Fig. 348

It is evident from Fig. 347 that the E.M.F. could be varied over a wide range without much effect on the current, if the circular arc were further to the left, so that CE were longer. Hence, machines with a large internal inductive drop $I.\omega L$ are comparatively insensitive to changes of excitation. This would be shown by flat V-curves.

If the inductive armature drop is small the V-curves will be more sharply pointed. Although we cannot vary ωL we can decrease the drop $I.\omega L$ by decreasing the current I, that is, by working on a smaller load. If Fig. 347 be drawn for a small input P_1, CE will be shorter and the arc will be nearer to the vertical OC. The result is shown in Fig. 348 in the lower curve, which has been drawn for a value of P_1 equal to 55 kilowatts. It is decidedly more acute than the upper curve.

We must now consider the power-factor of the power supplied by the generator to the bus-bars. The angle ϕ between the terminal pressure and the current can be found in Fig. 347 by drawing a line EF so as to make an angle α with EC, and then dropping a perpendicular from O on to EF. The triangle EOF is evidently identical with the triangle OBA in Fig. 337, and the angle COF is the required phase displacement ϕ. Moreover, we have from Fig. 347

$$OA = \frac{V}{2 \sin \alpha},$$

or

$$\frac{V}{2} = OA \,.\, \sin \alpha.$$

The triangle OAC is therefore isosceles, and AC makes an angle α with the horizontal. The three following equations are evident from the geometry of the figure:

$$\epsilon = 90^\circ + \alpha,$$

$$\phi + \gamma = 90^\circ - (\alpha + \delta),$$

and

$$180^\circ = \epsilon + \beta + \gamma + \delta.$$

By adding up each side we find that

$$\phi = \beta.$$

Hence, the angle by which the current vector lags behind the vector of terminal pressure is given by the angle between the lines CD and CE. If the excitation be adjusted to give the E.M.F. OD, the angle β vanishes and the current is in phase with the terminal pressure. This agrees with the previous calculation for this point, for which the input was 100,000 and the loss 8400, thus giving an output of 91,600 watts. Since the current was 91·6 amperes and the P.D. between the terminals 1000 volts, it follows that the power-factor $\cos \phi$ must have been unity.

It might, at first sight, appear preferable to excite the machine so that $\cos \phi = 1$ and the losses are a minimum. If, however, the external circuit be at all inductive and this one machine be so excited that its current has no wattless component, the wattless component of the external current must be borne entirely by the other machines. It would evidently be a much better arrangement to share the wattless current equally between the various machines, or rather, so as to make the power-factor the same for every machine.

If the excitation of our machine be greatly increased, the point E moves further up the arc, and the current lags far behind the terminal pressure vector. The machine delivers a heavy lagging wattless current, which causes the necessary armature drop $E_1 - V$. On the other hand, by weakening the field current, we cause the point E to move down towards B. The angle β or ϕ becomes negative, which means that the current vector leads ahead of the terminal pressure. The electromotive force E_1, which is equal to OE, becomes smaller than the terminal pressure V. This is due to the magnetising effect of the leading armature current. This case can occur if the load consists of a synchronous motor, since by strongly exciting the motor, its back E.M.F. can be made to exceed the E.M.F. of the generator. The terminal pressure of both machines will then have some value intermediate between these electromotive forces, and consequently greater than the E.M.F. of the generator.

The importance of the diagram in Fig. 347 is obvious when we consider that it enables us to read off directly the electromotive forces, the currents and the power-factors, and gives us, moreover, a clear insight into the mechanical and electrical changes caused by varying the excitation.

106. Phase-swinging of Alternators.

We shall consider an alternator connected to bus-bars, the P.D. and frequency of which may be taken as constant. For a prime mover we shall assume a turbine with a perfectly uniform turning-moment. If, owing to any sudden variation in the load or in the excitation, the machine be momentarily a trifle ahead of its normal position, the load P_1, which is practically proportional to the angle γ, will exceed the supply of mechanical power, and the machine will be retarded. It will not stop when it arrives at its normal position, but will swing past it like a pendulum. It will then be behind its normal position, and the corresponding value of P_1 will be less than the supply of mechanical power. The machine will therefore be accelerated and will pass again through its normal position. It is evident that the machine has an oscillation superposed upon the uniform rotation, and this oscillation is gradually damped out by the eddy currents which are caused by it.

Now, the rate at which the power or load P_1 increases with the angle γ was seen on page 331 to be $V.I_0.\cos(\gamma - \alpha)$ per radian. In actual practice the angle α, the tangent of which $= \frac{R_a}{\omega L}$, is very small, and the angle γ by which the terminal P.D. lags behind the E.M.F. does not exceed 10° or 20°. If the latter were larger it would mean that the self-induction of the armature was big and that the overload capacity of the machine was small. Very little error will be introduced by putting $\cos(\gamma - \alpha) = 1$. The torque is given by dividing the power by the mechanical angular velocity $2\pi.\frac{f}{p}$. Again, a mechanical displacement of 1 radian represents an electrical phase displacement of p radians, and, finally, 1 metre-kg per second is equal to 9·81 joules per second, i.e. 9·81 watts. With the aid of these relations we find that the torque due to the load on the alternator differs from its mean value by an amount, per radian of mechanical displacement, equal to

$$M_t = \frac{p.V.I_0}{9{\cdot}81.2\pi.\frac{f}{p}} = \frac{p^2.V.I_0}{9{\cdot}81.2\pi.f} \text{ met.-kg} \quad \dots\dots\dots\dots\dots(a).$$

If the moment of inertia of the revolving part of the machine be M_i met.-kg², then we know from mechanics that the time of a complete oscillation will be given by the equation

$$t = 2\pi\sqrt{\frac{M_i}{9{\cdot}81 M_t}} \text{ second} \quad \dots\dots\dots\dots\dots\dots\dots\dots(b).$$

If, for example, $M_i = 53{,}300$, $p = 32$, $f = 48{\cdot}5$, $V = 3000$, $I_0 = 670$, the time of a complete oscillation works out at $t = 0{\cdot}55$ second. Hence, the machine would swing through its neutral position about four times per second, twice going forward and twice moving backward.

We turn now from these natural oscillations of the alternator to the forced oscillations caused by the inequality of turning-moment of any reciprocating engine. To simplify the matter we may imagine the alternator to be replaced by a D.C. dynamo. We shall let t_1 represent the time which elapses between two successive maxima of the turning-moment. The turning-moment diagram will be irregular, but may be represented approximately by a sine curve, as shown in Fig. 349 *a*, in which the ordinates measured from *EH* represent the total torque of the engine, while, by measuring from *OD*, we get the amount by which the torque differs from its mean value. If $M_{t_1} = AG$, the maximum excess turning-moment, then we have for the maximum angular acceleration

$$a_1 = 9{\cdot}81 \frac{M_{t_1}}{M_i}.$$

This is the rate at which the angular velocity of the dynamo or flywheel is changing at any moment. It is a maximum when the turning-moment is a maximum, as shown in Fig. 349 *b*. So long as the acceleration is positive the angular velocity will increase, and will therefore attain its maximum value at the moment *B* (Fig. 349 *c*). At this moment it will exceed the normal angular velocity by an amount ω_1. To find ω_1 we must multiply the average value of the acceleration by the time $\frac{t_1}{4}$, since the velocity was normal at *A*. The mean value of a sine function is equal to its maximum value multiplied by $2/\pi$, so that

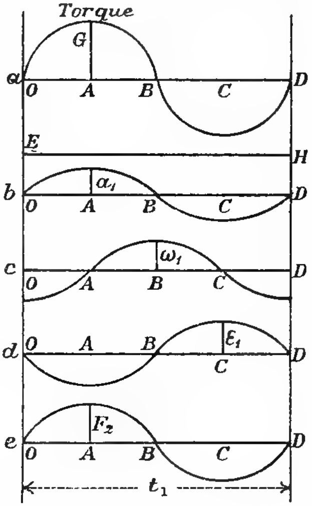

Fig. 349

$$\omega_1 = a_1 \cdot \frac{2}{\pi} \cdot \frac{t_1}{4} = \frac{2 \cdot 9{\cdot}81}{4\pi} \cdot \frac{t_1 \cdot M_{t_1}}{M_i} \quad \text{....................}(c).$$

So long as the velocity is above the normal, i.e. while ω is positive, the forward displacement of the flywheel increases. It will reach its maximum value of ϵ_1 radians at the moment *C*. To find the value of ϵ_1 we must multiply the average value of ω by the time $t_1/4$, thus

$$\epsilon_1 = \frac{2}{\pi} \cdot \omega_1 \cdot \frac{t_1}{4} = \frac{9{\cdot}81}{4\pi^2} \cdot \frac{t_1^2 \cdot M_{t_1}}{M_i} \quad \text{....................}(d).$$

This is due entirely to the steam engine, and is true for any load, electrical or otherwise. We must now consider the effect of the alternator on the angular displacement. We shall assume that the engine forces the alternator to oscillate with the frequency determined by the turning-moment diagram. When the displacement is forward the alternator load is increased, and when the displacement is backward the load is decreased, which is equivalent to an accelerating force exerted on the machine from the bus-bars. Now, by comparing Figs. 349 *a* and *d*, we find that **the machine has its extreme backward displacement at the moment when the turning-moment of the engine is a maximum.** At the same moment the accelerating torque supplied from the bus-bars (see equation *a*) is a maximum, so that these two forces are exactly in phase, having a forward maximum when the machine is in its most backward position and a backward or retarding maximum when the machine is in its extreme forward position. Every change of displacement from the normal position corresponds to a change in the power supplied to the bus-bars, and these variations may be so considerable as to render parallel operation impossible.

The variations of engine torque (Fig. 349 *a*) with a maximum amplitude M_{t_1} produce, if acting alone, a displacement of ϵ_1 (Fig. 349 *d* and equation *d*), but this displacement causes a synchronising torque in the alternator, which, as we have seen, acts in the same direction as the engine torque and therefore increases the displacement ϵ_1, this again increases the synchronising torque, which causes a further increase of the displacement. The ultimate result of this cumulative oscillation depends on several things, including the damping, which we shall neglect for the present. If the oscillation finally reaches a steady state with a displacement amplitude of ϵ instead of the initial ϵ_1, the electrical torque, which we may call M_{t_2}, will be equal to ϵM_t, since M_t was shown in equation (*a*) to be the torque per radian displacement. This final displacement ϵ is produced by the joint action of M_{t_1} due to the engine and M_{t_2} due to the alternator. Hence, instead of equation (*d*) we have

$$\epsilon = \frac{9{\cdot}81}{4\pi^2} \cdot \frac{t_1^2 (M_{t_1} + M_{t_2})}{M_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e).$$

If we compare equations (*b*) and (*e*), we shall find that

$$\frac{t_1^2}{t^2} = \frac{\epsilon M_t}{M_{t_1} + M_{t_2}} = \frac{M_{t_2}}{M_{t_1} + M_{t_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (f),$$

where t_1 is the time between two maxima of the engine torque, and t is the time of a complete oscillation ~~of the a~~lternator if left to itself.

We shall now consider three different cases.

Case 1. By making the flywheel heavy we can make the natural period of vibration t greater than the period t_1 of the forced oscillations due to the engine. This is the usual case in practice. Equation (*f*) will represent a proper fraction, and the effects of M_{t_1} and M_{t_2} must be additive as we saw in Fig. 349, where the maximum engine torque coincided with the extreme backward position. The effect of parallel running is to increase the amplitude of oscillation in the ratio

$$\frac{\epsilon}{\epsilon_1} = \frac{M_{t_1} + M_{t_2}}{M_{t_1}} = \frac{t^2}{t^2 - t_1^2} = \frac{1}{1 - \dfrac{t_1^2}{t^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (g).$$

The increase in the amplitude is evidently smaller when the natural period t is very long compared with the forced period t_1. The engine in the example on page 341 was a tandem compound engine with a single crank. It ran at 85 r.p.m., so that there were $\frac{85 \times 2}{60}$ impulses per second, or

$$t_1 = \frac{60}{85 \times 2} = 0{\cdot}353 \text{ second.}$$

We found the natural period t to be 0·55 second, which gives an increase of amplitude of

$$\frac{\epsilon}{\epsilon_1} = \frac{1}{1 - \left(\frac{0{\cdot}353}{0{\cdot}55}\right)^2} = 1{\cdot}7.$$

Case 2. If the natural period t of the alternator is exactly equal to the forced period t_1, equation (*f*) shows that the initial displacement due to the engine alone will be negligible compared with the increase of amplitude due to the synchronising forces. Equation (*g*) leads to the same result, viz. that ϵ/ϵ_1 will be infinitely big. This is a case of resonance, and the oscillation rapidly becomes so large that the machine falls out of step. This danger must be avoided by making the natural and forced frequencies decidedly unequal. The oscillations would, of course, only be infinitely large were there no damping, and there is always some damping.

Case 3. If the inertia of the rotating masses is small the natural period t will be smaller than the forced period t_1. Although unusual in the case of generators, this is met with in synchronous motors and rotary converters, where periodic variations of the load replace the disturbing effect due to an engine. From equation (*f*) we see that for t to be smaller than t_1, the difference between M_{t_2} and M_{t_1} must be taken instead of their sum. This means that when the alternator has its extreme backward position, the engine torque is not a maximum and assisting the synchronising torque of the alternator, but is a minimum. The forward acceleration at this moment is due to the synchronising torque of the armature, which is more than sufficient to counterbalance the deficiency of torque from the engine.

Equation (*g*) will now give a negative value for the ratio $\frac{\epsilon}{\epsilon_1}$ in which the amplitude of oscillation is increased by parallel running. This means that the oscillation is not only magnified, but is reversed in phase with regard to the engine torque.

As in the first case, so here, we see the advantage of making the natural period t differ as much as possible from the period t_1 of the forced oscillations. In the present case this difference is increased by decreasing the moment of inertia of the revolving system.

In the case of generators, however, the natural period t is generally greater than the forced period t_1 of the engine impulses, so that the parallel operation is only made possible in many cases by increasing the value of t. There are two ways in which this can be done. If choking coils are connected in the leads between the machine and the bus-bars, the short-circuit current I_0 is reduced. This reduces the synchronising torque per radian M_t (equation *a*) and consequently increases the period t (equation *b*). The introduction of choking coils is often the only way in which rotary converters can be prevented from hunting. The second way of increasing t is by increasing the rotating masses. This is a very effective method, since it reduces both the amplitude due to the engine alone and also the ratio in which this amplitude is increased by the synchronising forces.

A third method of reducing the amplitude of the oscillations is due to Leblanc, and consists in the utilisation of the damping action of the eddy currents produced by the oscillation. To increase the eddy currents the pole-shoes may be solid, or, if laminated, may have copper rods embedded in holes running axially through the pole-shoes, and connected to copper plates flanking the pole-shoes on either side. Each pole-shoe has usually about half a dozen of these copper bars, and the whole forms a short-circuited squirrel cage winding. When the poles swing, currents are induced in this so-called amortisseur winding due to its cutting the field produced by the armature currents. We know from Lenz's law that these induced currents will tend to stop the motion producing them, that is, they will tend to damp out the oscillations. This device has the further advantage of keeping the machine, say a rotary converter, in step on a sudden heavy overload of momentary duration. The damping winding acts as an induction motor on the sudden retardation of the machine, and diminishes its amplitude.

The comparison of the damping winding with an induction motor enables us to see, however, why the device is not always effective. The braking action is not due entirely to energy dissipated in the damping coils, but we have an actual induction machine, sometimes running below synchronous speed and taking power from the bus-bars to drive it as a motor, and at other times running as a generator above synchronous speed and supplying electrical power to the bus-bars. The maximum value M_d of the damping moment will occur when the machine is swinging through its neutral position, for it is then that the oscillating velocity is a maximum. It will be proportional to this velocity, and, since it always acts against the direction of motion, it will have a negative maximum at the moment B in Fig. 349 *c*. It is always 90° ahead of the synchronising torque M_{t_2}. The resultant total torque will be the resultant of the three torques M_{t_1}, M_{t_2} and M_d.

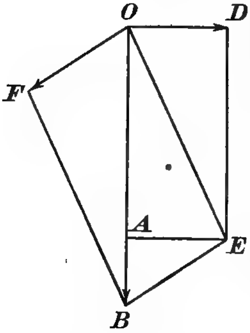

Fig. 350

We shall consider the effect of damping in the case generally met with, viz. where the flywheel is so heavy that the natural oscillations of the alternator are slower than the forced oscillations caused by the engine. In Fig. 350 OB is the total torque, neglecting damping; it is made up of the part $M_1 = AB$ due to the engine, and the part $M_{t_2} = OA$, which we assume to be the greater, due to the variation of power supplied by the alternator to the bus-bars. This total torque OB produces a displacement of such magnitude that the synchronising torque exerted by the alternator is equal to OA. Instead of using rectangular co-ordinates with abscissae representing time as in Fig. 349, we employ here a vector diagram, the length of the vectors representing the maximum values. If we assume for the moment that the addition of the damping still leaves the total torque OB unchanged, the resulting displacement, and therefore also the synchronising torque $M_{t_2} = OA$, will be the same as before. The damping torque will be represented by a vector OD 90° ahead of OB, of a length depending on the design of the machine, but proportional to OA, since OA depends on the maximum displacement and OD on the velocity with which the machine swings through the normal position.

Since we have assumed that the resultant torque is still OB, and therefore that M_{t_2} is still OA, and since OB must be the resultant of M_d, M_{t_2} and the excess engine torque M_{t_1}, we see that the latter cannot now be AB, but must be represented both in magnitude and phase by the vector OF. As a matter of fact, however, M_{t_2} will be the one constant magnitude of the whole diagram; hence, we see that our diagram should be reduced, so as to make OF represent the actual M_{t_1}, that is, in the ratio $\frac{AB}{OF}$. Both the synchronising torque M_{t_2} and the damping torque M_d are produced electrically, so that their resultant OE (reduced in the ratio AB/OF) will represent the fluctuations of power supplied to the bus-bars. Before the damping was present this fluctuation was due only to $M_{t_2} = OA$. It has therefore been decreased in the ratio

$$\frac{OE \cdot \frac{AB}{OF}}{OA} = \frac{OE}{EB} \cdot \frac{AB}{OA}.$$

It is evident that $OE : EB$ is nearer unity than $AB : OA$, so that the latter ratio determines whether the fluctuations of power will be decreased or increased by the damping. It will only be beneficial when A is below the mid-point of OB, that is when

$$\frac{M_{t_1} + M_{t_2}}{M_{t_1}} > 2 \quad \text{or} \quad M_{t_2} > M_{t_1}.$$

We saw in equation (*g*) that $\frac{M_{t_1} + M_{t_2}}{M_{t_1}}$ is equal to the ratio $\frac{\epsilon}{\epsilon_1}$ in which the amplitude of the oscillation is increased by the effect of parallel running, and that, for this to be larger than two, the moment of inertia of the moving parts must be small. The effect of damping will consequently be very advantageous in such cases as rotary converters, where there is no flywheel, apart from the armature of the machine itself, and generators driven by two- or three-crank engines with four or six impulses per revolution, in which case the flywheel can be made very small on account of the uniform turning-moment.

107. The Automatic Voltage Regulation of Alternators.

In recent years a number of generating stations have been fitted with automatic voltage regulators. The best known of these devices is the Tirrill regulator the essential details of which are shown diagrammatically in Fig. 351. The exciting current for the alternator G is supplied by the shunt dynamo D and led into the rotating field magnet system by means of brushes rubbing on slip rings. The resistance R which is in series with the field winding R_m of the dynamo is periodically short-circuited by the contact k, so that the field current is constantly pulsating. The rocking lever l_1 is subjected to two opposing forces, due respectively to the spring s and the downward pull on the soft iron core of the electromagnet m_1 the coil of which is connected across the terminals of the dynamo D. The core of the electromagnet m_2 is below its coil so that the upward attractive force is in opposition to the force of gravity. The coil of m_2 is connected across the alternator terminals or station bus-bars, the P.D. between which is to be maintained constant at all loads. The position taken up by the core of m_2 and therefore the position of the lever l_2 depends on the P.D. between the bus-bars; if the latter falls, the core falls, and the bottom contact at k rises.

When k is closed, R is short-circuited and the terminal P.D. of the dynamo increases, causing m_1 to exert an increased downward pull and thus opening

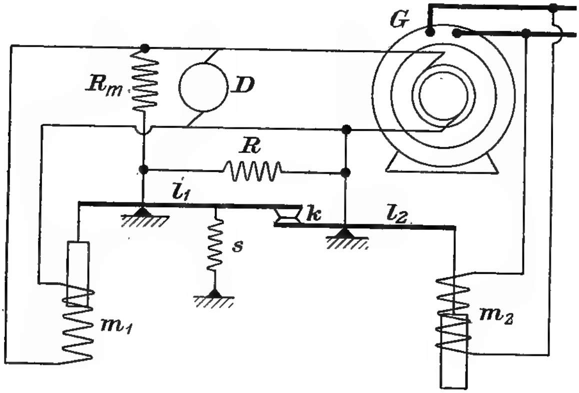

Fig. 351

the contact k against the tension of the spring s. The field rheostat being no longer short-circuited, the P.D. of the dynamo falls, the pull exerted by m_1 decreases causing the contact k to close again, upon which the cycle recommences. If now the P.D. of the bus-bars falls, due say to an increasing load on the station, m_2 allows its core to drop slightly, thus raising the bottom contact at k. This causes the next period during which k is closed to be longer than usual, with the result that the P.D. of the dynamo reaches a higher value. The upward movement of the bottom contact of k stretches the spring s and necessitates a greater current in m_1 before the contact k is opened. The P.D. of the dynamo therefore pulsates about a higher mean value. The field current of the alternator is therefore increased until the bus-bar voltage is restored to its normal value.

The operation will be better understood from a consideration of Fig. 352 in which the field current of the exciting dynamo is plotted to a time base. OA is the current when the resistance R is permanently inserted and OB that when it is permanently short-circuited. The curves c_1 and c_2 show how the current changes from one value to the other on opening or closing the contact *k*. The middle zigzag line represents the pulsations when the conditions are normal; each portion of the zigzag line is similar to the portions of the curves c_1 and c_2 lying between the same horizontal lines. As we have seen, if the bus-bar voltage falls, the time during which the contact is closed is greater and the mean value of the current rises; this is represented by the upper zigzag line. If the bus-bar voltage rises above its normal value, the time during which the contact *k* is closed is decreased and the conditions may be represented by the lower zigzag line.

Although it has no direct effect on the strength of the exciting current, it is interesting to note that the frequency of the pulsations is highest for the middle or normal zigzag line.

There are a number of methods by which an alternator may be compounded, that is, its excitation caused to vary by variation of the output current. The following example may be taken as typical of most of these devices. The main current, on leaving the alternator, passes through the primary windings of a transformer, the secondary windings of which are connected to brushes rubbing on slip-rings fitted to the exciting dynamo which is driven synchronously with or direct-coupled to the alternator. The exciter therefore acts both as a dynamo and as a rotary converter. The magnetising action of the alternating currents supplied to the armature of the exciter will increase the field due to its pole windings to an extent depending on the magnitude and phase of the alternating current. The terminal P.D. of the exciter and, therefore, the current supplied by it to the field winding of the alternator can thus be made to vary automatically in such a way that the terminal P.D. of the alternator remains approximately constant although the load may change both in magnitude and phase. Allowance has to be made for the drop of potential between the terminals and the bus-bars owing to the insertion of the series transformer.

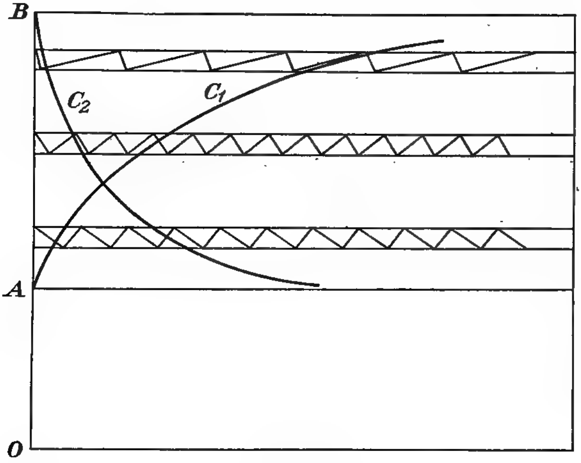

Fig. 352

CHAPTER XV

108. The principle of the synchronous motor.—109. Synchronous motor with constant excitation and variable load.—110. Synchronous motor with constant load and variable excitation.

108. The Principle of the Synchronous Motor.

There are three headings under which all alternating current motors can be roughly classified. The *synchronous motor*, which we shall consider in the present chapter, runs at exactly the same speed whether loaded or unloaded; the *asynchronous* or *induction motor* slows down slightly with increasing load; the *commutator motor* is similar to the D.C. motor and has similar characteristics. Synchronous motors are generally polyphase, induction motors either single phase or polyphase, and commutator motors generally single phase.

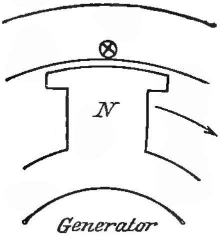

Fig. 353

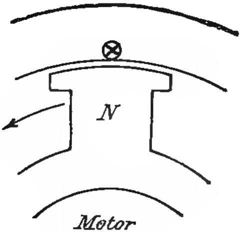

Fig. 354

Any alternating current generator will run as a synchronous motor, whether single phase or polyphase. In Fig. 353 is shown the north pole of a generator, which is rotated clockwise, inducing thereby an E.M.F. in the armature wire shown. If the external circuit is closed, the wire will carry a current which, neglecting phase displacement, will flow away from the reader. The armature current and magnetic field will produce a torque which must be overcome by the mechanical torque exerted by the engine. It follows from this that, if the field be excited as before and alternating current be supplied to the armature, the machine will rotate as a motor in the opposite direction.

A moment later the north pole in front of the wire will be replaced by a south pole. In order that the torque may still act in the same direction as before, the current must have reversed its direction during the interval. It is therefore necessary that the speed of the pole-wheel be in exact agreement with the frequency of the alternating current, even at the moment of switching on the motor. If the motor have p pairs of poles and rotate at n revolutions per second, then

$$n = \frac{f_1}{p},$$

where f_1 is the frequency of the A.C. supply. The synchronous motor must be brought up to the synchronous speed by some external means before the armature is connected to the mains. There are several ways of doing this. If the field current is supplied by a direct coupled D.C. dynamo, this can be used as a motor and driven by a battery or other D.C. source if available. This method is also available if the synchronous motor is used for driving a D.C. dynamo. Sometimes a small induction motor is supplied specially for starting up the synchronous motor. Another method, largely adopted in the United States, is to open the field circuit, and supply the armature with an A.C. at half or a quarter of the normal pressure by means of a transformer. The machine is brought up to speed by means of the eddy currents induced in the pole-shoes by the rotating field, as we shall better understand after studying the induction motor. This latter method is only applicable to polyphase motors. The necessity of bringing the synchronous motor up to speed in some way or other, before switching it on, prohibits its employment in a great number of cases.

The operation of switching in the motor when it has reached synchronous speed is exactly the same as connecting an alternator in parallel with others. To make this clear, we may assume that it is quite undecided whether to use the machine as a generator or as a motor. Such an assumption is quite practicable if the motor drives a D.C. generator connected across the terminals of a battery of accumulators. If the excitation of the dynamo be increased until its E.M.F. exceed that of the battery, the battery will be charged, and the D.C. machine will work as a generator. If, on the other hand, the excitation be weakened until the E.M.F. falls below that of the battery, the current will be reversed, and the battery will drive the D.C. machine as a motor.

To connect the A.C. machine to the A.C. supply, we follow the procedure of Section 104, and bring the machine up to the correct speed by means of the D.C. machine acting as a motor. The field current of the A.C. machine is then adjusted until its E.M.F. is equal to the supply pressure. The switch must be closed at the moment when the E.M.F. of the generator is exactly opposed to the P.D. of the mains, as indicated by the phase-lamps. If the moment is correctly chosen the main ammeter will not be affected, because of the exact opposition of E.M.F. and supply pressure. The power required to run the machine light will still be supplied by the D.C. machine.

To make the A.C. machine act as a generator its supply of mechanical power must be increased. This is accomplished by weakening the field current of the D.C. machine. As a general rule, the speed of a D.C. motor is increased by weakening its field. In the present case, however, the speed is fixed by the frequency of the A.C. supply, and a weakening of the field must result in a lowering of the E.M.F. This will enable the battery to send a larger current through the armature, in accordance with the equation $I_a = \frac{V - E}{R_a}$.

This larger current produces an increased torque, since the weakening of the field is very small compared with the resulting increase of current. The acceleration produced by this increase of torque brings the poles and armature winding of the A.C. generator into such a relative position that the electrical load or output is gradually increased, as shown in Section 102, until the increased mechanical torque is exactly balanced. The A.C. machine has thus been made to act as a generator by accelerating it, or rather, attempting to accelerate it, by an external mechanical torque.

We shall now reverse the procedure, that is, we shall strengthen the field of the D.C. machine, instead of weakening it. Since the speed is fixed, the effect of strengthening the field is to increase the E.M.F. until, when it exceeds the E.M.F. of the battery, the current reverses and the cells are charged by the D.C. machine acting as a dynamo. The A.C. machine must now be acting as a synchronous motor. The power supplied by the battery to the motor-generator set was first positive, then nil, and, finally, negative. This must cause a retardation or lagging of the machines, which brings the poles and armature windings of the alternator into such a relative position that the torque exerted by the alternator corresponds to the load on the D.C. dynamo. The A.C. machine has, therefore, been caused to act as a motor, simply by the retardation, or lagging behind, of the rotating part. In the next section we shall endeavour to make the action clearer by means of vector diagrams.

109. Synchronous Motor with Constant Excitation and Variable Load.

The vector diagram of the synchronous motor is very similar to that of the A.C. generator. In drawing the diagram (Fig. 355) we must consider the induced E.M.F. E as being a back E.M.F., acting more or less in the opposite direction to the current and applied terminal pressure V. The vectors $V = OK$ and $E = OB$ have a resultant OF, which must be the pressure used in driving the current through the armature against its resistance and self-induction. The resultant OF may therefore be resolved into two components, one OD to overcome the resistance, and the other OC to counterbalance the effect of self-induction. As before, we shall designate the angle by which OF lags behind OC by the letter α. The current I must be in phase with the component OD, and lags therefore behind the terminal pressure V by the angle ϕ. From the fundamental principles of a motor we know that the current must be more or less opposed to the back E.M.F., which we see is the case in Fig. 355. It is evident, moreover, that at no-load, that is, on first closing the switch, the vector E will lie along the dotted line, since there is no current and the resultant of E and V must consequently vanish. As the motor is mechanically loaded, it will lag more and more behind its neutral position, and the vector E will lag behind the dotted vector. With the load corresponding to Fig. 355 this angle of lag is γ. (Compare the closing sentences of the previous section.)

The power supplied to the motor is given by the equation

$$P_1 = V . I . \cos\phi,$$

where ϕ is the angle between the terminal pressure and the current. From the figure we see that $FH = V \cos \phi$, so that

$$P_1 = I.FH \text{ watts.}$$

A certain amount of this power will be lost in heating the armature winding, viz.

$$I^2.R_a = I.IR_a = I.FC \text{ watts.}$$

The mechanical output of the motor, including, however, friction and iron losses in the motor itself, will be given by the equation

$$P = P_1 - I^2.R_a = I\,(FH - FC) = I.CH.$$

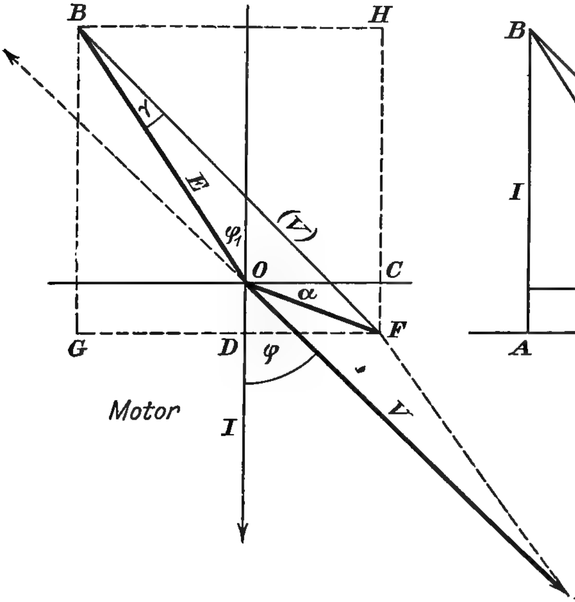

Fig. 355

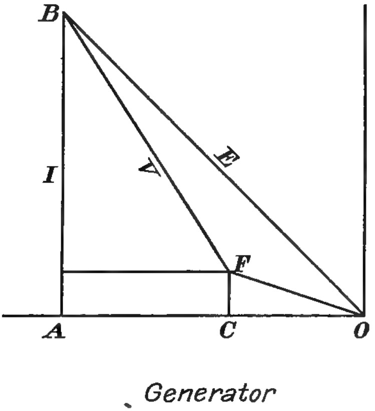

Fig. 356

If ϕ_1 be the angle between the vector of E.M.F. and the current vector produced backwards, we have

$$CH = E.\cos \phi_1.$$

This gives the following equation for the mechanical output of the motor:

$$P = E.I.\cos \phi_1 \quad \text{..........................(162).}$$

We might also have expected this result from our former consideration of the D.C. motor.

On comparing the triangle *FBG* with the triangle *OBA* in Fig. 337 a great similarity will be noticed. For the sake of comparison the vector diagram of the generator is drawn again in Fig. 356. The only difference between Figs. 355 and 356 is the interchange of *E* and *V*. The total power P_1 supplied to the motor in Fig. 355 is $FB.I.\cos \phi$, while the total power P_1 generated electrically by the alternator in Fig. 356 is $OB.I.\cos \phi_1$.

These are evidently equal. Now, on page 325 we established the equation (159) for the value of P_1 for a generator, and it is evident that it can be made applicable to a motor by putting E for V and V for E, giving us

$$P_1 = E.I_0.\sin(\gamma - \alpha) + V.I_0.\sin\alpha \ldots\ldots\ldots\ldots\ldots(163),$$

where P_1 is the power given to the motor, and the short-circuit current I_0 is given by the equation

$$I_0 = \frac{V}{\sqrt{R_a^2 + (\omega L)^2}}.$$

In a similar manner we see that the output P of the motor in Fig. 355 is equal to $OB.I.\cos\phi_1$, which is exactly equal to the output P of the generator, after deducting the $I^2.R_a$ loss in the armature, viz. $FB.I.\cos\phi$ in Fig. 356. Thus, the equation (160) found on page 327 for the generator will be true for a motor if E and V be interchanged. We have therefore

$$P = E.I_0.\sin(\gamma + \alpha) - \frac{E^2}{V}.I_0.\sin\alpha \ldots\ldots\ldots\ldots\ldots(164).$$

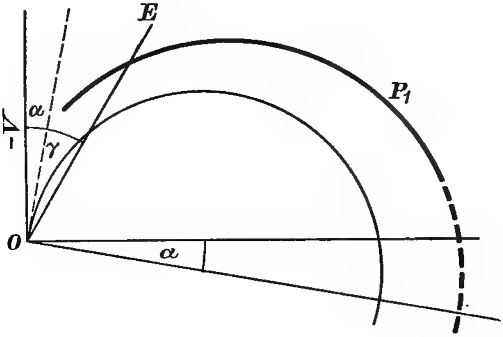

Fig. 357

γ is here the angle by which the E.M.F. lags behind the backward projection of the terminal pressure vector.

We must now draw curves to show graphically how the power varies with the angle γ. As in Figs. 340 and 341, so here in Figs. 357 and 358, circles are drawn passing through O and having their diameters inclined to the horizontal by the angle α. In Fig. 357 the diameter is set out below the horizontal, and in Fig. 358 above it. Its length in each case is $E.I_0$. Every chord drawn from the point O to any other point on the circle, is produced beyond the circle by an amount $V.I_0.\sin\alpha$. The heavy curve P_1, obtained in this way, is the locus of the extremities of lines drawn from O to represent the total power supplied to the motor for the corresponding value of γ. The vector of the terminal pressure V will be vertically downwards from O, and γ is the angle between E and $-V$, as shown in the figure.

Just as Fig. 357 represents equation (163) so Fig. 358 represents equation (164). In this case, however, a portion equal to $\frac{E^2.I_0.\sin\alpha}{V}$ is subtracted from every chord, giving a length representing to a certain scale the output of the motor (including friction, etc.).

The curves are drawn for the following values: $R_a = 1, \omega L = 6, V = 1200$ and $E = 1000$. The output is evidently a maximum when $\gamma = 90 - \alpha$. If the load be still further increased, the angle γ becomes larger but the output of the motor is thereby decreased, and the motor falls out of step. Since this would be equivalent to a short circuit of the supply mains it must be carefully avoided by working on the left-hand side of the diagram in Fig. 358. If the normal full load causes γ to be about 30°, the motor will have about 100 per cent. overload capacity.

In order that a given value of P may correspond to a position on the left-hand side of the figure, it is necessary for the circle, by means of which the figure is constructed, to have a large diameter. Hence $E . I_0$ must be large, since this is the diameter of the circle. The short-circuit current I_0 can be made large by keeping the self-induction of the armature as small as possible.

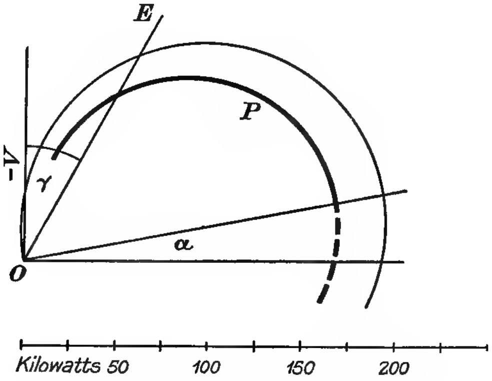

Fig. 358

What was said with regard to the synchronising power and phase swinging of alternators is equally applicable to synchronous motors. We now pass to the consideration of a motor on constant load, but the excitation of which is varied.

110. Synchronous Motor with Constant Load and Variable Excitation.

We shall now consider the effect of varying the excitation of a synchronous motor which is connected to mains with a constant P.D. The load on the motor, i.e. its mechanical output, is constant. If we take equation (164) on page 352, multiply each side by $-\frac{V}{I_0 . \sin \alpha}$, and then add $\left(\frac{V}{2 . \sin \alpha}\right)^2$, we get the following equation, in which we have put $\cos [90° - (\alpha + \gamma)]$ instead of $\sin (\alpha + \gamma)$,

$$E^2 + \left(\frac{V}{2 . \sin \alpha}\right)^2 - \frac{E . V}{\sin \alpha} . \cos [90° - (\alpha + \gamma)] = \left(\frac{V}{2 \sin \alpha}\right)^2 - \frac{P . V}{I_0 . \sin \alpha} \dots (165).$$

Since $\frac{V}{I_0}$ is the internal impedance of the machine, and is therefore constant, we see that the right-hand side of the equation is constant for all excitations, and can, for the sake of shortness, be represented by R^2, where R is some constant. As we saw on page 336, this equation represents a triangle with sides E, $\frac{V}{2 \sin \alpha}$ and R, in which the two former sides enclose an angle $90° - (\alpha + \gamma)$. This triangle is drawn in Fig. 359. From the point O a perpendicular OC is drawn equal to the terminal pressure V. On this line as base an isosceles triangle OAC is constructed with each side equal to $\frac{V}{2 . \sin \alpha}$ and the angle at A equal to 2α. With A as centre and radius equal to R, an arc DEB is struck. The triangle corresponding to equation (165) will have OA for a base, while its apex moves along the arc DEB. For any

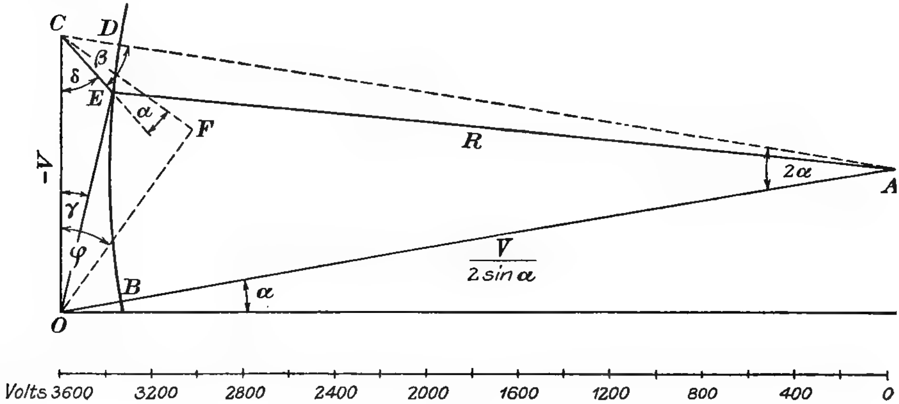

Fig. 359

point such as E, the line OE represents the back E.M.F., making the angle γ with the line OC, which is the backward projection of the terminal pressure vector. The figure is drawn for the values formerly assumed, viz. $R_a = 1$, $\omega L = 6$, $V = 1200$ and $E = 1000$, while the constant output P is 50,000 watts. As the excitation is weakened, OE gets smaller by the point E moving down the arc towards the critical point B. This E.M.F. OB is the minimum; that is to say, this is the weakest excitation with which the motor can give an output of 50,000 watts. If we reconstructed Fig. 358 for an E.M.F. equal to OB instead of 1000 volts we should find the maximum value of P, corresponding to $\gamma = 90° - \alpha$, to be 50,000 watts.

In addition to the angle γ we can find the angle ϕ between terminal pressure and current from the diagram in Fig. 359. A line CF is drawn making the angle α with CE, and a perpendicular OF is dropped upon it from O. It is seen at once that the triangle OFC corresponds exactly to the triangle BGF in Fig. 355. The angle COF is therefore equal to the phase displacement ϕ.

Adding up the three angles of the isosceles triangle AOC, we get

$$180° = 2\alpha + (\beta + \delta) + (90° - \alpha),$$

while, in the right-angled triangle OFC,

$$\phi = 90° - (\alpha + \delta).$$

By adding these two equations, we have

$$\phi = \beta.$$

When the point E coincides with D the angle β vanishes, and the current is consequently in phase with the terminal pressure. Seeing that the load is constant the current must have its minimum value at this point, which agrees with the fact that the armature drop CE will then have its minimum value CD. If a curve be plotted with the field current as abscissae, and the corresponding values of the main current taken by the motor as ordinates, **V**-shaped curves will be obtained, exactly similar to those shown in Fig. 348.

If the point E is below D, the current vector OF lags behind the terminal pressure, and the angle β represents the angle of lag ϕ. This is the case whenever the E.M.F. is less than the terminal pressure.

By increasing the excitation we can make the E.M.F. greater than the terminal pressure, which is an absolute impossibility in a D.C. motor, since it thereby becomes a generator. If OE exceeds OD, the angle β becomes negative, that is, the current vector leads ahead of the terminal pressure. We have previously mentioned that synchronous motors, when over-excited in this way, can be advantageously employed to neutralise the lagging current taken by other apparatus. In this way the load on the generators in the power station can be made to have unit power-factor.

CHAPTER XVI

111. The rotary field in a two-phase motor.—112. The rotary field in a three-phase motor.—113. Mesh or delta connection.—114. Star connection.—115. The measurement of power in polyphase circuits.—116. General principles of the rotor.

111. The Rotary Field in a Two-phase Motor.

The armatures of machines which generate two-phase alternating current contain two separate windings, or sets of coils, displaced from each other by 90°, the pole-pitch being considered 180°. When the coil-sides of one winding are situated under the middle of the poles, those of the other winding are midway between the poles (Figs. 360 and 361). The two windings differ merely in their instantaneous position, i.e. in their phase. For this reason the windings are generally spoken of as the phases of the generator, and we shall often use the expression in this sense.

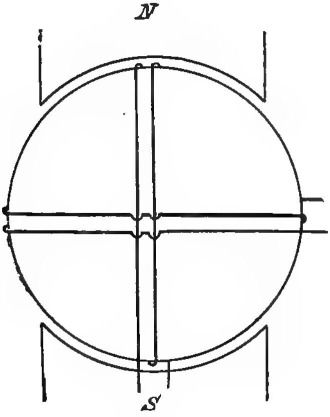

Fig. 360

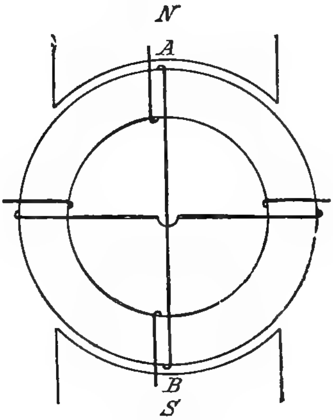

Fig. 361

The drum-winding in Fig. 360 is quite simple. The two-phase ring-winding in Fig. 361 is wound in such a manner that the end connections, after one coil is completed, pass right across the armature, as in the drum-winding. The practical identity of the two windings is evident. The wires on the outside of the ring at *A* and *B* are equivalent to the wires on the periphery of the drum, and the electromotive forces in all the wires of one phase can be added together.

The currents produced by the generator can be used to magnetise the stationary iron ring of an induction motor. This ring, which is called the stator, is built up of sheet-iron stampings, and is wound in exactly the same manner as the armature of the generator. The ends of the stator-windings are connected to the corresponding ends of the generator-windings (Fig. 362). If the armature of the generator rotates, the connections must, of course, be made by means of slip-rings.

If the armature be rotated in the direction indicated, i.e. clockwise, then an E.M.F. towards the observer will be induced in the wires under the south pole, and away from the observer in those wires which are under the north pole. This is shown by the arrows in the figure. If we neglect any phase displacement due to self-induction, the arrows will represent current at its maximum value. At the moment chosen in the figure there will be no current in the other phase-winding. By tracing out the currents in the stator, we find that the field produced will be as indicated by the dotted lines. The lines of force leave the stator at *N*, pass across the internal air space, or through an iron cylinder if one be placed within the ring, and enter the stator again at *S*. The internal surface of the ring will have a north pole at *N* and a south pole at *S*.

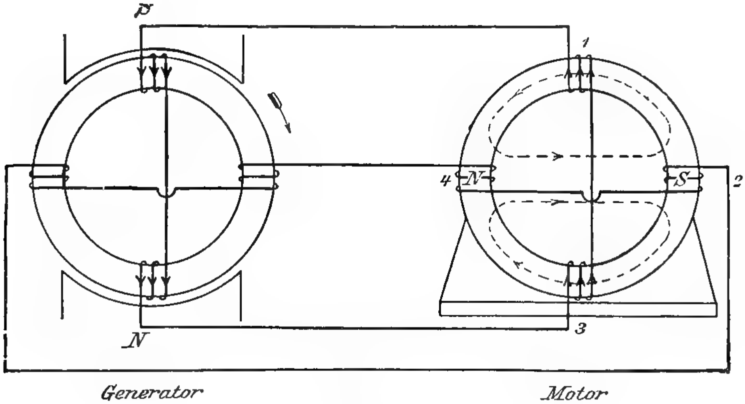

Fig. 362

After the armature of the generator has turned through 45° the conditions will be as shown in Fig. 363. The stator has not moved, but the armature coils of both phases of the generator are now equally induced. The field near the pole-tips will not be so strong as at the centre of the pole, and, moreover, the armature coils are only partly under the poles. The current will therefore be smaller than that in Fig. 362. We may assume that the current varies according to the sine law, so that its value at the moment indicated in Fig. 363 will be

$$i = \hat{i}.\sin 45^\circ = 0{\cdot}707\,\hat{i}.$$

The smaller current is indicated in the figure by lighter arrows.

If we trace out the current in the windings of the stator, we see that every coil is carrying current and that, although coils 1 and 2 belong to different phases, they are virtually equivalent to a single coil of double the number of turns. The current has the same direction in both, viz. outwards in the end connections facing us. Similarly coils 3 and 4 may be looked upon as a single coil, more especially as in actual motors the whole periphery is utilised and no gaps are left between the windings.

The magnetic field produced by these stator currents is shown by the dotted lines, the inner surface of the ring has still a north and a south pole, between which the flux crosses the internal space. The position of these poles has rotated through 45° from the position they occupied in Fig. 362. On turning the generator through another 45°, the stator coils 1 and 3 will have no current, while the current in coils 2 and 4 will have reached its maximum. There will then be a north pole at 1 and a south pole at 3.

We have thus arrived at the surprising result that the rotation of the armature of the generator causes the field of the motor to rotate, or causes the poles of the stator to rotate around the stationary iron ring. We can therefore imagine the stator to be replaced by a rotating field-magnet system with salient poles pointing inwards. The flux of such a field-magnet would continually cut through an iron cylinder placed within the stator ring (compare Fig. 382).

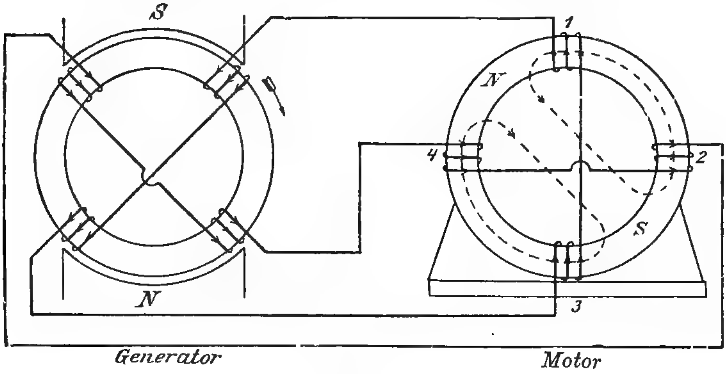

Fig. 363

This iron cylinder is called the rotor. It, also, is built up of sheet-iron stampings, carried on a central spindle or shaft. This shaft is supported at either end in bearings, so that the rotor can turn within the stator, with a small air-gap between the two. A short-circuited winding is contained in slots in the outer surface of the rotor. Currents are induced in this short-circuited winding by the rotating field produced by the stator, and these currents, as we know from Lenz's law, will tend to oppose the relative motion between field and conductors. The result is that the rotor is dragged round by the rotating field. The direction of rotation of the motor is therefore the same as that of the rotating field, and this depends on the way the stator phases are connected up to the generator. If the two connections to one of the phases be interchanged, it can be easily seen that the direction of rotation will be reversed.

112. The Rotary Field in a Three-phase Motor.

The armature of a three-phase generator contains three separate windings, displaced from each other by 120° (Fig. 364). Care must be taken when drawing the windings, that the beginnings 1, 2, and 3, of the three phases are 120° apart, and that, starting from these points, each phase-winding passes round the armature in exactly the same direction. In Fig. 364, for example, each winding commences by passing inwardly across the front end of the ring. These windings are connected by means of slip-rings with the corresponding terminals of the stator of the motor. The fact that the generator has been numbered clockwise and the motor in the opposite direction is immaterial, and has been adopted to avoid crossing the connections.

Phase 1, 1′ is shown in Fig. 364 under the middle of the pole and, if phase displacement due to self-induction be neglected, the current in this phase will have its maximum value. The instantaneous value of the current in the other phases will be

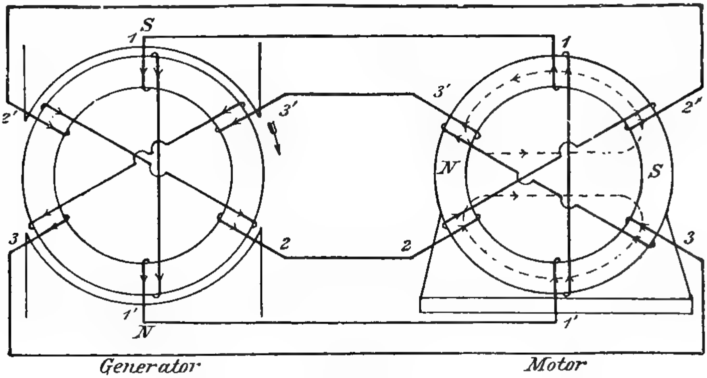

Fig. 364

$$i = \hat{i} . \sin 30° = 0{\cdot}5 \, \hat{i}.$$

By tracing out the current in the various coils of the stator, it is seen that the coil-sides 3′, 1 and 2′ carry current in the same direction, and may be grouped together with respect to their magnetic effect. The same is true of the coil-sides 2, 1′ and 3, although they belong to three separate windings. The resultant magnetic field will evidently be as indicated by the dotted lines, and the poles as shown by the letters *N* and *S*.

The state of affairs a twelfth of a period later is shown in Fig. 365. The generator has turned through 30°, and the phase-winding 3, 3′ is now in the neutral zone and therefore without current. In the other two phases the momentary value of the current is

$$i = \hat{i} . \sin 60° = 0{\cdot}866 \, \hat{i}.$$

The coil-sides 1 and 2′ of the motor may now be grouped together, and similarly 2 and 1′. The lines of force will be as indicated by the dotted lines, and the north pole of the stator will be situated about coil 3′. It is evident that the poles have rotated through 30° from the position they had in Fig. 364, and that the rotation of the generator causes a corresponding rotation of the motor field. If the stator has a 2-pole winding, as in the figure, the field makes a complete revolution in one period.

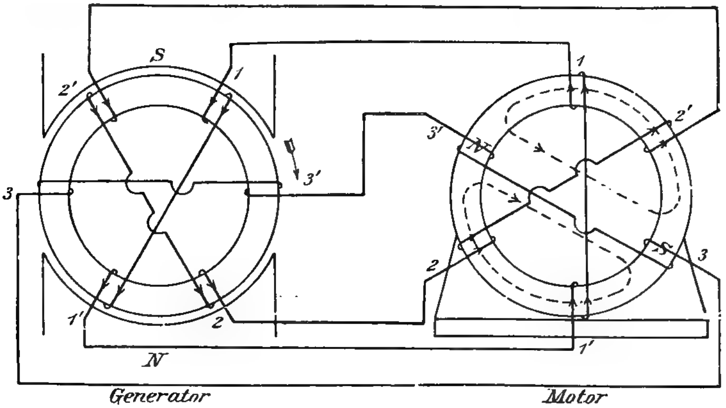

Fig. 365

113. Mesh or Delta Connection.

The six wires between the generator and the motor in Figs. 364 and 365 can be reduced to three by interconnecting the three phases. This can be done in two ways; one, known as mesh connection, we shall consider in this section, while the other, known as star connection, will be considered in the next section.

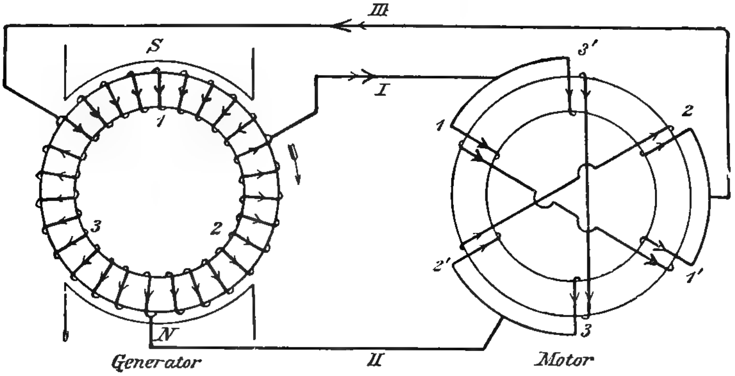

Fig. 366

In the mesh connection the end of each phase-winding is connected to the beginning of the next. The principle of this winding is seen very clearly in machines which are wound so as to supply both direct and alternating current. The three-phase current is taken from three slip-rings connected to points on the armature separated by 120°. In other respects the machine is identical with a D.C. dynamo. (See the left-hand part of Fig. 366.) The width of each coil-side is equal to $\frac{2}{3}$ of the pole-pitch. If the coil-side be made equal to $\frac{1}{3}$ of the pole-pitch, as represented diagrammatically in Figs. 364 and 365, the phases are connected as shown on the right-hand side of Fig. 366.

The admissibility of interconnecting the three phases in this way can be seen from the vector diagram in Fig. 367. The resultant electromotive force *R* of phases 2 and 3 is equal and opposite to that of phase 1. If then, we connect two of the phase-windings in series (Fig. 368), their joint E.M.F. is equal at every instant to the E.M.F. of the other phase, and the two voltmeters in the figure give the same reading. The equal instantaneous electromotive forces act in opposite directions around the ring, so that the whole ring can be connected up without any current circulating around it. With regard to the external circuit the two electromotive forces are, however, in parallel. This is evidently analogous with the two halves of a D.C. armature, or with two machines in parallel on the same bus-bars. It is quite clear that the P.D. between any two of the three external leads is equal to the E.M.F. of one phase-winding, neglecting, of course, any drop of pressure due to resistance when current is flowing.

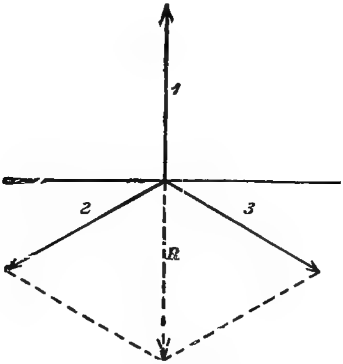

Fig. 367

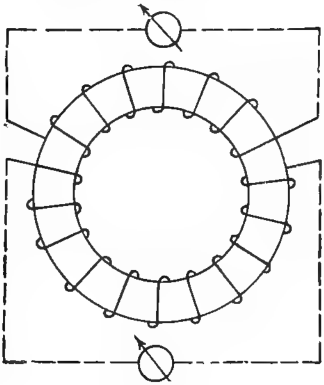

Fig. 368

The distribution of current in the mesh connection is apt to prove a difficulty on first consideration. We know that electricity cannot be heaped up or stored at any point of the circuit, so that each wire in turn must act as the return lead for the other two. To make the matter quite clear, we shall consider the currents in generator, leads and motor for the two cases previously considered.

At the moment represented in Fig. 366 phase 1 of the generator is directly under the pole, and its current will be a maximum, if we neglect any effect of self-induction. The current in each of the other phases will have half this value. The current in external lead *I* will be equal to $1{\cdot}5\,\hat{i}$

if $\hat{i}$ is the maximum current in each phase of the generator-winding. Of this external current a part $\hat{i}$ flows through phase 1, 1′ of the stator, while a current of half this value flows through phases 3′, 3 and 2′, 2 in series.

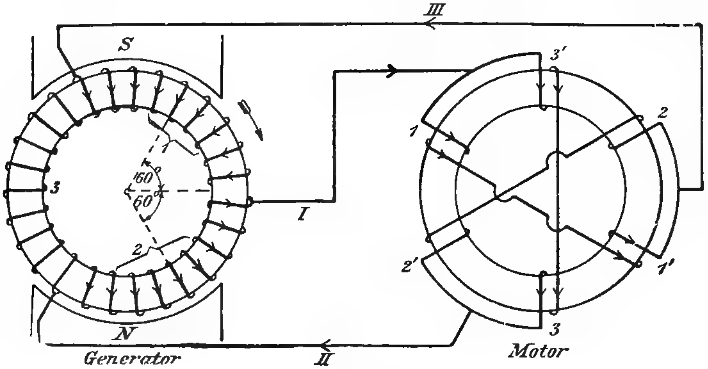

Fig. 369

Both in the generator and in the motor, the coil-side carrying the maximum current has on either side of it coil-sides carrying current in the same direction but of half the strength. The two stator currents combine between coil-sides 1′ and 2, and a current of $1{\cdot}5\,\hat{i}$ flows back to the generator through the lead marked *III* while lead *II* is momentarily without current.

A twelfth of a period later, the generator will have turned through 30°, as shown in Fig. 369. Phase 3 of the generator is now in the neutral zone and therefore without current, while phases 1 and 2 carry a current of

$$\hat{i}.\sin 60° = 0{\cdot}866\,\hat{i}.$$

These currents combine to produce a current of $1{\cdot}73\,\hat{i}$ in the external lead *I*. The vector diagram in Fig. 370 shows that this is the maximum value reached by the external current. Half of this current passes through the stator phase 3, 3′ and returns to the generator by lead *II*, while the other half passes through phase 1, 1′ and back to the generator through lead *III*. Hence, at the moment that the current in one of the external leads reaches its maximum value, one of the phases of both generator and motor is without current.

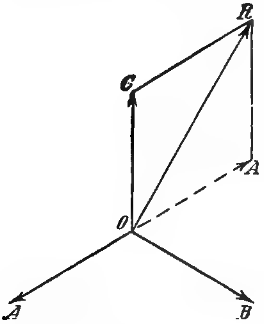

Fig. 370

We find, therefore, that the maximum external current is greater than the maximum current in one phase of the mesh connection, and that these maxima do not occur at the same time. This is very clear in the vector diagram (Fig. 370). We must remember, however, that the starting points of two phase-windings have not been joined, but the starting point of one to the end of the other. For this reason, when two current vectors are compounded to find the external current, one of them must be reversed. The external current *OR* is, for example, the resultant of *OC* and the dotted vector *OA*. It is 1·73 times as great as *OC*, and is 30° out of phase with it.

114. Star Connection.

Whereas in the mesh connection the end of each phase was joined to the beginning of the next phase, in the star connection the three starting points are connected together within the machine, and the three ends connected to the three slip-rings or terminals (Fig. 371). That such an interconnection of the three windings is permissible can be readily seen from Fig. 372. The vectors *OA* and *OB*, representing the current in two of the phases, have, as their resultant, the vector *OR*, which is exactly equal and opposite to the vector *OC* of the current in the third phase-winding. The total current flowing to the neutral point or common connection is therefore nil, or, in other words, the current flowing towards the neutral point is exactly equal, at every moment, to the current flowing away from this point. If the coil 1 of the generator armature shown in Fig. 371 is at the given moment exactly under the middle of a pole, the E.M.F. in it will be a maximum, and if we assume that the current is in phase with the E.M.F., the current in coil 1 will have its maximum value at this moment. This current will divide equally between the two other phases as shown in the figure.

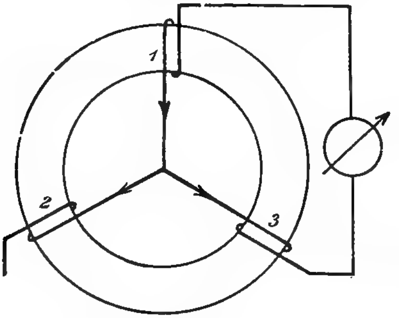

Fig. 371

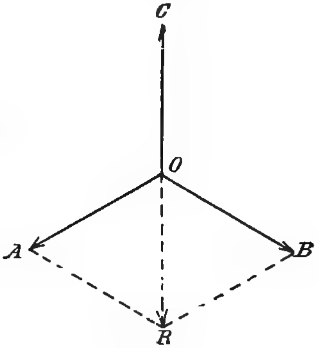

Fig. 372

It is obvious that, with this arrangement, the current in each external lead is identical both in magnitude and phase with the current in the corresponding winding. The P.D. between two external leads, however, will not be due to a single winding, but to two phase-windings connected in series, or, more accurately, in opposition. The voltmeter in Fig. 371 is connected across phases 1 and 3. At the moment indicated in the figure the induced E.M.F. in coil 1 is a maximum, while in coil 3 it is equal to

$$\hat{e}.\sin 30^\circ = 0{\cdot}5\,\hat{e}.$$

If the E.M.F. of coil 1 acts towards the neutral point, that of coil 3 will act away from it, since they are 120° out of phase. We must therefore add the two electromotive forces to find the voltmeter reading, which will therefore be equal to $1{\cdot}5\,\hat{e}$.

A twelfth of a period later the coil 2 will be in the neutral zone and have no induced E.M.F. Coils 1 and 3 will have equal electromotive forces induced in them, viz. $\hat{e}.\sin 60° = 0{\cdot}866\,\hat{e}$ (Fig. 373). As before, we see that the voltmeter will read the sum of these electromotive forces, that is, $1{\cdot}73\,\hat{e}$. It can be shown very easily that this is the maximum momentary value attained by the P.D. between two external leads. If, for example, the vectors OA and OB in Fig. 374 represent the electromotive forces induced in the coils 1 and 3, we must remember that these coils are connected in opposition, that is, their starting points are connected together, so that in passing through the armature from lead I to lead III (Fig. 373) we pass inwardly through coil 1, but outwardly through coil 3. To find the resultant E.M.F., which is the voltmeter reading, one of the vectors must therefore be reversed, giving OA and OB' with the resultant OR. This resultant is

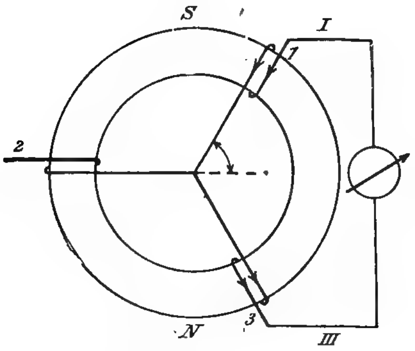

Fig 373

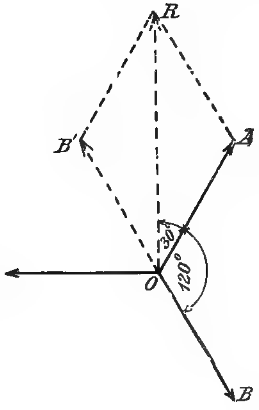

Fig. 374

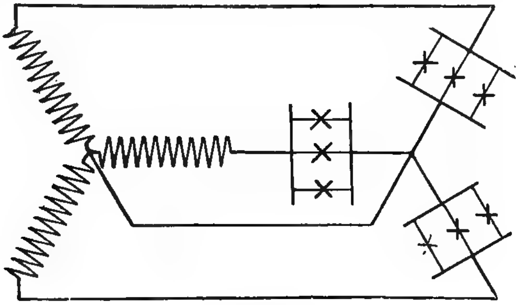

Fig. 375

1·73 times as large as either of the phase pressures, and is displaced from them with regard to phase, as shown in the figure.

It is quite allowable to have a star-connected generator driving a mesh-connected motor, or *vice versa*. For long-distance power transmission the star connection is preferable both for motors and generators, since the line pressure is then 1·73 times the pressure per phase of the machines. If the load consists of resistances such as lamps, they can be arranged either in star or mesh connection. If the former arrangement be adopted, care must be taken that the number of lamps is the same in each branch.

This limitation can be removed and the branches loaded unequally without affecting the constancy of pressure, so necessary for glow-lamps, by the use of a fourth lead connecting the neutral points of generator and load (Fig. 375). This is analogous to the neutral or middle wire in the 3-wire system which is now so largely used for direct-current lighting.

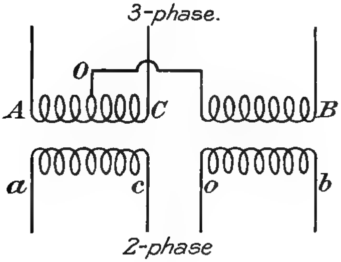

Fig 376

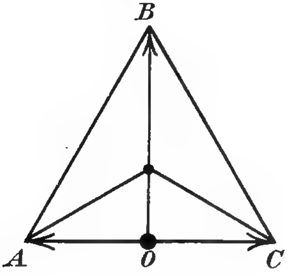

Fig. 377

The so-called Scott system for transforming energy from a two-phase to a three-phase system or *vice versa* embodies a special type of star connection. Two single-phase transformers are employed connected as shown in Fig. 376, the tapping or star point O being at the middle of the winding AC. The wires ac are connected to one phase of the two-phase system and the wires ob to the other, whilst the three-phase system is connected to the points A, B and C. Regarding the two phase side as the primary, the electromotive forces induced in AC and OB will be 90 degrees out of phase. With respect to the star point O, the E.M.F. induced in OA is 180 degrees out of phase with that induced in OC, since when the E.M.F. is directed towards the star point in one it is directed away from it in the other. This is shown in Fig. 377 which is a vector diagram of the electromotive forces. Neglecting any drop in the windings, the triangle ABC represents the vector diagram of the potential differences between the three phase lines. For this to be symmetrical, the triangle must be equilateral, and therefore

$$OB = AC \cos 30^\circ = \frac{\sqrt{3}}{2} . AC.$$

Hence the ratio of the turns $\frac{OB}{ob}$ must be $\frac{\sqrt{3}}{2}$ of the ratio of the turns $\frac{AC}{ac}$.

The transfer of energy can take place, of course, in either direction.

115. The Measurement of Power in Polyphase Circuits.

We shall consider three-phase power supply in this section. The measurement of two-phase power is merely a duplication of the single-phase case, and need not be further considered. The following symbols will be used independently of the method of connection employed (star or mesh).

V_1 = the P.D. per phase (i.e. per limb in star connection),
I_1 = the current per phase (i.e. per limb in mesh connection),
ϕ = the angle between V_1 and I_1,
V = the P.D. between external lines,
I = current in each external line,
P = the total power in watts.

The power per phase for either connection is equal to

$$V_1 . I_1 . \cos\phi,$$

and the total power is three times this amount.

For star connection we have

$$V = 1{\cdot}73\, V_1,$$

$$I = I_1,$$

and $$P = 3V_1 . I_1 . \cos\phi = 3 . \frac{V}{1{\cdot}73} . I . \cos\phi = \sqrt{3} . V . I . \cos\phi.$$

For mesh connection we have

$$I = 1{\cdot}73 I_1,$$

$$V = V_1,$$

and $$P = 3V_1 . I_1 . \cos\phi = 3V . \frac{I}{1{\cdot}73} . \cos\phi = \sqrt{3} . V . I . \cos\phi \quad ...(166).$$

The formula obtained by using the line current and line pressure is thus exactly the same, no matter what the connection employed.

This is a suitable point at which to compare the percentage losses of power and pressure in the lines of three-phase and D.C. transmissions. We assume that the same power P has to be transmitted the same distance in each case. We assume also that the effective P.D. between the lines is the same in each case, and also the loss of power in the lines. We proceed to determine the relative amounts of copper necessary in each case. Let

R_1 be the resistance of each single D.C. line,
R_2 ,, ,, ,, ,, 3-phase ,,
A_1 ,, section ,, ,, D.C. ,,
A_2 ,, ,, ,, ,, 3-phase ,,

We have:

For D.C.	For 3-phase.
$P = V.I,$	$P = \sqrt{3}.V.I.\cos\phi,$
$I = \frac{P}{V}.$	$I = \frac{P}{\sqrt{3}\,.\,V.\cos\phi}.$
Loss in the two wires.	Loss in the three wires.
$2\,.\,I^2\,.\,R_1 = 2\frac{P^2\,.\,R_1}{V^2}.$	$3\,.\,I^2\,.\,R_2 = \frac{P^2\,.\,R_2}{V^2\,.\cos^2\phi}.$

Since these losses have to be the same in each case, we equate them, thus

$$2R_1 = \frac{R_2}{\cos^2\phi},$$

or

$$\frac{R_1}{R_2} = \frac{1}{2\cos^2\phi}.$$

Since the lengths are the same in both cases, the cross-sections must be inversely proportional to the resistances. Hence

$$\frac{A_2}{A_1} = \frac{R_1}{R_2} = \frac{1}{2\cos^2\phi}.$$

Again, as the lengths are equal, the volume of copper in each case is proportional to the sum of the cross-sections of the wires, so that

$$\frac{V_2}{V_1} = \frac{3A_2}{2A_1},$$

or

$$\frac{V_2}{V_1} = \frac{3}{2}\cdot\frac{R_1}{R_2} = \frac{0{\cdot}75}{\cos^2\phi}.$$

Hence, if $\cos\phi = 1$, i.e. if the current is in phase with the pressure, the three-phase system requires only 75 per cent. of the copper necessary for the D.C. transmission. If the load is at all inductive, it is evident that this advantage rapidly disappears. These conclusions depend entirely on our assumption that the effective pressure between the lines is the same in each case. The same method is applicable, of course, whatever assumptions we make as to the pressure.

We shall now consider the actual measurement of the power in a three-phase circuit. With star connection and an accessible neutral point, the measurement is very simple. The pressure coil of the wattmeter is connected between the neutral point and the end of one phase-winding, while the current of this phase is passed through the main coil of the wattmeter. The wattmeter reading will give the power in one phase, and if the three phases are equally loaded, the total power will be three times this amount. If the loads on the phases are unequal, the measurement must be carried out in each phase and the three results added together. If three wattmeters are available, the three readings can be taken simultaneously; if only one wattmeter is used, it must be connected successively in the three phases.

If the neutral point is not available, another method, known as the two wattmeter method, must be adopted. This method is equally applicable to star and mesh connection, and is represented in Fig. 378. The current coils of the wattmeters are introduced into two of the leads, and their pressure coils are connected between the same lead and the lead which has no wattmeter coil in it. If v_1, v_2 and v_3 are the instantaneous values of the phase pressures, and i_1, i_2 and i_3 the corresponding values of the currents, then the value of the power at the given moment will be

$$p = v_1 i_1 + v_2 i_2 + v_3 i_3.$$

Since the sum of the currents flowing away from the generator at any moment is zero, we have

$$i_2 = -(i_1 + i_3).$$

If this value be substituted for i_2 in the above equation for p, we get

$$p = v_1 i_1 - v_2 (i_1 + i_3) + v_3 i_3,$$

or

$$p = (v_1 - v_2) . i_1 + (v_3 - v_2) . i_3.$$

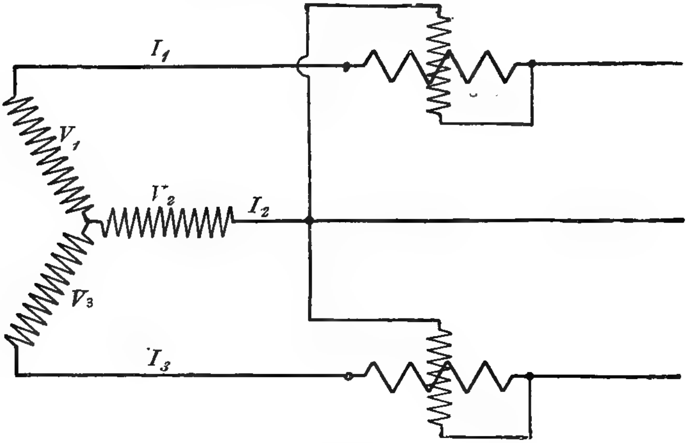

Fig. 378

Now, $(v_1 - v_2)$ is the momentary P.D. across the pressure coil of the upper wattmeter, while $(v_3 - v_2)$ is the momentary P.D. across the pressure coil of the lower wattmeter. The product $(v_1 - v_2) . i_1$ gives the momentary value of the torque on the upper wattmeter, while $(v_3 - v_2) . i_3$ gives that on the lower one. The reason for the differences of the phase pressures appearing in the brackets and not the sums, is that the two phase-windings are connected in opposition across the pressure coils of the wattmeters, so that to pass from one pressure terminal to the other we go through two phase-windings, one towards the neutral point and the other away from the neutral point.

In consequence of the inertia of the moving systems of the wattmeters, they take up stationary positions corresponding to the mean value of the power. The total three-phase power will therefore be given by the sum of the two wattmeter readings. One must be careful, however, to observe the signs of the two terms in the last equation for p. As a rule, they are both positive, and the wattmeters, if connected up as shown, will deflect in the same direction. In this case their readings must be added.

If, however, the load be very inductive, one of the wattmeters will deflect in the opposite direction. The direction of the deflection can be reversed by reversing the pressure coil connections of this wattmeter. The total power is then equal to the difference between the two wattmeter readings. The reason for this reversal is evident from Fig. 379. The vector V' is the resultant of V_1 and $-V_2$, and represents the pressure across the upper wattmeter. Similarly, V'' is the pressure across the lower wattmeter. The currents in the main wattmeter coils are represented by the vectors I_1 and I_3. The angle of lag in each phase is ϕ. It is evident that an increase of the lag ϕ will increase the reading of the upper wattmeter, which is equal to $V'.I_1.\cos(30-\phi)$, until $\phi = 30°$, beyond which it will again decrease. In the same way, the reading of the lower wattmeter is equal to

$$V''.I_3.\cos(30+\phi),$$

and will gradually decrease as the load becomes more inductive, until when $\phi = 60°$ the wattmeter will read zero. If the lag increases still further, the reading of the wattmeter will reverse.

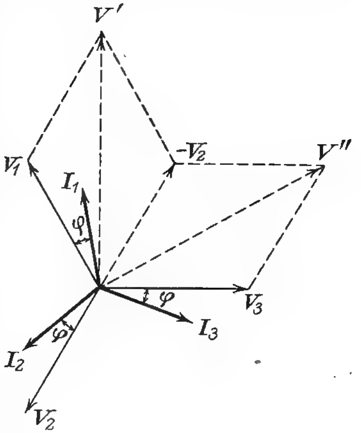

Fig. 379

The measurements can be carried out very conveniently with one wattmeter, by means of a board containing six mercury cups, which can be connected up by means of copper bridges as shown in Figs. 380 and 381. In the first figure the current coil of the wattmeter is in series with phase 2, while in the second figure phase 2 is unbroken and the wattmeter is placed in phase 1. In either case, the pressure coil is connected between the current coil and the unbroken line 3. The small mercury commutator is for the purpose of reversing the pressure coil if necessary.

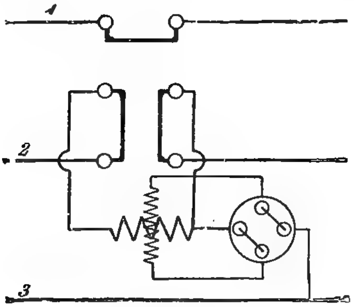

Fig 380

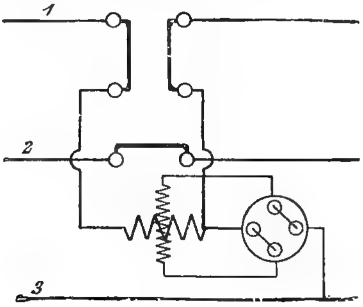

Fig. 381

The duplicate readings can be avoided by the use of special wattmeters consisting of two separate wattmeters with a common spindle, so that the reading depends on the sum or difference of the two torques, as the case may be.

116. General Principles of the Rotor.

Having considered the production of the rotating magnetic field and the winding of the stator, we shall now study the effect of the rotating field on the rotor. The rotor resembles a D.C. armature in that it is built up of stampings of sheet iron, assembled on the shaft or on a spider or boss carried on the shaft. The winding is carried in slots in the usual way, and may consist of a permanently short-circuited arrangement of bars or coils, or may be wound with a three-phase winding very similar to that on the stator. In the latter case the three ends are brought to three slip-rings, between which starting resistances can be introduced.

We have seen that the field of a stator, with a two-pole winding, makes one complete revolution during every period. In order to follow this rotating field more easily, we shall imagine, for the time being, that the stator is replaced by a magnet ring with salient poles, excited with direct current. This is shown in Fig. 382, where the poles are supposed to rotate around the rotor in an anti-clockwise direction. The currents induced in the rotor windings will be exactly the same as if the poles were stationary and the rotor turned in the clockwise direction. These currents are indicated in the figure, and are away from the observer under the north pole, and towards him under the south pole.

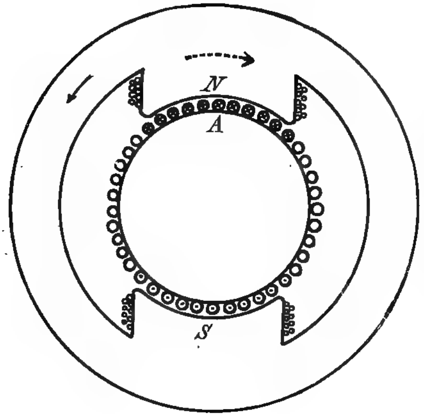

Fig. 382

We assume, moreover, that the field magnet represents not merely the field produced by the stator currents, but the resultant flux Φ, produced by the combined action of stator and rotor currents. In this case the induced E.M.F. and the rotor current will both have their maximum values under the middle of the poles.

We know from Lenz's law that the currents induced in the rotor wires will tend to oppose the relative motion between themselves and the rotating field, that is, the rotor will be dragged round in the same direction as the field. To make this clearer, we may imagine ourselves to be swimming down a rotor wire at *A* and looking out towards the north pole, then we know from Ampère's rule that the north pole would be driven in the clockwise direction as shown by the dotted arrow. There is, of course, an equal and opposite force acting on the rotor wire at *A*, and as this is free to move, the motor will rotate in an anti-clockwise direction.

If the motor is unloaded and the bearings are entirely frictionless, the rotor revolves at the same speed as the primary field. There would then be no cutting of lines, no induced E.M.F. and no current in the rotor. When a load is applied to the motor spindle, a certain rotor current is essential to the production of the necessary torque. This current can only be produced by the rotor running at a lower speed than the field, so that the latter can cut the rotor wires and thus induce the necessary current in them. This decrease of rotor speed is known as the slip. Under ordinary conditions the resistance of the rotor coils is so small that a very small E.M.F., and therefore a very small slip, is sufficient to produce the necessary current. The slip, that is, the difference between the speeds of field and rotor, is sometimes no more than 1 or 2 per cent. on full load.

The fact that the rotor runs at approximately the same speed as the primary field excludes the employment of two-pole windings with the high

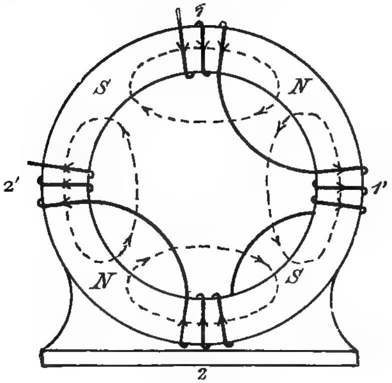

Fig. 383

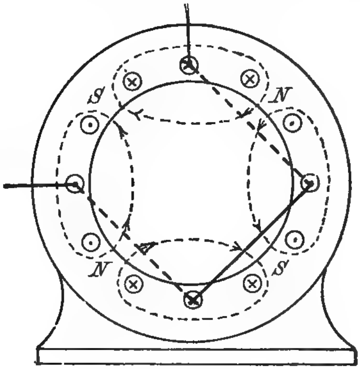

Fig. 384

frequencies commonly used. To ensure a steady illumination without any appreciable flicker, the frequency is generally about 50 per second. A two-pole motor driven from these mains would have a speed of nearly 3000 revolutions per minute. In systems where the load consists almost entirely of motors, and little account need be taken of lighting, the frequency is sometimes taken as low as 25. Even then a two-pole motor would make 1500 revolutions per minute, which is generally too high for motors above about 10 H.P. The speed is reduced by using multipolar windings. In a four-pole winding the simple two-pole winding, which we have already considered, is compressed into a half of the total periphery, and connected in series with a similar winding on the remaining half of the stator. In Fig. 383, for example, the coils 1 and 1′ of a three-phase stator are no longer diametrically opposite, as they were in Fig. 362 on page 357, but are now separated by 90°. The other coils 2, 2′ belonging to the same phase are connected in series with them. For the sake of clearness, only the one phase has been drawn in Fig. 383, but the direction of the currents in the other phases is evident. Whenever the current in any coil is a maximum, the coils on either side of it carry currents in the same direction, but of half the magnitude. If the current in the phase shown in the figure is a maximum at the given moment, the dotted lines will represent the lines of force and show the positions of the poles.

The rotating field will now only make half a revolution during each period. If p be the number of pairs of poles, and f_1 the frequency of the supply, then the field will make n_1 revolutions per second, where

$$n_1 = \frac{f_1}{p}.$$

If the speed of the rotor be n, the difference will be $n_1 - n$ revs. per second. The effect on the rotor is the same as if it rotated backward at this speed in a stationary field. The frequency of the currents induced in the rotor will therefore be

$$f = (n_1 - n)\,p \quad \text{..........................(167)}.$$

If the rotor is wound, it must be arranged to have the same number of poles as the stator. The winding, however, can be either two or three-phase, quite irrespective of the number of phases of the stator-winding.

A four-pole drum-winding is shown in Fig. 384, where one phase only is shown connected up, the currents in the other phases being represented by crosses and dots. It is assumed that the current has its maximum value in the phase shown connected up, while the other currents have half this value. In actual practice drum-windings are always used; we have only used the ring-windings to make the rotating field as simple as possible.

The direction of rotation of the rotor is determined by that of the primary field. It can be reversed by changing two of the supply leads on the stator terminals.

The polyphase motor differs in this respect from the single-phase induction motor. Such a motor can be obtained by disconnecting one of the supply leads to a three-phase star wound stator. The motor will not start from rest, but, if the rotor be started in either direction, the speed will increase in that direction until it reaches the speed corresponding approximately to the primary frequency. Such motors are generally started as two-phase motors by means of an auxiliary winding on the stator. The action of the single-phase induction motor is very complicated, and we shall postpone its further consideration until Section 132.

CHAPTER XVII

117. Stator and rotor windings.—118. The magnetic flux of an induction motor.—119. The effect of the iron in the magnetic path.—120. Torque of an induction motor.—121. Calculation of slip.—122. Calculation of the E.M.F.

117. Stator and Rotor Windings.

For the sake of simplicity we have always assumed that all the wires constituting a coil-side are contained in a single slot, or that the width of the coil-side is so small that there are considerable gaps between the wires of different phases. As a matter of fact, however, the wires of each coil-side are distributed between several slots, so that the coil-side has considerable width. In this way the field strength changes gradually from point to point round the stator, and sudden jumps are avoided. Moreover, the number of ampere-wires in each slot is kept small in order to decrease the self-induction and leakage, as will be seen in Section 131. As a general rule each stator coil-side is distributed between from 2 to 5 slots, while each coil-side of the rotor occupies from 3 to 7 slots. The rotor and stator must have a different number of slots, as otherwise there is a tendency for the motor to remain stationary and act as an ordinary transformer, instead of starting.

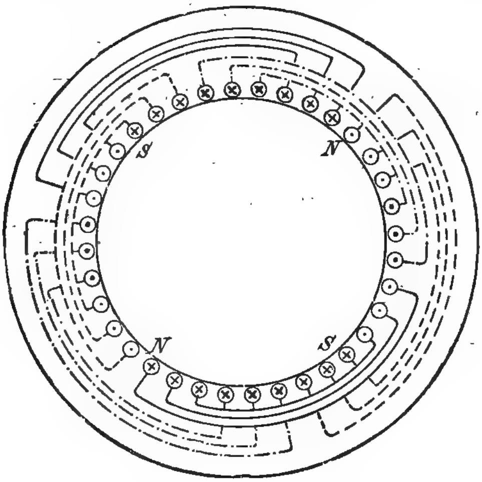

Fig. 385

A four-pole distributed coil-winding is represented in Fig. 385. The three phases are distinguished by full, dotted, and chain-dotted lines, of which the latter is assumed to be carrying its maximum current, while the others have half this current. The strength of the current is indicated approximately by drawing the crosses and dots heavily or lightly. It is easily seen that the mid-points of the magnetic poles are situated at the junction of opposite currents, so that the neutral axes lie always at the centres of groups of wires carrying current in the same direction.

In these coil-windings the breadth of each coil-side is equal to $\frac{1}{3}$ of the pole-pitch. Each coil is completely wound with the requisite number of turns before proceeding to the corresponding coil under the next pair of poles.

Another type of winding which can be used for both stator and rotor is the creeping bar-winding (wave-winding). In this winding the step is always in one direction, that is, it is a simple wave-winding, and, after going completely round the armature, arrives at a point near that from which it started. On proceeding with the winding, those wires which belong virtually to the same coil-side will lie together. If the total number of wires be Z the winding-step will be given by the formula

$$y = \frac{Z \pm 2}{2p}.$$

y must be an odd number, and Z must be divisible by 3, since the whole winding is to be divided into three phases. The principle of the winding is much clearer when a large number of wires is chosen, on account of a small unavoidable dissymmetry which is exaggerated when the total number of wires is small. The values chosen for Fig. 386 are $Z = 54$, i.e. 18 wires per phase, and $p = 2$, which gives a four-pole winding. For the step, we have

$$y = \frac{54 \pm 2}{4} = 14 \text{ or } 13.$$

For a singly re-entrant winding, 14 is impossible, so that $y = 13$. The winding is connected up as a star-winding in Fig. 386, the external leads coming to wires 6, 24 and 42. The current is assumed to have its maximum value in the first phase, which, starting from the neutral point, passes along *a* to wire 1, where it goes down to the further end of the stator or rotor, crosses to $1 + 13 = 14$, up which it comes to the front, and so on, in accordance with the following table:

1—14	10
/	/
27—40	23—36
/	/
53—12	49— 8
/	/
25—38	21—34
/	/
51—10	47— 6

If we continued in this way we should obtain an ordinary direct-current winding. It would be closed on itself and by tapping three points, separated from each other by $\frac{2}{3}$ of the pole-pitch, we should obtain a mesh-connected armature. By reference to Fig. 366 it is evident that the breadth of the coil-side would be equal to $\frac{2}{3}$ of the pole-pitch. In Fig. 386, however, we have chosen star connection; the end of wire 6 is led to a terminal or slip-ring, and wire 19 is taken as the beginning of a new phase. Whether star- or mesh-connected, the breadth of the coil-side is the same, viz. $\frac{2}{3}$ of the pole-pitch.

To make the diagram as clear as possible, the slots are shown in two tiers. A thick line is drawn around each coil-side of the first phase, a thin line around each coil-side of the second phase, and a dotted line for the third phase. Since the sum of any two of the currents is at any moment equal to the third, the currents in phases *b* and *c* must flow towards the neutral point, so that, in following the winding of these phases from the neutral point, we shall continually oppose the direction of the current. The winding of phase *b* is given by the following table:

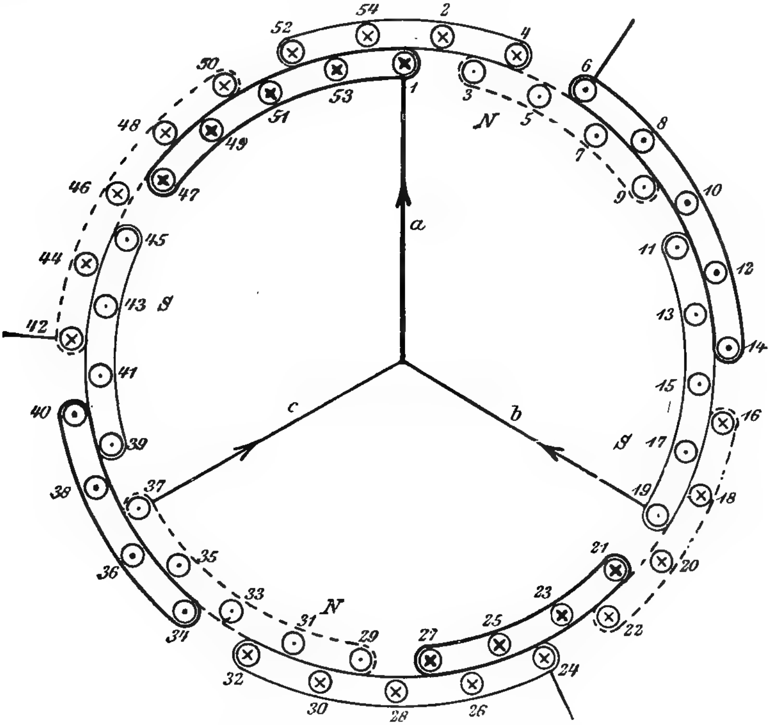

Fig. 386

19—32
/
45— 4
/
17—30
/
43— 2
/
15—28

28
/
41—54
/
13—26
/
39—52
/
11—24

Similarly, the winding of phase *c* is as follows:

37—50
/
9—22
/
35—48
/
7—20
/
33—46

46
/
5—18
/
31—44
/
3—16
/
29—42

The lines which have been drawn so as to enclose the various coil-sides make it very evident that the phases overlap. It is for this reason that the breadth of each coil is $\frac{2}{3}$ of the pole-pitch; without overlapping the breadth could not exceed $\frac{1}{3}$ of the pole-pitch. Such a wide coil is magnetically and electrically unfavourable. It can be seen from Fig. 386 that at some points of the stator, adjacent wires carry opposing currents, which is obviously an inefficient arrangement. The overlapping or consequent breadth of the coil-side is therefore avoided in actual practice by making the winding slightly different from that shown in Fig. 386. Instead of winding the whole of phase *a* straight forward from 1 to 6 as shown in the table on page 374, we stop when a half is completed, viz. 1, 14, 27, 40, 53, 12, 25, 38. In the ordinary way we should have gone from 38 to 51, but we now pass from 38 to 2, at least, the actual connection is equivalent to this. From 2 we proceed in the usual way to 15, 28, etc., and pass through 54 and 52. The wires constituting the coil-side are now 53, 1, 52, 54, 2, and they cover about $\frac{1}{3}$ of the pole-pitch.

The method actually adopted is not exactly that explained above, for, after winding half the phase, the direction of the winding is reversed. This is merely to get symmetrical end-connections, and makes no difference in the principle of the winding. Another method by which the want of symmetry in Fig. 386 can be avoided consists in making the wires per phase a multiple of the number of poles, and neglecting entirely the condition $2py = Z \pm 2$. In Fig. 387, for example, there are 4 wires, or 2 slots, per pole per phase. For the four poles this gives a total of 24 slots with 48 wires. This slot-step is exactly equal to the pole-pitch, viz. 6 slots. If we start from the neutral point *O* and follow round phase *I*, we pass from *O* to 1, down 1, across the back to 7, up 7, across the front to 13, and so on. The regular step would take us from the bottom of slot 19 to the top of slot 1, which is already filled. We therefore make the connection a little longer so as to pass to the top of the next slot 2. This longer connection is drawn slightly heavier. The winding then proceeds in the regular manner through 2 (top), 8 (bottom), etc., until we come to the bottom of 20, from whence we pass by a special connection to the bottom of 2, and reverse the direction round the winding. After passing through 2, 20, 14, 8 a special long connection, shown by the heavy line, connects across to the bottom of 1, whence by 19, 13, 7 we come to the end of the winding. This winding allows the end-connections to be arranged very conveniently; all those at the further end of the armature are exactly alike, and span the pole-pitch. This is also true of the front end, except, in the present case, for two slightly longer connections, which are adjacent to the beginning and the end of the winding. In general, if there are q' wires per coil-side, the winding will go round the stator or rotor $\frac{q'}{2}$ times in each direction, and the number of long end-connections will be $q' - 2$.

The other phases are wound in exactly the same way. The beginning of phase *II* should be separated from that of phase *I* by $\frac{2}{3}$ of the pole-pitch,

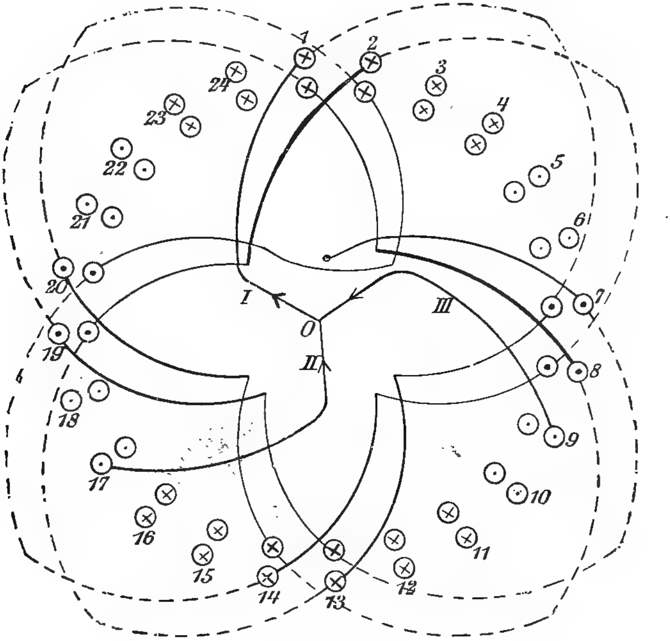

Fig. 387

but we can naturally start from the diametrically opposite but corresponding wire 17 instead of from 5. This is only true, of course, because this happens to be a four-pole winding.

118. The Magnetic Flux of an Induction Motor.

In the present section we shall seek to prove that the magnetic field of an induction motor has an approximately sinusoidal distribution. Moreover, we must calculate the magnetic flux produced by the stator and rotor currents with the different types of windings. Fig. 388 represents diagrammatically a coil-winding with a large number of slots per coil-side. For the sake of clearness, the stator periphery is shown straight, and the coil-sides are slightly separated from each other. The current in one phase is just at its maximum value, while in the other two it has half that value, as indicated by the crosses and dots.

We see now that D is the centre of a north pole, while at the points A and G the north pole ceases and south poles commence. The strength of the field falls off on both sides of D, and corresponds at every point to the number of ampere-turns which are effective at that point. This is represented by the length of the arrows in the figure.

In order to calculate the excitation at various points on the periphery we must observe that any line of force is produced by the ampere-wires which it encircles. If there are q' wires in each coil-side, i.e. q' wires per pole per phase, then

at A the excitation is 0,
,, B ,, ,, $q'.\hat{i}$ ampere-wires,
,, D ,, ,, $\frac{q'.\hat{i}}{2}+q'.\hat{i}+\frac{q'.\hat{i}}{2}$
$= 2q'.\hat{i}$.

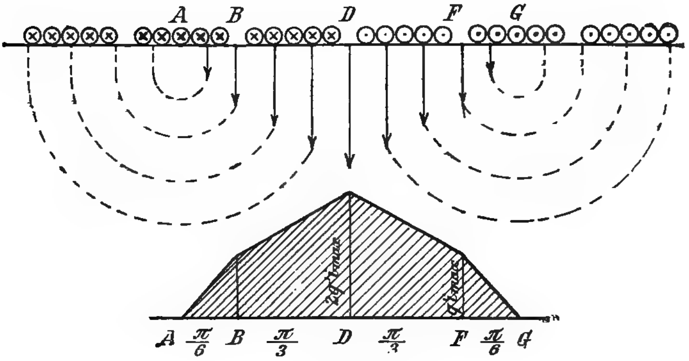

Fig. 388

If we plot the excitation at every point as ordinates on a base representing the pole-pitch π, we obtain the lower part of Fig. 388. Between A and B, and between B and D, the increase is uniform and is represented by straight lines. The mean excitation is obtained by finding the area of the shaded figure and dividing it by the base π. In this way we get

$$X_{\text{mean}} = \frac{1}{\pi}\cdot\left(\frac{\pi}{6}\cdot q'.\hat{i} + 3\cdot\frac{\pi}{3}\cdot q'.\hat{i}\right) = 1{\cdot}166\,.q'.\hat{i}.$$

Since the strength of the field at the various points of the periphery is proportional to the excitation at those points, it follows that the field curve has the same shape as the excitation curve, and is roughly a sine curve.

We turn now to the other extreme case, in which one phase is without current, while the current in the other two phases is equal to

$$\hat{i}.\sin 60^\circ = 0{\cdot}866.\hat{i} \text{ (Fig. 389).}$$

The wires between D and H form a coil-side of $2.q'.0{\cdot}866.\hat{i} = 1{\cdot}73.q'.\hat{i}$ ampere-wires. This is the effective excitation between points B and D, and is represented by the equal arrows between these points. Outside these points the excitation falls off uniformly, as shown by the arrows and by the shaded figure. By calculating the area of this figure and dividing by the base, we get

$$X_{\text{mean}} = \frac{1}{\pi} \cdot 2 \cdot \frac{\pi}{3} \cdot 1{\cdot}73 \cdot q' \cdot \hat{i} = 1{\cdot}155 \cdot q' \cdot \hat{i}.$$

This differs but little from the value found above for the other extreme moment; hence, the mean excitation and the mean flux density in the air-gap are approximately constant. Taking the mean of the two limiting values we have

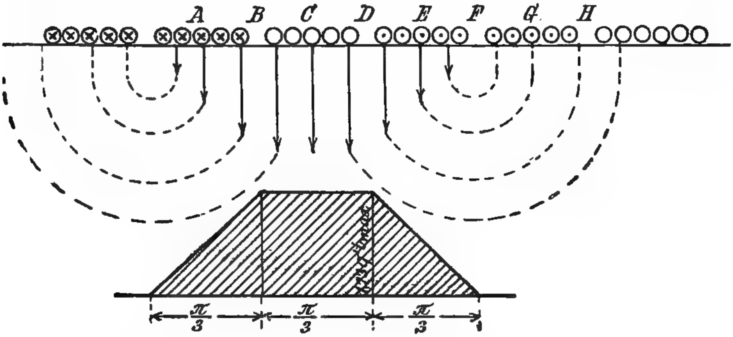

Fig. 389

$$X_{\text{mean}} = \frac{1{\cdot}166 + 1{\cdot}155}{2} \cdot q' \cdot \hat{i} = 1{\cdot}16 \cdot q' \cdot \hat{i} \quad \ldots\ldots\ldots\ldots(168).$$

The flux produced is proportional to the excitation. The rotating field has therefore an approximately constant number of lines of force, but a maximum flux density which varies, according to Figs. 388 and 389, in the ratio of 2 to 1·73. The two-phase motor is inferior to the three-phase in this respect, since the fluctuations in its maximum flux density are much greater, as can be seen on reference to Section 111.

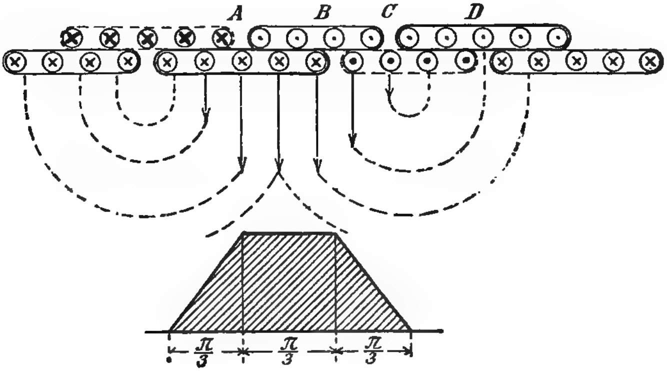

Fig. 390

The mean value of the excitation can be determined in a similar manner for a wave-winding of the type shown in Fig. 386 (Fig. 387 is analogous to the coil-winding just considered). In Fig. 390 the current in one phase has its maximum value, while in the other phases it is equal to $\frac{1}{2}.\hat{i}$. The conductors between A and B neutralise each other. The field strength between these points is therefore constant and corresponds to the ampere-wires between B and D. The number of these ampere-wires is

$$q'.\hat{i} + 2.\frac{q'}{2}.\frac{\hat{i}}{2} = 1{\cdot}5\,q'.\hat{i}.$$

If we make the same assumption as before as to the large number of slots per coil-side, the flux density will decrease uniformly from B to a point between B and D, where the north pole ends and the south pole begins. If the excitation is plotted on a base representing the pole-pitch, the shaded figure in Fig. 390 is obtained.

The same is repeated for the other extreme case, when the current in one phase is passing through zero (Fig. 391). The wires between A and D may now be looked upon as belonging to one large coil-side, in the middle of which the field strength is zero. From this point the field increases uniformly on either side, until point B is reached, at which the excitation will be $2.\frac{q'}{2}.\hat{i}.\sin 60° = 0{\cdot}866\,.q'.\hat{i}$. From B to A the field still increases uniformly, but at a smaller rate. The excitation at A is due to all the ampere-wires between A and D, viz. $2q'.\hat{i}.\sin 60° = 1{\cdot}73.q'.\hat{i}$.

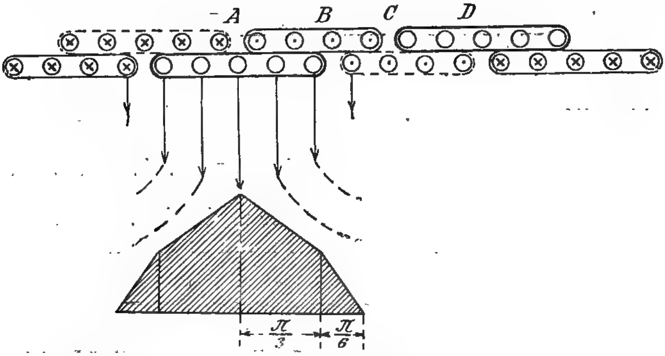

Fig. 391

On comparing Figs. 390 and 391 with Figs. 388 and 389, we find that the shaded curves have exactly the same shape, but that they have been exchanged, the flat-topped curve being now obtained for the conditions of phase which previously gave the pointed curve. The ordinates of the present curves are only 0·866 of the previous ordinates, so that equation (168) becomes

$$X_{\text{mean}} = 0{\cdot}866.1{\cdot}16.q'.\hat{i} = 1{\cdot}005.q'.\hat{i} \quad \text{...........(169)}.$$

We shall now consider the excitation produced by the currents in a squirrel cage rotor. The upper curve in Fig. 392 represents the field which cuts the rotor bars. It varies from point to point around the rotor according to the sine law. The E.M.F. induced in the various rotor bars, and the rotor current i_2, will also be a sine function of the position on the periphery. This is indicated in the figure by the crosses and dots being correspondingly thickened. The magnetising effect of these rotor currents will be zero at B, and will increase on either side up to the points A and C, where it will be a maximum.

It hardly requires to be proved that the curve of the field produced by the rotor currents is also a sine curve, displaced 90° from the field cutting the rotor bars*. If the total number of conductors on the rotor be Z_2, the conductors per pole will be $\frac{Z_2}{2p}$, and the mean current taken around the rotor is $\frac{2}{\pi} . \hat{i}$. The excitation at C is therefore

$$X_{\max} = \frac{2}{\pi} . \hat{i} . \frac{Z_2}{2p}.$$

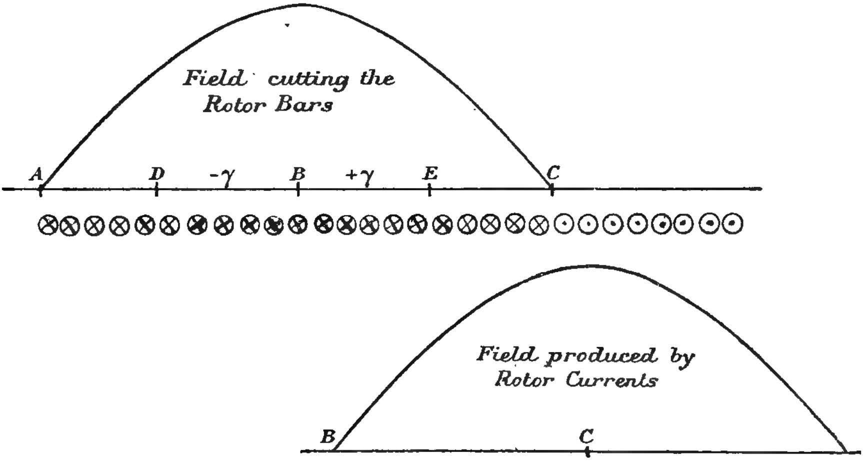
Field cutting the Rotor Bars
A
D
-γ
B
+γ
E
C
Field produced by Rotor Currents
B
C

Fig. 392

Since we have just seen that the field has a sine curve distribution $X_{\text{mean}} = \frac{2}{\pi} . X_{\max}$. In order to compare the squirrel cage with the other type of winding, we may imagine it to be divided into three parts or phases, and put

$$Z_2 = 3 . 2p . q'.$$

We have then

$$X_{\text{mean}} = \frac{2}{\pi} . \frac{2}{\pi} . \hat{i} . \frac{3 . 2p . q'}{2p} = \frac{12}{\pi^2} . q' . \hat{i} = 1{\cdot}22q' . \hat{i} \quad \ldots\ldots(170).$$

---

* The average current in the band of conductors DE is

$$\frac{1}{2\gamma} \int_{-\gamma}^{+\gamma} \hat{i} . \cos \alpha . d\alpha = \hat{i} . \frac{\sin \gamma}{\gamma}.$$

and the number of conductors in this band is $\frac{Z_2}{2p} . \frac{2\gamma}{\pi}$. The ampere-wires producing th field at D and E are therefore $\frac{Z_2}{p . \pi} \hat{i} . \sin \gamma$. Hence the field is proportional to $\sin \gamma$ where γ is measured from B.

In general, whatever the type of winding, we may write

$$X_{\text{mean}} = c \, . \, q' \, . \, \hat{i} \quad \dots\dots\dots\dots\dots\dots\dots\dots (171),$$

in which the coefficient c has the following values:

$c = 1\cdot005$ for a coil-width of $\frac{2\pi}{3}$,
$1\cdot16$ „ „ „ $\frac{\pi}{3}$,
$1\cdot22$ „ squirrel cage rotors.

If the area of the pole-face be A_g, the flux per pole can be found from the formula

$$\Phi = B_{\text{mean}} \, . \, A_g = \frac{0\cdot4\pi \, . \, X_{\text{mean}} \, . \, A_g}{l_g} = \frac{0\cdot4\pi \, . \, c \, . \, q' \, . \, \hat{i} \, . \, A_g}{l_g} \quad \dots (172).$$

We have here neglected the reluctance of the iron and considered merely that of the double air-gap of length l_g.

119. The Effect of the Iron in the Magnetic Path.

We shall now make use of the formulae which we have just established, to calculate the magnetising current of an actual 150 H.P. three-phase motor made by the Oerlikon Company of Zürich. The following particulars of this motor, together with Fig. 393, are taken from Arnold's *Konstruktionstafeln.*

Terminal pressure ... ... $V = 3300$ volts.
Primary frequency ... ... $f_1 = 50$.
Number of pairs of poles ... ... $p = 6$.
Axial length ... ... $L = 32\cdot5$ cm.
Diameter of rotor ... ... $D = 90$ cm.
The double air-gap ... ... $l_g = 0\cdot15$ cm.
Total number of stator conductors ... $Z_1 = 2016$.
Combined length of a stator and rotor tooth $= 6$ cm.
Ratio of tooth pitch to tooth width ... $= 2\cdot5$.

From these data we get the following:

Pressure per phase.................. $V_1 = \frac{3300}{\sqrt{3}} = 1910$ volts.

Conductors per phase .... $z_1' = \frac{Z_1}{3} = 672$.

Conductors per pole per phase... $q' = \frac{Z_1}{3 \, . \, 2p} = 56$.

Polar area .... $A_g = \frac{\pi D \, . \, L}{2p} = 763$.

Let Φ_0 be the total flux per pole produced by the stator at no-load. Since the pressure per phase V_1 is almost exactly equal and opposite to the induced electromotive force E_1, we can apply formula (145) found on p. 297 for coil-windings, viz.

$$V_1 = E_1 = 2{\cdot}12\Phi_0 . f_1 . z_1' . 10^{-8}.$$

Hence $$\Phi_0 = \frac{V_1 . 10^8}{2{\cdot}12 . f_1 . z_1'} = \frac{1910 . 10^8}{2{\cdot}12 . 50 . 672} = 2{\cdot}69 . 10^6.$$

If we assume that 2 per cent. of this flux is lost in leakage and that 98 per cent. of it passes through the rotor, the flux Φ in the latter will be

$$\Phi = 0{\cdot}98\Phi_0 = 2{\cdot}64 . 10^6.$$

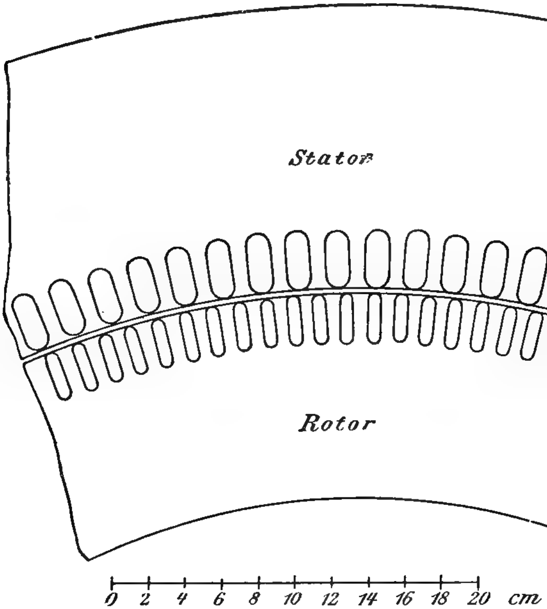

Fig. 393

From equation (172) on page 382, we have

$$\Phi = \frac{0{\cdot}4\pi . c . q' . \hat{i}}{l_g} . A_g,$$

in which $c = 1{\cdot}16$ for coil-windings. Introducing the effective value $I_0 = \frac{\hat{i}}{\sqrt{2}}$, we get

$$I_0 = \frac{\Phi . l_g}{\sqrt{2} . 0{\cdot}4\pi . c . q' . A_g} = 4{\cdot}5 \text{ amperes.}$$

This would be the magnetising current if the iron had no reluctance. The actual current taken by this motor on open circuit was 6 amperes, which is 33 per cent. more than the above value.

The main part of the reluctance of the iron lies in the teeth, on account of the flux density in them being very high, and the idea will suggest itself at once of adding something to the length of the air-gap to allow for this. Since the ratio of the tooth pitch to the thickness of the tooth is 2·5, and about 15 per cent. of the cross-section of the iron is lost in insulation, the flux density in the teeth will be 2·5/0·85 = 2·95 times that in the air-gap. The length of air-gap equivalent to the teeth will be found by multiplying the length of the teeth by 2·95 and dividing by the permeability μ. The mean flux density in the air-gap is

$$B_{\text{mean}} = \frac{\Phi}{A_g} = \frac{2{\cdot}64 \,.\, 10^6}{763} = 3450.$$

The mean flux density in the teeth is 2·95 times as high, and is therefore 3450 . 2·95 = 10,200. The magnetisation curve for the iron is given in Fig. 394, and from this we find that a value of B equal to 10,200 requires 2·6 ampere-turns per cm, i.e.

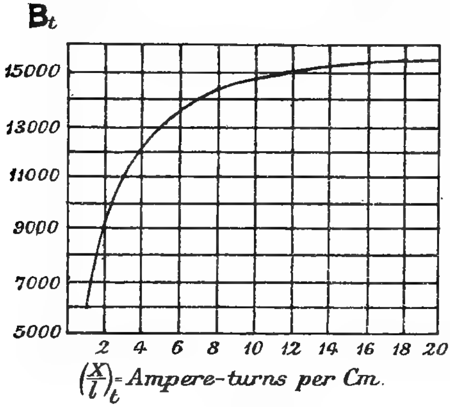

Fig. 394

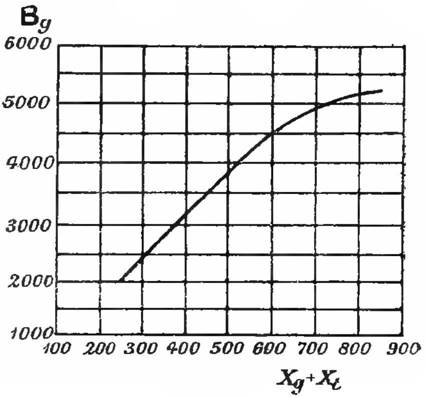

Fig. 395

$$\frac{X}{l} = 2{\cdot}6, \quad \text{and } H = 0{\cdot}4\pi \,.\, \frac{X}{l} = 3{\cdot}25.$$

Hence
$$\mu = \frac{B}{H} = \frac{10{,}200}{3{\cdot}25} = 3140.$$

The combined length of a stator and rotor tooth is 6 cm, so that the path of a line of force has a length of 12 cm in the teeth. This means an equivalent addition to the air-gap of $\frac{12 \,.\, 2{\cdot}95}{3140} = 0{\cdot}011$ cm. This will increase the value of l_g from 0·15 to 0·16, and the value of I_0 will increase in the same proportion, viz. from 4·5 to 4·8 amperes. This is still very much smaller than the actual value, and proves that the above method of allowing for the reluctance of the iron is incorrect. The reason for this difference between the calculated and actual current is to be found, mainly, in the fact that the iron has a particularly high permeability at the mean value of the flux density, viz. 10,200 lines per cm. We have assumed that the permeability was equally good at every point, which is far from being the case. A more reasonable method would be to take the mean permeability at the various flux densities, instead of the permeability at the mean flux density. Even then the result is little better.

The only reliable method of accurately determining the magnetising current is to draw a curve giving the relation between the flux density at any point of the air-gap and the ampere-turns necessary to produce this flux density. Then, knowing the excitation acting at each point of the periphery we can find from this curve the corresponding flux density. To plot the curve we choose several values of B_g and calculate the ampere-turns necessary for the gap alone from the formula

$$X_g = 0{\cdot}8 . B_g . l_g,$$

or since in our case $l_g = 0{\cdot}15$ cm,

$$X_g = 0{\cdot}8 . B_g . 0{\cdot}15 = 0{\cdot}12 B_g.$$

We then calculate the corresponding flux density in the teeth by multiplying the density in the gap by the ratio of the cross-sections. In our case, then, we have

$$B_t = 2{\cdot}95 B_g.$$

The ampere-turns per cm of path for the flux density B_t in the teeth is found from the magnetisation curve for the stampings (Fig. 394). We have then

$$X_t = \left(\frac{X}{l}\right)_t . l_t.$$

For l_t we must put the total length of path in the teeth, that is, two stator teeth and two rotor teeth, so that

$$X_t = \left(\frac{X}{l}\right)_t . 2 . 6 = 12 \left(\frac{X}{l}\right)_t.$$

Adding X_g and X_t, we have the total excitation X necessary to produce the assumed flux density. This calculation has been carried out for values of B_g from 2000 to 5200, and the results are given in the following table:

B_g	$X_g = 0{\cdot}12 B_g$	$B_t = 2{\cdot}95 B_g$	$\left(\frac{X}{l}\right)_t$	$X_t = 12\left(\frac{X}{l}\right)_t$	$X_g + X_t$
2000	240	5,900	1·1	13	253
3000	360	8,850	1·9	23	383
4000	480	11,800	3·6	43	523
4500	540	13,300	5·4	65	605
5000	600	14,800	10·7	128	728
5200	624	15,300	18·5	222	850

In Fig. 395 we have plotted the values of $X_g + X_t$ as abscissae, and the corresponding values of B_g as ordinates.

We must now assume a value of the no-load current and find the flux produced by it. If the result comes out far from the actual flux, we shall have to repeat the calculation with another value of the current. Seeing that our approximate calculation gave 4·8 amperes, we shall estimate the current at 5·5 amperes, so that

$$I_0 = 5{\cdot}5 \text{ amperes},$$

and

$$\hat{i}_0 = \sqrt{2}.I_0 = 7{\cdot}78 \text{ amperes}.$$

Half the pole-pitch is taken as a base line, and the excitation is plotted at every point, giving the curve X in Fig. 396, similar to the curve obtained in Fig. 388 for the same conditions. As before, it can be shown that the maximum excitation is

$$2q'\hat{i}_0 = 2.56.7{\cdot}78 = 870.$$

For an abscissa of $\pi/6$ the excitation has half this value. The scale of ampere-turns is plotted on the left-hand vertical axis. Points have been marked on this axis corresponding to the values of $X_g + X_t$ in the above table, viz.

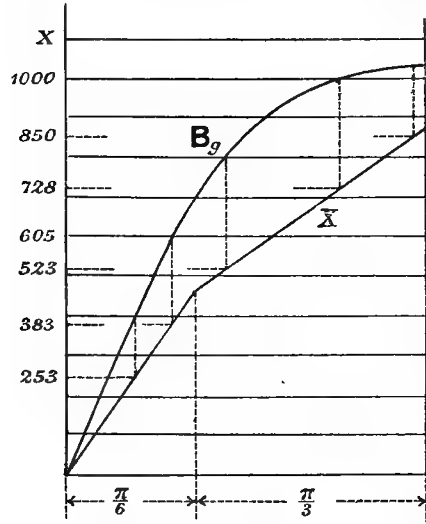

Fig. 396

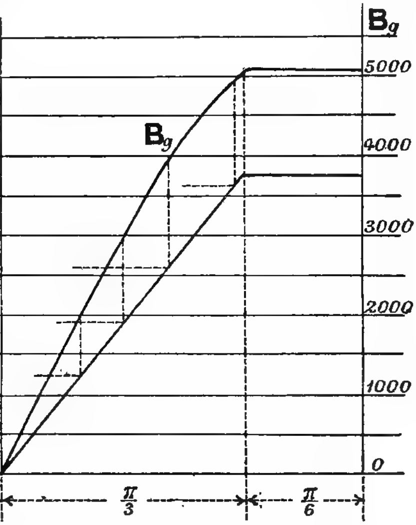

Fig. 397

$$X = X_g + X_t = 253, 383, 523, 605, 728, 850,$$

and horizontal lines have been drawn through these points to meet the curve X. Ordinates have been drawn through the points of intersection, and made equal to the corresponding values of B_g in the table, viz.

$$B_g = 2000, 3000, 4000, 4500, 5000, 6200.$$

The scale for B_g has been arbitrarily chosen and is plotted on the right of the figure. The curve B_g is drawn through the points found in this way, and represents the field distribution for a current of 5·5 amperes. The average flux density is determined by measuring the area enclosed between this curve and the base, and dividing it by the base.

The result obtained with the aid of a planimeter was

$$B_{g\text{mean}} = 3635.$$

The same construction has been carried out in Fig. 397 for the other extreme condition, corresponding to Fig. 389. As found in connection with the latter figure, the maximum excitation is here

$$1{\cdot}73 q' \hat{i}_0 = 1{\cdot}73 . 56 . 7{\cdot}78 = 750.$$

The result obtained from Fig. 397 was

$$B_{g\,\text{mean}} = 3583.$$

The mean of these two values of the flux density in the gap is

$$B_{g\,\text{mean}} = \frac{3635 + 3583}{2} = 3609.$$

The total flux entering the rotor per pole is therefore

$$\Phi = B_{g\,\text{mean}} . A_g = 3609 . 763 = 2{\cdot}75 . 10^6.$$

This value is so near the required flux of $2{\cdot}64 . 10^6$ lines that we may safely assume proportionality between current and flux. The magnetising current required for a flux of 2·64 million lines will therefore be

$$I_0 = \frac{5{\cdot}5 . 2{\cdot}64 . 10^6}{2{\cdot}75 . 10^6} = 5{\cdot}3 \text{ amperes.}$$

Now, we have neglected throughout the reluctance of the stator and rotor cores. In addition to this, we have taken the flux density in the air-gap as uniform, whereas the slots will cause it to vary, thus increasing the maximum density and the necessary excitation. Finally, we must remember that the observed no-load current of 6 amperes contains an energy component to cover the friction and iron losses. We cannot be surprised, therefore, that the calculated current of 5·3 amperes is about 10 per cent. smaller than the measured current of 6 amperes. Our last result is certainly more reliable than that obtained by taking the permeability for an average value of the flux density.

120. The Torque of an Induction Motor.

To calculate the torque we shall assume that the rotating field is exactly sinusoidal, with the same flux per pole as the actual field. We shall confine our attention in the first place to a coil-side of the rotor-winding, with a breadth 2γ and with its centre displaced by an amount α_1 from the point where the field is zero, both γ and α being expressed in terms of the pole-pitch (Fig. 398). The field being sinusoidal the E.M.F. induced in such a coil-side, and therefore the current in it, is proportional to the sine of the angle $\alpha_1\pi$. The instantaneous value of the rotor current is therefore

$$i_2 = \hat{i}_2 . \sin \alpha_1\pi.$$

This current is the same for every wire in the coil-side, but the field in which each wire is situated is different. To find the value of the torque at this moment, we must determine the average strength of the field over the arc 2γ. The flux density at any point is equal to $\mathbf{B} . \sin \alpha\pi$, where $\alpha\pi$ is the angle between this point and the point of zero field strength. The average over the arc 2γ is found as follows:

$$B = \frac{1}{2\gamma}\int_{(\alpha_1-\gamma)}^{(\alpha_1+\gamma)} \mathbf{B} . \sin \alpha\pi . d\alpha = \mathbf{B} . \frac{\sin \gamma\pi}{\gamma\pi} . \sin \alpha_1\pi.$$

To calculate the torque we shall require the product of the mean field strength and the momentary value of the current. We have

$$i_2 \,.\, B = \mathbf{B} \,.\, \hat{i}_2 \,.\, \frac{\sin \gamma\pi}{\gamma\pi} \,.\, \sin^2 \alpha_1\pi .$$

In this equation i_2 is the momentary value of the current, and B the mean value of the field strength over the coil-side at that moment.

Now, the coil-side moves steadily through the field, i.e. the angle $\alpha_1\pi$ varies from 0° to 360°, and, since the value of the product $i_2 . B$ is proportional to the square of the angle $\alpha_1\pi$, its mean value can be found in exactly the same way as the mean power was determined in Section 73. Analogous to equation (115) on page 237, we have

$$(B \,.\, i_2)_{\text{mean}} = \frac{\mathbf{B} \,.\, \hat{i}_2}{2} \,.\, \frac{\sin \gamma\pi}{\gamma\pi} .$$

The force on the coil-side in dynes is found by multiplying together the strength of the field, the absolute current in the wire, and the total length of effective wire in the coil-side. Since, however, the same mean force is given by every coil-side during a period, the total tangential force on the rotor is equal to f, where

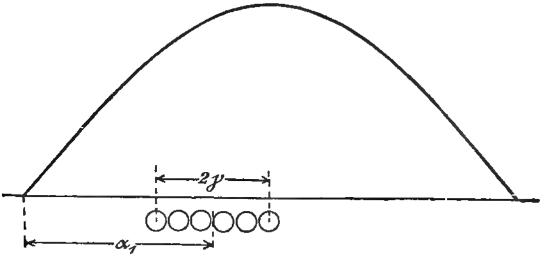

Fig. 398

$$f = \frac{\mathbf{B} \,.\, \hat{i}_2}{2 \,.\, 10} \,.\, \frac{\sin \gamma\pi}{\gamma\pi} \,.\, Z_2 \,.\, L \text{ dynes} \quad \ldots\ldots\ldots\ldots\ldots\ldots(173).$$

This force acts at an arm equal to the rotor radius r. Since the cylindrical surface of the rotor is equal to the product of the polar surface A_g and the number of poles $2p$, we have

$$2\pi r \,.\, L = 2p \,.\, A_g ,$$

or

$$r = \frac{p \,.\, A_g}{\pi \,.\, L} .$$

To convert the torque or turning moment from dyne-cm into kg-metres, we must divide by 981,000 . 100. Thus we get

$$M_t = \frac{f \,.\, r}{9{\cdot}81 \,.\, 10^7} = \frac{2p \,.\, \mathbf{B} \,.\, A_g \,.\, Z_2 \,.\, \hat{i}_2}{\pi \,.\, 40 \,.\, 9{\cdot}81 \,.\, 10^7} \,.\, \frac{\sin \gamma\pi}{\gamma\pi} \quad \ldots\ldots\ldots\ldots(174).$$

Since

$$\frac{2}{\pi} . \mathbf{B} = B_{\text{mean}},$$

$$\frac{2}{\pi} . \mathbf{B} . A_g = \Phi.$$

Substituting, further, the effective current I_2 for $\hat{i}_2/\sqrt{2}$, we get

$$M_t = 3{\cdot}6 . p . \Phi . Z_2 . I_2 . 10^{-10} . \frac{\sin \gamma\pi}{\gamma\pi} \text{..............(175).}$$

The breadth γ becomes 0 for a squirrel cage, is equal to $\frac{1}{3}$ for a coil-winding, and $\frac{2}{3}$ for a winding like that on the left of Fig. 366.

The ratio $\frac{\sin \gamma\pi}{\gamma\pi}$ entered into the coefficient k in the equations (143) to (146) for the electromotive force. We come, therefore, to the conclusion that, with various rotor-windings, the torque is proportional to the coefficient k for the E.M.F. induced in the rotor.

The same result together with the formula for the torque, can be obtained directly by imagining that the rotor is fixed, and that the whole power supplied to it is dissipated in heating the rotor or a resistance connected in series with it. Let Φ be the actual field cutting the rotor bars, as the resultant of the stator and rotor currents. This field cuts the stationary winding with the primary frequency $f_1 = p . n_1$, where n_1 is the revolutions made by the field per second. The current in the rotor will be in phase with the E.M.F. induced by this field. If this E.M.F. in each phase be E_2, and the total number of rotor conductors Z_2, then the total power supplied to the rotor will be

$$P_2 = 3E_2 . I_2 = 3 . k . \Phi . f_1 . z_2' . 10^{-8} . I_2 = k . \Phi . f_1 . Z_2 . 10^{-8} . I_2 \text{...(176).}$$

The power is supplied to the rotor by a field which rotates with an angular velocity

$$\omega = 2\pi . n_1 = 2\pi . \frac{f_1}{p} \text{*}.$$

Now the power in met.-kg per second is equal to the product of the torque and the angular velocity. Remembering that equation (176) gives the value of P_2 in watts, and that a met.-kg per second is equal to 9·81 watts, we have

$$P_2 = M_t . \omega . 9{\cdot}81.$$

Substituting for P_2 and ω, we get

$$M_t = \frac{k . p . \Phi . Z_2 . I_2 . 10^{-8}}{2\pi . 9{\cdot}81} \text{....................(177).}$$

We have assumed here that the rotor is fixed, but this is quite immaterial so long as the values of Φ and I_2 remain the same. If we substitute for k the values found for it on page 297, we get the same values for the torque as those found above. This is therefore a check on the above result.

It is interesting to compare the torque of a three-phase induction motor with that of a D.C. motor. For the latter we found, in equation (98) on page 200, that

$$M_t = \frac{p . \Phi . Z . I_a . 10^{-8}}{a . 2\pi . 9{\cdot}81} = 3{\cdot}25 . p . \Phi . Z . \frac{I_a}{2a} . 10^{-10}.$$

---

* In this case we must use the mechanical angular velocity, and not the phase or vectorial angular velocity $2\pi f_1 = 2\pi . p . n_1$.

$\frac{I_a}{2a}$ is the current in each armature conductor. From equation (175) for a squirrel cage rotor we get a coefficient of 3·6 instead of 3·25 because, the coil-side having no width, $\frac{\sin \gamma\pi}{\gamma\pi} = 1$, but the equation is otherwise the same. Hence, other things being equal, the squirrel cage three-phase motor gives a 10 per cent. greater torque than the direct current motor.

121. Calculation of Slip.

We saw in Section 116 that the completely unloaded rotor runs synchronously with the field, i.e. at the speed

$$n_1 = \frac{f_1}{p}.$$

We saw, moreover, that, as the motor is loaded, the speed decreases to a value n such that the rate at which the lines of force are cut is sufficient to induce the necessary E.M.F. E_2 in the rotor. In this way a current is produced in the rotor, and this current produces the necessary torque, in accordance with the equations established in the previous section.

The slip, or the frequency f, with which the field cuts the rotor-winding, corresponds to the difference between the speed of the rotating field and that of the rotor, and

$$f = (n_1 - n)\,p.$$

The small E.M.F. which is thus induced in the rotor winding is given by the equation

$$E_2 = k.\Phi f z_2'.10^{-8}.$$

By Φ we understand the field which actually cuts the rotor conductors, so that there is no difference of phase between E_2 and the rotor current I_2. If the resistance of each phase of the rotor-winding be R_2, we have

$$E_2 = I_2.R_2.$$

The power wasted in heating the three-phase windings of the rotor is therefore

$$3.I_2^2.R_2 = 3.E_2.I_2 = k.\Phi f Z_2.10^{-8}.I_2 \;\ldots\ldots\ldots\ldots(178).$$

Z_2 has been substituted for $3z_2'$. By dividing this equation by equation (176) we get

$$\frac{3I_2^2.R_2}{P_2} = \frac{f}{f_1} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(179).$$

Hence, the slip, expressed as a percentage of the synchronous speed, is equal to the percentage of the power supplied to the rotor which is lost in heating the rotor-winding. This explains why motors are always made with a small slip. It is evident that the slip can be made to have almost any value by suitably altering the rotor resistance.

It is interesting to compare the behaviour of a three-phase motor with that of a direct-current shunt motor. For the latter we have the equation

$$V.I_a = E.I_a + I_a^2.R_a.$$

In this equation $V.I_a$ is the power supplied to the armature, while $E.I_a$ is the power transformed into mechanical output. These two powers are in the ratio of V to E, that is, the ratio of the speed n_0 at perfect no-load to the speed n when loaded. Hence

$$\frac{E \,.\, I_a}{V \,.\, I_a} = \frac{n}{n_0}.$$

and

$$\frac{n_0 - n}{n_0} = \frac{V \,.\, I_a - E \,.\, I_a}{V \,.\, I_a} = \frac{I_a^2 \,.\, R_a}{V \,.\, I_a}.$$

The percentage drop of speed from no-load to full load is therefore equal to the percentage loss of power in the armature winding.

Let us now consider the case in which the slip is 100 per cent. This occurs at the moment of starting, before the motor begins to rotate. The rotating field cuts the rotor-winding at the high primary frequency f_1 and induces a very large E.M.F. in it. The current in both rotor and stator might then be so great as to damage the motor and cause a complete breakdown. It is for this reason that squirrel cage rotors are generally confined to small motors. They can be used, however, for large motors, if they are brought up to the normal speed before switching the stator on the supply mains, or if they are started up together with the generator. Large squirrel cage motors are largely used in the United States; they are generally started at a reduced terminal pressure, a special transformer being supplied for the purpose.

The majority of motors above about 5 H.P., and many smaller ones, have rotor-windings into which resistances are introduced at starting. These windings can be connected either in star or mesh. Although a mesh-connected rotor-winding is closed on itself, it is electrically open until the three corners of the mesh are connected together, either through the external starting resistance or by short-circuiting the slip-rings. The starting resistances are most conveniently arranged in star connection.

Liquid resistances are often used for starting induction motors. The resistance is gradually reduced by dipping the electrodes further into the liquid. The starting resistance does not merely reduce the current taken by the motor, but materially increases the starting torque (compare Section 126).

It is evident from equation (179) that the slip is increased and the speed consequently decreased by putting resistance in the rotor circuit. This is analogous to putting resistance in the armature circuit of a shunt motor, and leads to the same large waste of power. The speed varies, moreover, with every change of load.

It has been tried to vary the speed by altering the number of poles of the stator, but the necessary connections are very complicated. If two motors are coupled, as is generally the case in traction, they can be made to run at half speed by supplying the primary of one from the secondary of the other. This so-called cascade arrangement has been adopted on an Italian railway. It cannot be denied, however, that the constant speed of the induction motor and the lack of an efficient speed regulation make it very unsuitable for a great many applications.

The consideration of a motor with its rotor fixed and considerable resistance in series with the rotor-winding is of special interest. The whole power supplied to the motor is dissipated as heat in the rotor and starting resistance, and the motor is nothing more than a transformer. If the current is the same, the total power supplied to the rotor is the same as when the motor was running. The heat produced in the starting resistance is equal to the mechanical output of the motor when working normally at the same current. The study of the induction motor is much simplified by looking upon it as a transformer, to which the laws and equations which we have established for transformers can be applied. The fact that the field is here a constant rotating one, and not a simple alternating one, introduces no material difficulty.

122. Calculation of the Electromotive Force.

In the foregoing sections we have assumed that the rotating field had a sine-wave distribution. The formula for the electromotive force, on this assumption, is

$$E = k.\Phi.f.z'.10^{-8}.$$

From p. 297 we find that

$k = 2{\cdot}12$ for ordinary three-phase coil windings,
$1{\cdot}84$ for a winding with a coil-side equal to $\frac{2}{3}$ of the pole-pitch,
and $2{\cdot}22$ for a squirrel-cage, since the coil-side has then no width.

For Φ we have taken the mean of the two symmetrical fluxes coinciding with maximum and with zero current in one of the phases (see Section 118).

As a matter of fact, the flux has not a sine-wave distribution, nor is it symmetrical, except at the special moments considered in Section 118. The flux-density does not increase uniformly, as it would do with a perfectly distributed winding. The coil-side is often distributed between three, or even two, slots, so that the curve of flux distribution is a stepped diagram.

We will calculate the E.M.F. induced by this flux distribution, but will confine ourselves to the case of a three-phase motor.

Fig. 399 represents the upper half of a 2-pole stator, with the unusually large number of 9 slots per pole per phase. At the given moment the upper nine slots have their maximum current and the others half this value, and the flux linking the former phase has its maximum value. We shall investigate the E.M.F. induced in this phase winding, which is distinguished in the figure by heavier dots. If the currents are sine functions of the time, the flux linking the coils must also be a sine function of the time, no matter how it may be produced by the joint action of the three phases.

We then have

$$E = 4{\cdot}44\Phi.f.T.10^{-8},$$

where Φ is the maximum value of the flux linking the T turns. In our case, however, the flux linking the various coils will vary from slot to slot, and we must find the maximum linkages for each slot and add together the results obtained. Let there be g slots per coil-side, and let us assume in the first place that there is only one wire per slot. We shall neglect the effect of the varying permeability of the iron depending on the flux-density, and also the effect of the different lengths of path in the stator and rotor cores. The flux per tooth will therefore be proportional to the ampere-turns acting on the flux in that tooth, and may, so far as we are concerned with merely relative values, be put equal to the ampere-turns.

If the maximum current be taken as 1 ampere, the 9 slots on either side will be carrying 0·5 ampere. Then from the figure we have

(*a*)	Tooth	Effective ampere-turns or flux in tooth	Turns linked	Number of linkages
	1	1	1	1
	2	3	3	3^2
	3	5	5	5^2
	.	.	.	.
	.	.	.	.
	x	g	g	g^2

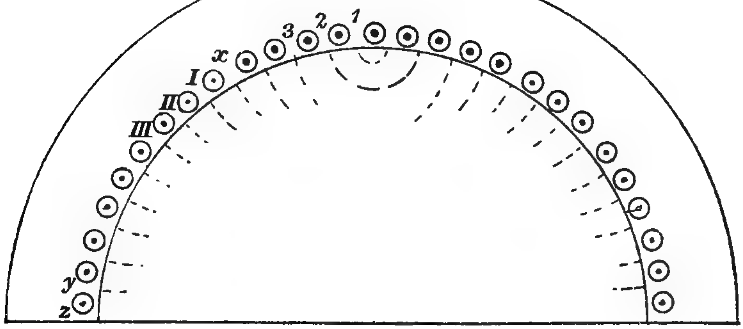

Fig. 399

The other slots carry only half the current, but the flux links all the g slots of our upper phase, so that we have

(*b*)	Tooth	Effective ampere-turns or flux in tooth	Turns linked	Number of linkages
	I	$g + \frac{1}{2}.2$	g	$g\,(g + 1)$
	II	$g + \frac{1}{2}.4$	g	$g\,(g + 2)$
	III	$g + \frac{1}{2}.6$	g	$g\,(g + 3)$
	.	. .	.	.
	.	. .	.	.
	y	$2g - \frac{1}{2}.2$	g	$g\,(2g - 1)$

The number of turns in this series is $(g - 1)$.

(*c*) Finally, we have the tooth z. It has a flux due to $2g$ ampere-turns, and this flux links all the g turns of our phase, hence the linkages are $2g^2$.

The sum of the series (*a*) can be shown to be $\frac{g^3}{6} + \frac{g^2}{2} + \frac{g}{3}$, and the sum of the series (*b*) $\frac{3}{2}g^3 - \frac{3}{2}g^2$. Since these linkages (*a*) and (*b*) occur also in the lower half, these values must be doubled, whereas (*c*) occurs only once.

Hence the total linkages of the phase winding at the moment of maximum current are equal to

$$\frac{2g}{3}(5g^2+1).$$

To turn from relative to actual values of the linkages we know that the flux in the tooth z is not $2g$, as we have assumed, but $\mathbf{B}$ multiplied by the cross-section of the air-gap for one tooth, viz. $\frac{A_g}{3g}$, where A_g is the cross-section per pole. Moreover, there is not one wire in each slot, but $\frac{q'}{g}$, where q' is the number of wires in a coil-side. Remembering that there are p pairs of poles connected in series, we get

$$\Sigma\Phi T = \frac{2g}{3}(5g^2+1)\frac{A_g\,.\,\mathbf{B}}{3g}\,.\,\frac{1}{2g}\,.\,\frac{q'}{g}\,.\,p$$
$$= c_1\,.\,p\,.\,A_g\,.\,q'\,.\,\mathbf{B},$$

where c_1 is put for $\frac{5g^2+1}{9g^2}$.

Similar calculations with an even number of slots per coil-side lead to the same result.

The following table gives the values of c_1 calculated for various values of g:

g	c_1
1	0·667
2	0·583
3	0·568
4	0·563
∞	0·555

For the E.M.F. we have

$$E = 4{\cdot}44 c_1\,.\,p\,.\,A_g\,.\,q'\,.\,\mathbf{B}\,.\,f\,.\,10^{-8}.$$

For very distributed windings ($g = \infty$) this formula agrees almost exactly with equation (145). Hence, as g is never less than 2, we may always safely assume that the field is distributed according to the sine law. It must be remembered that the above figures have been obtained by ignoring the varying permeability of the iron of the stator and rotor teeth.

(For further calculations on this subject, see Görges, *E.T.Z.*, 1907, p. 1.)

CHAPTER XVIII

123. Rotor current, torque, and output in relation to the slip, neglecting leakage.—124. The circle-diagram, neglecting primary losses.—125. Output, torque and slip from the circle-diagram.—126. Normal load, starting torque and maximum torque.—127. The circle-diagram, corrected for primary copper loss.—128. The corrected values of output, rotor current and slip.—129. The most convenient form of circle-diagram.—130. Practical example.—131. The leakage factor.

123. Rotor Current, Torque, and Output in relation to the Slip, neglecting Leakage.

We shall assume, for the sake of simplicity, that there is no magnetic leakage, so that no lines of force pass across the slots in stator or rotor. There is then only one flux which crosses the air-gap and links both stator and rotor conductors. This flux is produced by the combined action of the stator and rotor currents. We shall assume that this flux Φ has a positive maximum at A and a negative maximum at B, so that A is the mid-point of a north pole and B the mid-point of a south pole. The lines of force pass vertically downwards, and the vector OX may represent the ampere-turns or resultant excitation producing this flux.

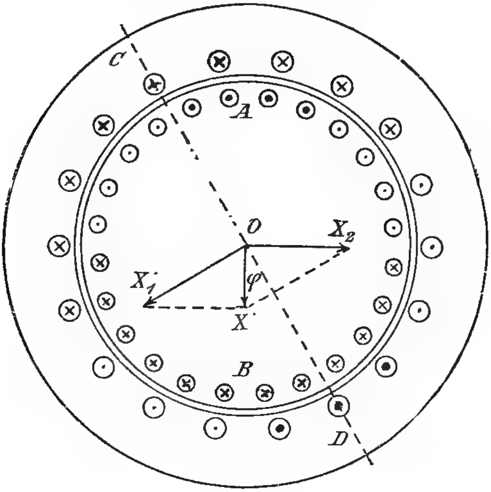

Fig. 400

If the field be rotating in the clockwise direction, it will induce a current in the rotor with the maximum values at A and B.

By means of the right-hand rule, it is seen that the current at A is towards the observer, while at B it is away from him. Each coil-side of the three-phase rotor-winding is distributed between 4 slots. The points and crosses in the slots at A and B are drawn heavy to indicate the maximum current. In the other two phases the current has half this value.

The vector OX_2 of the excitation due to the rotor currents alone must be drawn horizontally from left to right, since this is the direction of the flux which the rotor currents alone would produce.

In order to give the resultant excitation OX, it is necessary for the stator excitation to be equal to OX_1 both in magnitude and phase. The stator currents have their maximum values at C and D, with crosses at C and dots at D. The stator has 3 slots per pole per phase. The current will have its maximum value in the phase CD, and half that value in other phases.

From Fig. 400 we see

(1) that the primary and secondary currents are in approximate opposition, as in the ordinary transformer;

(2) that the current induced in the rotor is strongest at the point where the field cutting the rotor is strongest; and

(3) that the rotor-current vector is at right angles to the rotor-field vector.

When, now, we remember that the rotating field cuts the stator as well as the rotor, we see that the E.M.F. induced in the stator-winding will be a maximum at points A and B, whereas the maximum stator current is at C and D.

We see then

(4) that the stator current lags behind the stator pressure by the angle ϕ in the figure.

This comparatively large angle of lag is due to the fact that the induction motor has to produce its own field, and that, since the magnetic path necessarily includes the air-gap, there must be a large wattless or magnetising current. Although the air-gap is sometimes reduced to a quarter of a millimetre or less, it is only in large motors that $\cos \phi$ can be got up to 0·9, and only in extremely large ones that this value of the power-factor can be exceeded. The wattless current, as we have already seen, causes a needless loss in the wire in generator, mains, and motor, besides having a bad effect on the pressure regulation of the former. For this reason, induction motors must be constructed relatively larger than direct-current and synchronous motors. Finally, the large magnetising current is especially unfavourable when the motor is running light.

We shall now calculate the most important values for an ideal motor free from magnetic leakage. If the ohmic pressure drop in the stator is small, the induced E.M.F. in the stator will be almost equal to the constant pressure V_1 applied to each phase. It also depends on the magnetic flux Φ, which, in the absence of leakage, links both stator and rotor. The induced E.M.F. in the stator is given by the equation

$$E_1 = k . \Phi f_1 z_1' . 10^{-8}.$$

If the ohmic drop in the stator-winding be neglected the value of E_1 is equal to that of V_1, and is therefore constant, and, as is evident from the above equation, the flux Φ must also be constant.

If this field, which we have seen to be common to both stator and rotor, cuts the rotor conductors with a frequency f corresponding to the slip, an E.M.F. will be induced in the rotor-winding, in accordance with the equation

$$E_2 = k.\Phi f z_2'.10^{-8}.$$

We shall assume, for the sake of simplicity, that the windings of stator and rotor are similar, so that both have the same value of k.

The rotor current I_2 will be equal to E_2/R_2, and is therefore proportional to the slip. We can express this by the equation

$$I_2 = c_1.f,$$

where c_1 is constant.

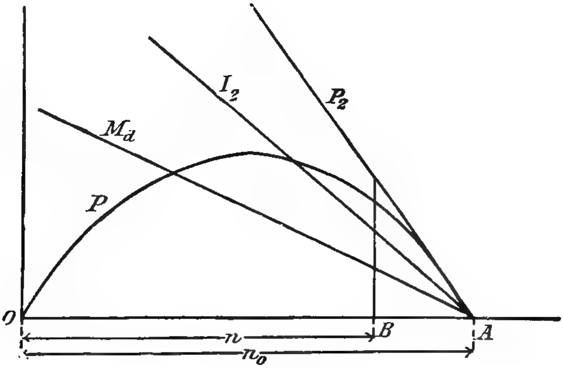

Fig. 401

The torque is proportional to the product of the rotor current I_2 and the flux Φ (equation 177). In our case the latter is constant, while the current is proportional to the slip. By introducing a constant c_2 we can write

$$M_t = c_2.f.$$

The power transmitted to the rotor is also porportional to the product $I_2.\Phi$ (equation 176). Since the flux is constant, this power is therefore proportional to the slip, and we can put

$$P_2 = c_3.f,$$

where c_3 is another constant.

The mechanical power P, including the power wasted in friction, can be found by subtracting the copper loss in the rotor from the total power transmitted to the rotor. Now, the copper loss is proportional to the square of the current, and therefore to the square of the slip. If c_4 be another constant and R_2 the resistance of each phase of the rotor, we have

$$P = P_2 - 3.I_2^2.R_2 = c_3.f - c_4.f^2.$$

The rotor current, torque, transmitted power and mechanical power are plotted in Fig. 401 on a base line representing speed. OA is equal to the speed n_0 at no-load and OB is equal to the speed at a certain load. The difference $OA - OB = AB$ is equal to the revolutions per second lost by the slip. The first three of the above quantities are proportional to the slip, and will therefore be represented by straight lines through A. The mechanical power, on the other hand, is given by the ordinates of the parabola in the figure. It must be zero, both at O where the rotor is at rest, and at A where the motor is running light. It will reach its maximum value for a speed n equal to $\frac{n_0}{2}$, i.e. for a slip of 50 per cent. The normal working position is much further to the right, where the slip is small and the efficiency consequently high.

An actual motor differs from this ideal motor in almost every respect, but especially with regard to the starting torque and overload capacity.

At starting, the speed is nil and the torque, according to Fig. 401, is very large. As a matter of fact, the effect of leakage in an actual motor is to make the starting torque extremely small. Resistance has to be inserted in the rotor circuit to enable the motor to start against the normal load.

This can be explained by the large leakage flux caused by the excessive currents in stator and rotor when the slip is 100 per cent. The flux which cuts the rotor conductors is thereby so reduced that there is very little torque in spite of the large rotor currents. Hence, the starting resistance not only reduces the otherwise excessive current taken by the motor, but also enables the motor to exert a large starting torque. This will be more clearly understood after studying the circle-diagram of Heyland, to which we now proceed.

124. The Circle-diagram, neglecting Primary Losses*.

We shall consider, in the first place, the general effect of the leakage, before passing on to the vector diagram and an accurate determination of the various quantities involved.

When there is no load on the motor, the total stator field is made up of two fields, one, the flux which crosses the air-gap and enters the rotor, the other, the primary leakage flux Φ_{l_1}. Both fluxes are produced by the no-load stator current, and are therefore in phase with each other, and inversely proportional to the magnetic reluctance of the air-gap path and the primary leakage path.

When the motor is loaded, the flux Φ_g, which crosses the air-gap, is produced by the combined action of the stator and rotor currents.

It was formerly the custom to consider this air-gap flux as linking the rotor-winding (Fig. 402) and inducing therein an E.M.F. sufficient to maintain the rotor currents against the resistance and self-induction of the rotor-winding. The ohmic pressure drop in the rotor would then be equal to the resultant of two electromotive forces, produced by the fluxes Φ_g and Φ_{l_2} respectively. This second flux is the rotor leakage flux, and it is not in phase with the air-gap flux. As a matter of fact, however, these two fluxes do not exist separately in the rotor, but combine to form the resultant rotor flux Φ. It is therefore preferable to consider that only a part Φ of the air-gap flux Φ_g passes round the rotor-windings, while a part Φ_{l_2} is forced along the leakage paths by the back ampere-turns of the rotor current (Fig. 403).

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* The historical development of the circle-diagram is very interesting. Heyland published the diagram in the *E.T.Z.* on the 11th Oct. 1894, and gave further developments on pages 649 and 823 for the year 1895. In the *E.T.Z.*, 1896, page 63, Behrend developed the diagram analytically, but made a small error in the determination of the rotor current. The convenient determination of the slip and losses was given by Heyland in the *E.T.Z.* for 1896, p. 138. (See also Heyland's "Eine Methode zur experimentellen Untersuchung an Induktionsmotoren," published in Voit's *Sammlung*, vol. II. 1900.) Emde corrected Behrend's error in a letter to the *E.T.Z.*, 1900, p. 781, which opened an interesting discussion. In the *Z. für E.*, Vienna, for 1899, Ossanna gave the diagram, corrected for stator loss. (See also an article by Ossanna in the *E.T.Z.*, 1900, p. 712, and also by Thomälen in the *E.T.Z.*, 1903, p. 972.) It is interesting to note, however, that Ossanna's circle was really included in Heyland's first publication.

In Fig. 404, let the vector LD represent the flux Φ which passes through the rotor-winding. The E.M.F. induced in the rotor-winding by this flux will lag 90° behind it, and will be in phase with the rotor current, since, in accordance with Fig. 403, this E.M.F. has to cover merely the ohmic pressure drop and not the self-induction. Hence $LB = I_2$ is perpendicular to LD.

The secondary leakage flux $\Phi_{l_2} = LE = DA$ is drawn in the opposite direction to the rotor current, because, from the point of view of Fig. 403, this flux is produced by a current which exactly balances the back magnetomotive force of the rotor current.

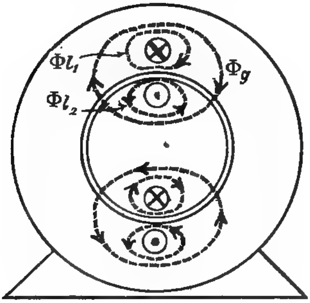

Fig. 402

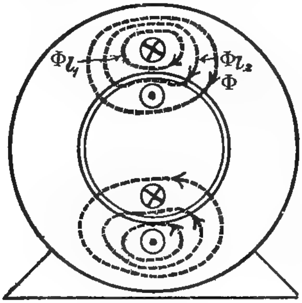

Fig. 403

The flux Φ_g in the air-gap is the sum of the rotor flux Φ and the secondary leakage flux Φ_{l_2}. We have, therefore, $\Phi_g = LA$.

The magnetomotive forces of the stator and rotor currents are nearly opposed to one another. Their resultant, however, must be such as to produce the flux Φ_g in the air-gap. If c be a constant and R the reluctance of the air-gap, and if, moreover, both stator and rotor have the same number of turns, the law of the magnetic circuit gives us the equation

$$\Phi_g = \frac{c \,.\, I_g}{R}.$$

In this equation I_g is the magnetising current which is necessary to maintain the flux Φ_g across the gap. It will be the resultant of the primary and secondary currents. If the scale to which the current is plotted be suitably chosen, the length LA can be made to represent the current I_g as well as the flux Φ_g. If LB represents I_2 to this same scale, the primary current I_1 must be represented by the line AB. For the sake of compactness, we have drawn the triangle of forces instead of the parallelogram.

The primary leakage flux Φ_{l_1}, which links the stator-winding without crossing the air-gap, is in phase with the primary current, and is represented by the line AO in continuation of the line BA. The total flux Φ_0 in the stator is represented by LO. **With a constant terminal pressure, and consequently, on our present assumptions, a constant induced E.M.F. in the stator, the primary flux $\Phi_0 = LO$ is constant at all loads.**

Let R_{l_1} and R_{l_2} be the reluctances of the primary and secondary leakage paths, and let

$$\tau_1 = \frac{R}{R_{l_1}}; \quad \tau_2 = \frac{R}{R_{l_2}}.$$

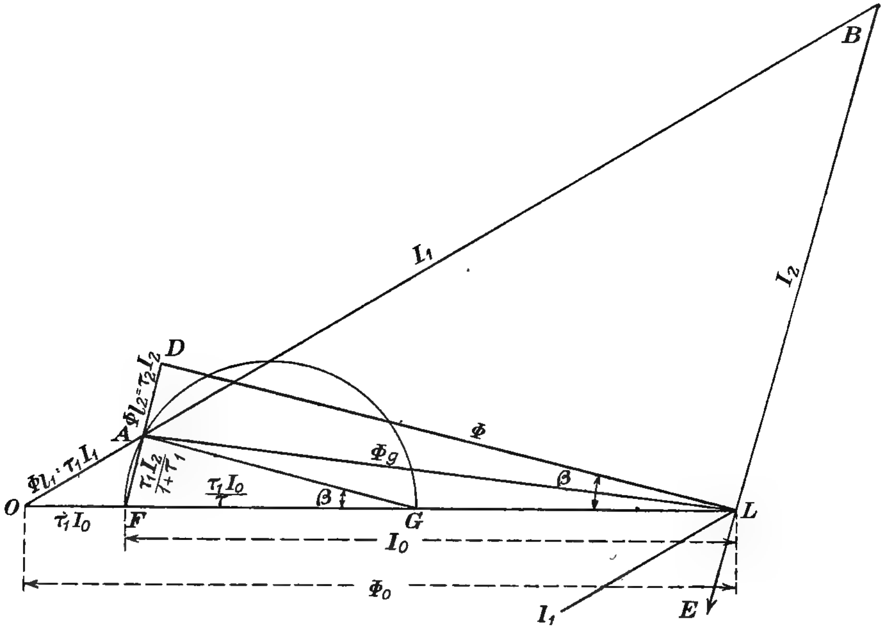

Fig. 404

Then, from the law of the magnetic circuit, we have

$$AO = \Phi_{l_1} = \frac{c \,.\, I_1}{R_{l_1}} = \frac{c}{R} \,.\, \tau_1 \,.\, I_1 = \tau_1 \,.\, I_1,$$

$$DA = \Phi_{l_2} = \frac{c \,.\, I_2}{R_{l_2}} = \frac{c}{R} \,.\, \tau_2 \,.\, I_2 = \tau_2 \,.\, I_2.$$

It is to be noted that we have so chosen the scale of current in Fig. 404, that the same length, viz. AL, represents both I_g and Φ_g, which, since $\Phi_g = \frac{c \,.\, I_g}{R}$, is equivalent to putting $\frac{c}{R} = 1$. Now, the point A divides the line OB in the ratio of τ_1 to 1, and as DAF is parallel to BL, it follows that

$$FA = \frac{\tau_1}{1 + \tau_1} \,.\, I_2.$$

It is also evident that the point F divides the primary flux LO in the ratio $\tau_1: 1$, that is, in the ratio of the reluctance of the air-gap to that of the primary leakage path. Since the flux will divide between two parallel paths in inverse proportion to their reluctances, OF must be the primary leakage flux at no-load and FL the corresponding flux in the air-gap. If I_0 be the no-load stator current, it follows from the law of the magnetic circuit that

$$FL = \frac{c \, . \, I_0}{R} = I_0,$$

whence
$$OF = \tau_1 . FL = \tau_1 . I_0.$$

If we draw AG parallel to DL, i.e. perpendicular to FA, we get

$$FG = FL \, . \frac{FA}{FA + AD} = I_0 \, . \frac{\dfrac{\tau_1}{1+\tau_1} \, . \, I_2}{\dfrac{\tau_1}{1+\tau_1} \, . \, I_2 + \tau_2 \, . \, I_2}.$$

By simplifying this and putting

$$\tau = \tau_1 + \tau_2 + \tau_1 \tau_2,$$

we get
$$FG = \frac{I_0 \cdot \tau_1}{\tau}.$$

Hence, both F and G are fixed points on the constant line LO, and the point A moves on a semicircle on GF as diameter. If, now, the whole figure be enlarged in the ratio $\tau_1: 1$, and allowance be made for the fact that the rotor conductors are not equal in number to those on the stator, but are only $\frac{Z_2}{Z_1}$ of the latter, we have the following values:

$$OF = I_0, \qquad FG = \frac{I_0}{\tau} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (180),$$

$$OA = I_1, \qquad FA = \frac{1}{1+\tau_1} \, . \, \frac{I_2 Z_2}{Z_1} \quad \ldots\ldots\ldots\ldots (181).$$

The end of the primary current vector moves on a semicircle on the diameter $\frac{I_0}{\tau}$ (Fig. 405).

It is evident that the conditions at various loads are largely dependent on the leakage coefficients τ_1 and τ_2 and the resulting coefficient τ. The values of τ_1 and τ_2 lie between 0·07 and 0·02. If τ_1 and τ_2 are assumed to be equal, the value of τ can be found from the table on page 406.

125. Output, Torque and Slip from the Circle-diagram.

From equation (176) on page 389 we know that the power supplied to the rotor is

$$P_2 = k . \Phi f_1 Z_2 . 10^{-8} . I_2.$$

Since the point F in Fig. 404 divides the line LO in the ratio $\tau_1: 1$, it follows that $FL = \frac{\Phi_0}{1+\tau_1}$. From the same figure we have

$$\Phi = LD = FL \, . \cos\beta = \frac{\Phi_0}{1+\tau_1} \cos\beta \quad \ldots\ldots\ldots\ldots (182),$$

and from equation (181),

$$I_2 . Z_2 = (1 + \tau_1) . FA . Z_1 = (1 + \tau_1) \, FA . 3 . z_1'.$$

If, now, we put $FA = \frac{AH}{\cos\beta}$ (Fig. 405), and substitute these values of $I_2.Z_2$ and Φ in the above equation for the power supplied to the rotor, we get

$$P_2 = 3.k.\Phi_0 f_1 z_1'.AH.$$

On the assumption that there are no losses in the stator, the terminal pressure V_1 will be represented by a vector perpendicular to Φ_0 and also to I_0, as shown in the figure. The ordinate AH is therefore the energy component $I_1 \cos\phi$ of the primary current. The product $k.\Phi_0 f_1 z_1'$ is equal to the electromotive force E_1, and therefore also to the primary terminal pressure V_1. On substituting these values, we get

$$P_2 = 3.V_1.AH = 3.V_1.I_1.\cos\phi \text{ ....(183).}$$

This proves that the power supplied to the rotor is equal to that supplied to the motor itself, which is, of course, a necessity, since we have neglected stator losses.

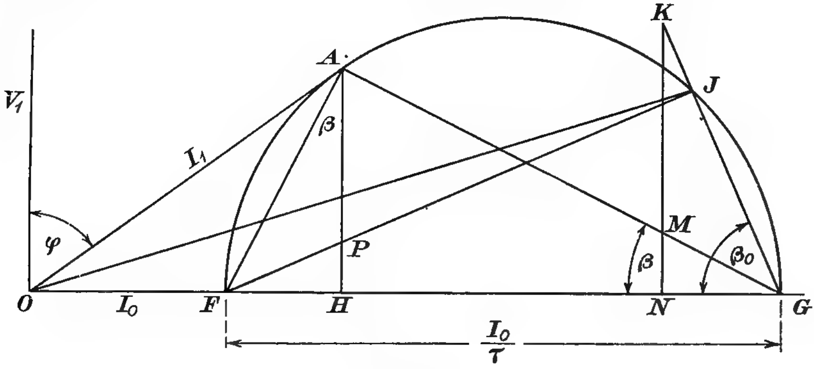

Fig. 405

The power P_2 is transmitted to the rotor by means of a field which rotates with an angular velocity $\omega = 2\pi f_1/p$, and it is immaterial whether the rotor is fixed or rotating. The torque is obtained by dividing the power by the angular velocity, hence

$$M_t = \frac{P_2}{\omega . 9{\cdot}81} = \frac{P_2 . p}{2\pi . f_1 . 9{\cdot}81} \text{ met.-kg.}$$

If we substitute for P_2 its value from equation (183), we get

$$M_t = \frac{3 . V_1 . p}{2\pi . f_1 . 9{\cdot}81} . AH = \frac{V_1 . p}{20{\cdot}5 . f_1} . AH \text{ met.-kg ....(184).}$$

Not only does the Heyland diagram, however, enable us to read off the power transmitted to the rotor and the torque exerted, but it also gives us a direct reading of the slip corresponding to any value of the stator current. Equation (179) on page 390 expresses the important fact that the percentage slip is equal to the percentage power lost in the rotor-windings, or

$$\frac{f}{f_1} = \frac{3 . I_2^2 . R_2}{P_2}.$$

By squaring each side of equation (181) on page 401, we get

$$\frac{I_2^2 . Z_2^2}{(1+\tau_1)^2 . Z_1^2} = FA^2,$$

and from the figure it is evident that

$$\frac{FA}{HF} = \frac{GF}{FA} \quad \text{or} \quad FA^2 = \frac{I_0}{\tau} . HF.$$

It follows from these two equations that

$$I_2^2 = \frac{(1+\tau_1)^2 I_0}{\tau} . \frac{Z_1^2}{Z_2^2} . HF.$$

By substituting this for I_2^2 in the above formula for $\frac{f}{f_1}$, and puttin $P_2 = 3V_1 . AH$ and $\frac{HF}{AH} = \tan\beta$, we have

$$\frac{f}{f_1} = C . \tan\beta \quad \text{.............................(185)},$$

where

$$C = \frac{(1+\tau_1)^2}{\tau} . \frac{Z_1^2}{Z_2^2} . \frac{I_0 . R_2^*}{V_1} \quad \text{.......................(186)}.$$

The percentage slip is therefore proportional to the tangent of th angle β. To determine it graphically, the line *GK* in Fig. 405 is drawn so a to make an angle β_0 with the horizontal, where

$$\tan\beta_0 = \frac{1}{C} \quad \text{.............................(187)}.$$

Let this line cut the semicircle at *J*; then, when the load is such tha the primary current becomes *OJ*, the slip *f* will be equal to f_1, because

$$\frac{f}{f_1} = C . \tan\beta_0 = 1.$$

The slip will then be 100 per cent., and the motor is therefore at rest *OJ* is the so-called short-circuit current, i.e. the current taken by the stato when the rotor is fixed, or at the moment of starting.

A point *K* is taken on the line *GJ* so that the perpendicular *KN* on t the base is exactly 100 mm long. This perpendicular cuts the line *GA* i the point *M*. When the primary current is *OA*, we have

$$\frac{f}{f_1} = C . \tan\beta = \frac{\tan\beta}{\tan\beta_0} = \frac{MN}{KN},$$

and since *KN* is 100 mm the length of *MN* in mm is equal to the per centage slip.

We could determine the mechanical power, or output of the rotor, b decreasing the power supplied to the rotor by an amount correspondin to the slip. A simpler way of proceeding is to join FJ, cutting the ordinate AH in the point P. The angle FPH is equal to β_0, and we have

---

* The resistance $\frac{Z_1^2 . R_2}{Z_2^2}$ is the secondary resistance reduced to its equivalent primar value, as we have already seen in connection with transformers. If the windings o stator and rotor are dissimilar, $k_1 Z_1$ and $k_2 Z_2$ must be substituted for Z_1 and Z_2 in equa tions (181) and (186) (compare p. 382).

$$\frac{f}{f_1} = \frac{\tan\beta}{\tan\beta_0} = \frac{HF}{AH} \Big/ \frac{HF}{PH} = \frac{PH}{AH}.$$

Since the percentage slip is equal to the percentage loss in the rotor,

$$\frac{3 \cdot I_2{}^2 \cdot R_2}{P_2} = \frac{f}{f_1} = \frac{PH}{AH}.$$

We have already seen that AH is a measure of the power transmitted to the rotor, whence PH must represent to the same scale the loss in the rotor, and the difference AP, the mechanical power or output. It is evident from equation (183) on page 402 that the mechanical output P may be expressed as follows:

$$P = 3 \cdot V_1 \cdot AP \text{ watts.}$$

This output consists of the frictional loss and the useful output.

126. Normal Load, Starting Torque and Maximum Torque.

The induced E.M.F. lags, roughly speaking, 90° behind the magnetising current, and the E.M.F. vector should be drawn vertically downwards in Fig. 406. If the stator losses be neglected, the terminal pressure vector will be exactly opposite to the E.M.F. vector, and will make an angle ϕ with the primary current, as shown in Fig. 406. The angle of lag will be evidently a minimum when the current vector is a tangent to the semicircle, and the motor should be so designed that the normal working conditions correspond approximately with this position. The maximum value of the power-factor is given by the following equation:

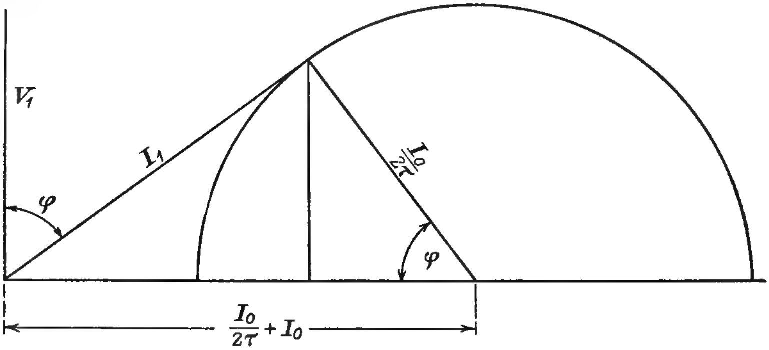

Fig. 406

$$(\cos\phi)_{\max} = \frac{\frac{I_0}{2\tau}}{\frac{I_0}{2\tau} + I_0} = \frac{1}{2\tau + 1} \qquad \text{..................(188).}$$

Hence, if the primary losses be neglected, the maximum value of th power-factor depends only on the leakage coefficient τ.

The table given below shows the maximum power-factors attainab with different values of the leakage coefficients, on the assumption tha $\tau_1 = \tau_2$. We see that a value of $\cos\phi = 0{\cdot}9$ can only be obtained by makin τ_1 and τ_2 as small as 0·03, which is about as small as it is possible to reach It is further evident that, for a given value of τ, the value of the primar current I_1, corresponding to the maximum power-factor, has a definit relation to the no-load current I_0. This is expressed in the equation

$$I_1^2 = \left(\frac{I_0}{2\tau} + I_0\right)^2 - \left(\frac{I_0}{2\tau}\right)^2.$$

From which we get,

$$\frac{I_0}{I_1} = \sqrt{\frac{\tau}{1+\tau}}.$$

From this equation we can determine the most advantageous value the ratio I_0/I_1. The values have been calculated and are given in the tab below. We see that under the best conditions the no-load current li between $\frac{1}{5}$ and $\frac{1}{3}$ of the normal full-load current. The relatively large no-loa current is caused by the air-gap between stator and rotor, and, although wattless current, is one of the greatest disadvantages of an induction moto Even when the leakage of stator and rotor is reduced to 2 per cent. eac the no-load current is still 0·197 of the normal current. For example, 500 H.P. motor of the Alioth Company†, which, on account of its size, shoul give very good results, had a no-load current of 9·3 amperes, and a full-loa current of 50 amperes. This gives a value of the ratio $I_0 : I_1$ equal to 9·3 : 5 or 0·186. It is safe to conclude that this motor had leakage coefficients and τ_2 of 2 per cent., since the normal current vector of 50 amperes woul almost certainly be tangential to the Heyland circle.

We turn now to the *overload capacity* of the motor, still keeping to th assumption that the motor runs at its normal full load with the smalle possible value of ϕ. The primary current I_1 is the tangent to the semicircl and $I_1\ (\cos\phi)_{\max}$ is a measure of the normal torque (equation (184)). Th maximum possible torque is given to the same scale by the radius $\frac{I_0}{2\tau}$. Th overload capacity, or the ratio of the greatest possible to the norm torque, is therefore

$$\frac{\text{maximum torque}}{\text{normal torque}} = \frac{I_0/2\tau}{I_1 \,.\, (\cos\phi)_{\max}} = \frac{I_0}{I_1} \cdot \frac{2\tau + 1}{2\tau}.$$

---

* Heyland makes $\cos\phi = 1$ by fitting a commutator with a small number of segmen to the rotor and supplying the magnetising current to the rotor instead of the stato Since the frequency of the E.M.F. induced in the rotor is very small, a small pressure sufficient to drive the magnetising current through the rotor-windings. This sm pressure is usually tapped off from the stator coils and applied to the brushes. T wattless magnetising current is practically negligible compared with the whole curre supplied to the stator, so that the motor has a power-factor of unity. (See the *E.T.* 1901, p. 633; 1902, pp. 28 and 533; 1903, pp. 51, 72, 95 and 213.)

† See *E.T.Z.*, 1901, p. 547.

The last column of the following table has been calculated from this formula:

$\tau_1 = \tau_2$	$\tau = \tau_1 + \tau_2 + \tau_1\tau_2$	$(\cos\phi)_{\max} = \frac{1}{2\tau + 1}$	$\frac{I_0}{I_1} = \sqrt{\frac{\tau}{1+\tau}}$	Overload ratio $= \frac{I_0}{I_1} \frac{2\tau+1}{2\tau}$
0·07	0·145	0·776	0·355	1·58
0·06	0·124	0·802	0·33	1·65
0·05	0·103	0·83	0·305	1·78
0·04	0·082	0·86	0·275	1·95
0·03	0·061	0·892	0·24	2·2
0·02	0·040	0·925	0·197	2·63

It is evidently very difficult to obtain an overload capacity of 2·6 times that obtained with the maximum value of cos ϕ. It is possible, of course, to design the motor so that the normal primary current vector falls below the tangent. The value of cos ϕ is not greatly decreased, and the overload capacity, with regard to the normal torque, is increased.

A point of special interest and importance is the *torque at starting*. If the resistance of the rotor is negligibly small, equations (186) and (187) give us the following result:

$$\tan\beta_0 = \frac{1}{C} = \frac{\tau \,.\, Z_2^2 \,.\, V_1}{(1+\tau_1)^2 \,.\, Z_1^2 \,.\, R_2 \,.\, I_0} = \infty \,.$$

The angle β_0 is then equal to 90°, and the points J and G in Fig. 405 fall together. OG is therefore the theoretical stator current at starting, on the assumption that the rotor resistance is negligibly small. Under these conditions the starting torque is evidently zero.

Even when the actual resistance of the rotor is put into the equation for β_0, the line OJ is still very nearly perpendicular, and the point J, the ordinate of which gives the starting torque, is still very low. It is evident, however, that the resistance of the rotor circuit can be increased to any desired extent by putting starting resistance across the slip-rings. As can be seen from the above equation, the angle β_0 will be thereby decreased, and the ordinate of the point J, which gives the starting torque, increased. To make the motor start with its maximum torque, the angle β_0 must equal 45°, so that $\tan\beta_0 = 1$. If the combined resistance of rotor and starting resistance per phase be R_2, the conditions for this maximum starting torque are as follows:

$$\tan\beta_0 = \frac{\tau \,.\, Z_2^2 \,.\, V_1}{(1+\tau_1)^2 \,.\, I_0 \,.\, Z_1^2 \,.\, R_2} = 1,$$

or

$$R_2 = \frac{\tau \,.\, Z_2^2 \,.\, V_1}{(1+\tau_1)^2 \,.\, I_0 \,.\, Z_1^2} \,.$$

With this resistance per phase the starting torque of the motor will have its greatest possible value.

127. Ossanna's Circle-diagram.

Up to this point we have assumed that the resistance of the stator winding and the consequent losses could be neglected. On this assumptio the terminal pressure V_1 can be represented by a vector equal and opposit to that of the induced E.M.F., and making an angle of 90° with the no-loa current I_0. As a matter of fact, however, the terminal pressure V_1 has no only to overcome the back E.M.F. E_1, but has also to drive the current I through the resistance R_1. In Fig. 407 the terminal pressure vector V is drawn vertically, as before, and the primary current vector $OP = I$ lags behind V_1 by the angle ϕ. Along OP is drawn the primary copper dro $OP_2 = I_1 . R_1$. The vector E_1 which counterbalances the back E.M.F. is th side of a parallelogram, of which OP_2 is the other side and V_1 the resultan diagonal. E_1 is evidently smaller than V_1, and the magnetic flux and mag netising current must be correspondingly reduced. ~~We may assume as a~~

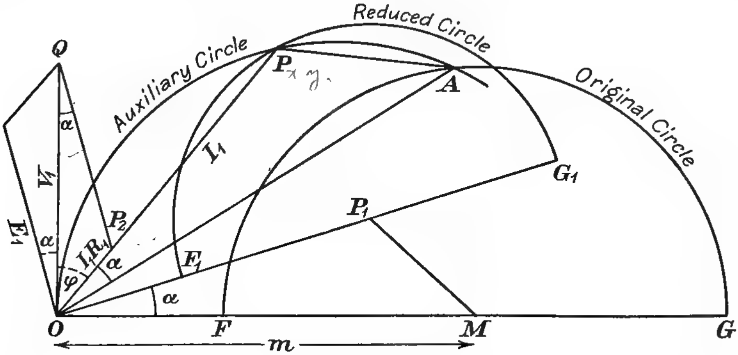

Fig. 407

approximation that the reduction of magnetising current is proportion to the reduction of flux. The reduced magnetising current OF_1 is at righ angles to E_1, and $OF_1 : F_1G_1 = \tau$. The end P of the primary current vecto lies on the circle on the diameter F_1G_1 with the mid-point P_1. The positio and size of this circle varies with every change in the load, but we shall us it to find the locus of the point P. To this end we must find the equatio to the circle on F_1G_1, that is, we must express the co-ordinates of the centr P_1, and the radius of the circle, as functions of the co-ordinates of P.

Let $OF = I_0$ be drawn at right angles to V_1 to represent the no-loa current for a stator without resistance, that is, the current which woul produce a flux to give an E.M.F. equal to V_1. Further, let $FG = \frac{I_0}{\tau}$, $FM =$ $OM = m$, and $P_1F_1 = r_1$, and let

x and y be the co-ordinates of the point P,
x_1 ,, y_1 ,, ,, ,, ,, P_1,
x_2 ,, y_2 ,, ,, ,, ,, P_2.

On the assumption that the no-load currents are proportional to the electromotive forces, we have

$$OP_1 : OM = E_1 : V_1$$

and, since the angles OQP_2 and MOP_1 are equal, these triangles must be similar. It follows from the figure that

$$\frac{x_1}{V_1 - y_2} = \frac{m}{V_1}, \qquad \frac{y_1}{x_2} = \frac{m}{V_1}, \qquad \frac{r_1}{r} = \frac{OF_1}{OF} = \frac{E_1}{V_1}.$$

Moreover,

$$x_2 = x.R_1, \quad y_2 = y.R_1, \quad E_1{}^2 = V_1{}^2 + I_1{}^2.R_1{}^2 - 2V_1I_1R_1 \cos\phi.$$

Since $I_1.\cos\phi = y$ and $I_1{}^2 = x^2 + y^2$, we have

$$x_1 = \frac{m.V_1 - myR_1}{V_1}, \qquad y_1 = \frac{mxR_1}{V_1},$$

$$r_1{}^2 = \frac{r^2}{V_1{}^2}\{V_1{}^2 + (x^2 + y^2) R_1{}^2 - 2V_1R_1y\}.$$

Now, the point P lies on the circle with the centre P_1, and its co-ordinates must therefore fulfil the equation

$$(x - x_1)^2 + (y - y_1)^2 = r_1{}^2.$$

We can substitute the values already found for x_1, y_1 and r_1, and put

$$m^2 - r^2 = s^2 \text{ and } z^2 = V_1{}^2 + s^2.R_1{}^2.$$

We thus get the equation

$$x^2 + y^2 - \frac{2m.V_1{}^2}{z^2}.x - \frac{2s^2.V_1.R_1}{z^2}.y = -\frac{s^2.V_1{}^2}{z^2}.$$

This is obviously the equation of a circle, the centre of which is given by the co-ordinates

$$p = \frac{m.V_1{}^2}{z^2} \quad \text{..............................(189)},$$

$$q = \frac{s^2.V_1.R_1}{z^2} \quad \text{..............................(190)}.$$

If R be the radius of this circle (Fig. 408),

$$R^2 = p^2 + q^2 - \frac{s^2.V_1{}^2}{z^2} \quad \text{......................(191)}.$$

This circle is known as the **Ossanna circle.**

If the values of p and q be substituted in equation (191), and the last term be multiplied top and bottom by $z^2 = V_1{}^2 + s^2.R_1{}^2$, we get

$$R^2 = \frac{m^2.V_1{}^4}{z^4} + \frac{s^4.V_1{}^2.R_1{}^2}{z^4} - \frac{s^2.V_1{}^2.(V_1{}^2 + s^2.R_1{}^2)}{z^4},$$

whence

$$R^2 = \frac{m^2.V_1{}^4}{z^4} - \frac{s^2.V_1{}^4}{z^4},$$

or, since $m^2 - s^2 = r^2$,

$$R = \frac{r.V_1{}^2}{z^2} \quad \text{..............................(192)}.$$

From the circle with the radius R, the angle ϕ corresponding to any primary current I_1 can be directly determined.

If the line OA be drawn in Fig. 407 so as to make the angle α with OP, then OA will make the same angle $\alpha + \phi$ with V_1 in the original diagram as OP makes with E_1 in the reduced diagram. Since both diagrams are based on the same value of τ, we have

$$\frac{OP}{OA} = \frac{OF_1}{OF} = \frac{E_1}{V_1}.$$

Hence, for many purposes, it is sufficient to take the point A in the original diagram, and to imagine the diagram reduced in the ratio $OP : OA$, and, at the same time, turned up by the angle α. The end of the primary current vector will thus be brought to the point P on the Ossanna circle. Further, since the triangles OPA, QP_2O are similar, the angle OAP must be equal to ϕ. It follows from this that the points A and P lie on a circle to which OQ is a tangent at O, and which has its centre on the line OG. Hence, the point P on the Ossanna circle corresponding to the point A on the original circle must lie on the circle through the points O and A, which has its centre on the base line OG. This is the auxiliary circle in Fig. 407.

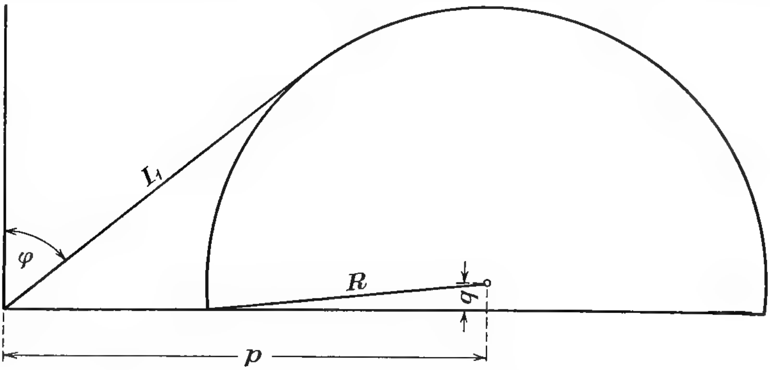

Fig 408

It must be carefully noted that neither of the three circles in Fig. 407 is the Ossanna circle. We proceed now to give a graphical construction, whereby the Ossanna circle may be determined. The points F and G in Fig. 409 are the ends of the original vectors of no-load current and short-circuit current respectively. F' and G' are the corresponding points on the Ossanna circle. We know, from what we have just seen, that F' must lie on a semicircle on OF, and G' on a semicircle on OG, the centre of which is M_1. The triangles $OF'F$ and $OG'G$ are therefore right-angled triangles.

Now, both at no-load and on short-circuit with a rotor of negligible resistance, the motor acts merely as a choking coil, so that the primary current OF' or OG' must form one side of a right-angled triangle of which the hypotenuse is $\frac{V_1}{R_1}$. In Fig. 409 OD is made equal to $\frac{V_1}{R_1}$ and the angles $OF'D$, $OG'D$ are therefore right angles.

Hence, the points F' and G' of the Ossanna circle lie not only on the semicircle on OD, but also on the lines DG and DF. This gives us a very simple way of finding the points G' and F'.

If M is the centre of the original circle, we have from equations (189) and (190) that

$$\frac{q}{m-p} = \frac{V_1/R_1}{m}.$$

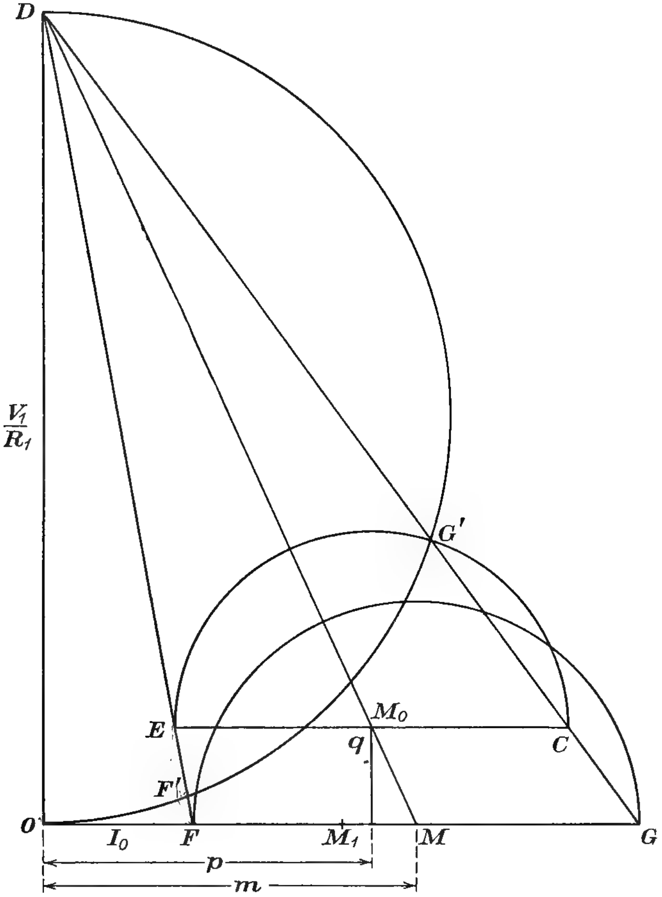

Fig. 409

Hence the centre M_0 of the Ossanna circle must lie on the line MD, and as we have found two points on the circle, viz. G' and F', the circle can now be drawn.

If we draw a semicircle on OG with centre M_1 it must just touch the Ossanna circle at G', since it touches the original circle at G. Hence, the points G', M_0 and M_1 must lie on a straight line.

It can also be shown that the points C and E, at which the Ossanna circle intersects the lines DG and DF, are on the same level as the centre M_0. Finally, the points OEG' and $OF'C$ lie on two straight lines.

This construction can only be conveniently carried out in the case of small motors, in which the primary resistance plays an important part. In large motors with a good efficiency, the length $OD = \frac{V_1}{R_1}$ would be enormous, compared with the remainder of the diagram. In such a case the only effect of the primary resistance is to raise the centre of the semicircle almost vertically through a distance q, while the diameter is hardly changed from $\frac{I_0}{\tau}$.

In a 600 H.P. Oerlikon motor*, for example,

$$V_1 = 1900, \qquad I_0 = 36, \qquad R_1 = 0{\cdot}4, \qquad \tau = 0{\cdot}117;$$

from these data we find

$$r = \frac{I_0}{2\tau} = 154,$$

$$m = I_0 + \frac{I_0}{2\tau} = 190,$$

$$s^2 = m^2 - r^2 = 12{,}400,$$

$$z^2 = V_1{}^2 + s^2.R_1{}^2 = 3600.10^3 + 2.10^3.$$

If we neglect the second term in the expression for z^2, the error is only 2 in 3600, and we can then put $z^2 = V_1{}^2$. This makes $p = m$ in equation (189), and M_0 vertically above M in Fig. 409. It also makes the new radius R equal to the original radius r in equation (192). The distance through which the centre is raised is

$$q \neq \frac{s^2.V_1.R_1}{z^2} = \frac{s^2.R_1}{V_1} = 2{\cdot}6 \text{ amperes.}$$

The smallness of this correction may be seen from the fact that the length OG represents 344 amperes.

128. Ossanna's Circle-diagram—*continued.*

In Fig. 410 we have drawn both the original circle, neglecting the stator resistance, and the Ossanna circle, the former dotted and the latter as a full line. The auxiliary circle through O is also drawn as a full line. We saw in the previous section that the triangles OPA, QP_2O were similar. If we drop the ordinate PBK, the angle OPK is equal to ϕ. The triangles OBP, OPA and QP_2O are therefore similar, and we have

$$\frac{PB}{I_1R_1} = \frac{I_1}{V_1}.$$

The power lost in stator resistance is

$$3I_1{}^2.R_1 = 3V_1.PB.$$

The power supplied to the stator is

$$P_1 = 3V_1.I_1.\cos\phi = 3V_1.PK.$$

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* See *E.T.Z.*, 1900, No. 52.

By subtracting one from the other we get the power transmitted to the rotor, viz.

$$P_2 = 3V_1 . BK.$$

The length F_1P in Fig. 407 is a measure of the rotor current, in accordance with equation (181). To find F_1P in Fig. 410 we must reduce FA in the ratio $\frac{E_1}{V_1} = \frac{OP}{OA}$, or

$$\frac{I_2 Z_2}{(1+\tau_1) Z_1} = F_1P = FA . \frac{OP}{OA} .$$

To make this reduction a horizontal is drawn through B to meet FA in C. The ordinate through C cuts the original circle in D. Then we have

$$\frac{FD^2}{FA^2} = \frac{FV}{FH} = \frac{FC}{FA} = \frac{OB}{OA} .$$

Also

$$\frac{OB}{OA} = \frac{OB}{OP} \cdot \frac{OP}{OA} = \frac{OP^2}{OA^2} .$$

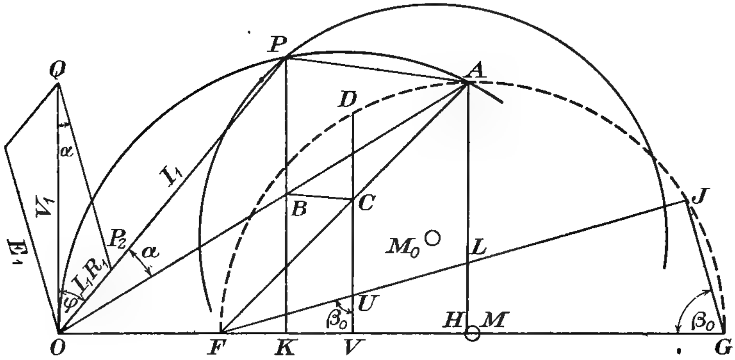

Fig. 410

From these two equations, it follows that

$$FD = FA . \frac{OP}{OA} = \frac{I_2 . Z_2}{(1+\tau_1) . Z_1} .$$

The line GJ is drawn at an angle β_0 with the horizontal, where β_0 is the same as in equations (186) and (187) on page 403. Then we have

$$UV = \frac{FV}{\tan \beta_0} ,$$

and

$$FV = \frac{FD^2}{2r} \doteq \frac{FD^2}{I_0/\tau} = \frac{I_2^2 . Z_2^2}{(1+\tau_1)^2 . Z_1^2} \div \frac{I_0}{\tau} .$$

From this equation, together with equation (186) on page 403, we get

$$UV = \frac{I_2^2 . R_2}{V_1} .$$

The loss in the rotor is therefore

$$3I_2^2 . R_2 = 3V_1 . UV.$$

Since $CV = BK$, the mechanical power given out by the rotor must be

$$P = 3V_1 . CU.$$

Further, since the relative slip is equal to the relative loss in the rotor,

$$\frac{f}{f_1} = \frac{UV}{CV} = \frac{LH}{AH}.$$

For the convenient determination of the slip the method of Fig. 405 can be employed, using the point A in the original circle.

129. The most Convenient Form of the Circle-diagram.

Ossanna's method of correcting the circle-diagram, as described in the foregoing section, has the advantage of strict accuracy, but the decided disadvantage of being extremely cumbersome. For this reason another method, due to Heyland, is generally employed, in which the losses in the stator and rotor are subtracted from the ordinates of the original circle-diagram. The simplicity of the method more than compensates for the small error thus introduced.

It was shown in Section 127 that the radius of the corrected circle should be almost exactly $\frac{I_0}{2\tau}$, and that its centre should be nearly vertically above the centre of the original circle by a small distance q. We shall now neglect this small distance, so that the corrected circle and the original circle will be identical. The vector of the primary terminal pressure will be vertical, and the power supplied to the motor will be (Fig. 411)

$$P_1 = 3V_1 . I_1 . \cos\phi = 3V_1 . AH.$$

From this supplied power we must subtract the primary copper loss. The line GK is drawn at an angle α_1 below the horizontal, so that

$$\tan\alpha_1 = \frac{I_0 . R_1}{V_1} . \left(2 + \frac{1}{\tau}\right) \quad \text{.......................(193)}.$$

A circle is drawn with its centre at M_1 where the vertical through M meets GK, and with a radius M_1G. This circle meets the line GM_1 produced at the point K vertically below F; it also cuts the line AG in the point P. A horizontal is drawn through P to meet the ordinate AH at the point C. It is evident from the geometry of the figure that the angles α_1, α_2 and α_3 are all equal, and that the triangles AHF, PCA are similar. From this it follows that

$$\frac{AC}{AP} = \frac{HF}{FA} \text{ and } \frac{AP}{FA} = \tan\alpha_3.$$

Multiplying these equations together, we have

$$AC = HF . \tan\alpha_3 = HF . \tan\alpha_1 \quad \text{..............}(a).$$

In the triangle OAF, we have

$$I_1^2 = FA^2 + I_0^2 + 2I_0 . HF,$$

and since

$$FA^2 = GF . HF = \frac{I_0}{\tau} . HF,$$

$$I_1^2 = \frac{I_0}{\tau} . HF + I_0^2 + 2I_0 . HF.$$

Solving this equation for HF, we get

$$HF = \frac{I_1^2 - I_0^2}{I_0\left(2 + \frac{1}{\tau}\right)}.$$

Substituting this in the above equation for AC, and using equation (193), we have

$$AC = \frac{I_1^2 - I_0^2}{I_0\left(2 + \frac{1}{\tau}\right)} . \tan \alpha_3 = \frac{(I_1^2 - I_0^2) . R_1}{V_1}.$$

We shall now neglect the term $I_0^2 . R$, which is always very small, more especially as we can allow for it later when correcting for the no-load losses. For the primary copper loss, we have therefore

$$3 . I_1^2 . R_1 = 3 V_1 . AC.$$

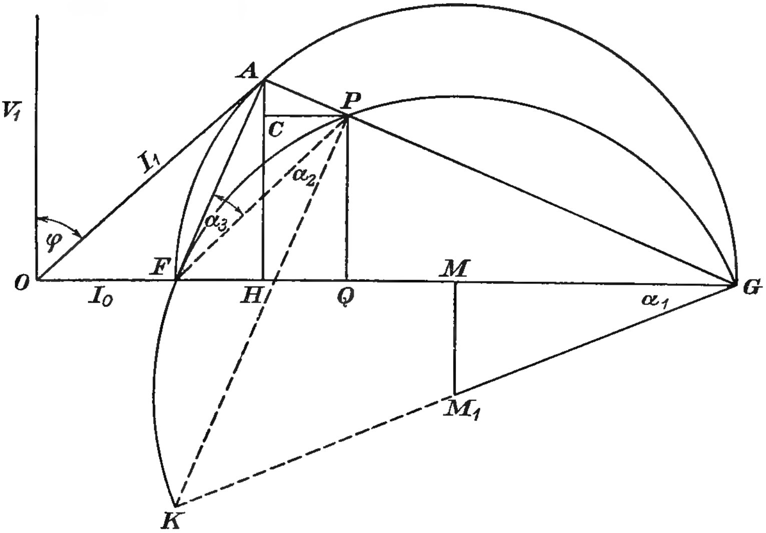

Fig. 411

By simple subtraction we find the power transmitted from the stator to the rotor to be

$$P_2 = 3 . V_1 . CH = 3 . V_1 . PQ.$$

To determine the effect of the rotor copper loss we draw the line GL in Fig. 412, so that

$$\tan \gamma_1 - \tan \alpha_1 = \frac{(1 + \tau_1)^2 . I_0 . Z_1^2 . R_2}{\tau . Z_2^2 . V_1} = C \quad \text{...........(194)}.$$

This is the same expression as we used in Section 125 (equation (186)) for determining the slip. The vertical through M cuts this line in the point M_2. With the centre M_2 and the radius M_2G we draw the circle $GRFL$. As before, the angles γ_1, γ_2 and γ_3 are equal. A horizontal through R meets AH in S. Then

$$\frac{RA}{FA} = \tan \gamma_3 \text{ and } \frac{AS}{RA} = \frac{HF}{FA}.$$

Multiplying these equations together, we have

$$AS = HF \tan \gamma_3 = HF \tan \gamma_1 \quad \text{....................}(b).$$

We saw above that $AC = HF \tan \alpha_1$, whence

$$CS = AS - AC = HF \tan \gamma_1 - HF \tan \alpha_1 = HF (\tan \gamma_1 - \tan \alpha_1).$$

If we substitute for HF its equivalent $\frac{FA_2}{I_0/\tau}$, and for $\tan \gamma_1 - \tan \alpha_1$ its value from equation (194), we get

$$CS = \frac{FA^2}{I_0/\tau} \cdot \frac{(1+\tau_1)^2 . I_0 . Z_1^2 . R_2}{\tau . Z_2^2 . V_1} = \frac{(1+\tau_1)^2 . FA^2 . Z_1^2}{Z_2^2} \cdot \frac{R_2}{V_1}.$$

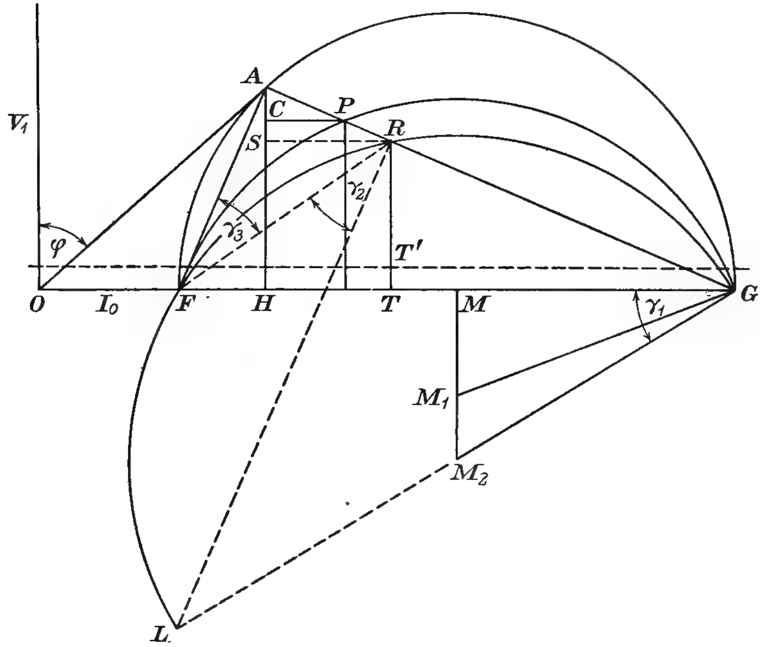

Fig. 412

From equation (181) on page 401, we know that

$$\frac{(1+\tau_1)^2 . FA^2 . Z_1^2}{Z_2^2} = I_2^2,$$

so that

$$CS = \frac{I_2^2 . R_2}{V_1}.$$

We get, therefore, for the copper loss in the rotor

$$3 I_2^2 . R_2 = 3 V_1 . CS.$$

Subtracting this from the power transmitted to the rotor, we have

$$P = 3 . V_1 . SH = 3 . V_1 . RT.$$

This is the power which is transformed into mechanical power in the rotor. It includes the no-load losses, which, however, can be allowed for by drawing a horizontal line at a suitable distance above the base line. This line, which is shown dotted, is then taken as the foot of the output ordinates. For the sake of simplicity we add the stator hysteresis loss to the friction loss, and remember that we have still a part of the stator copper loss, viz. $I_0^2 . R_1$, to allow for. The sum of these losses is practically equal to the power taken to run the motor light at the normal voltage. Calling this P_0, we have

$$P_0 = 3 . V_1 . T'T.$$

The ordinate RT' is then a measure of the nett mechanical output of the motor.

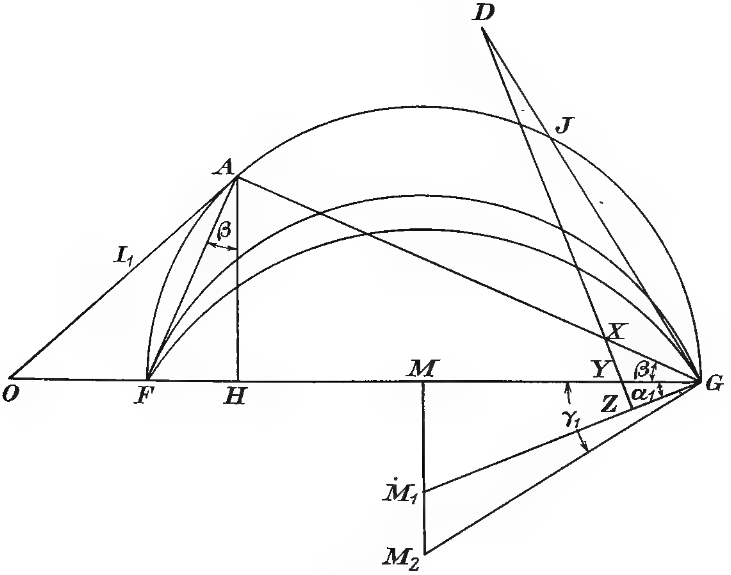

Fig. 413

We must now construct a scale of slip in our diagram. For this purpose the line GJ is drawn at right angles to GM_2 (Fig. 413). This line cuts the original circle in the point J. Since GJ is a tangent to the lowest circle, it does not intersect it; in other words, the point R in Fig. 412 coincides with the point G, and the mechanical output is nil. The motor is therefore stationary, and the point J corresponds to a slip of 100 per cent. The vector OJ represents the primary starting current.

To find the slip for any primary current I_1 we join GA, and drop a perpendicular from any point D in GJ produced, on to GM_1. Since the sides of any triangle are proportional to the sines of the opposite angle, we have

$$\frac{XY}{GY} = \frac{\sin\beta}{\sin[90° - (\alpha_1 + \beta)]} = \frac{\sin\beta}{\cos(\alpha_1 + \beta)}.$$

Similarly, since the angle at D is equal to $\gamma_1 - \alpha_1$, we have

$$\frac{GY}{DY} = \frac{\sin(\gamma_1 - \alpha_1)}{\sin(90^\circ - \gamma_1)} = \frac{\sin(\gamma_1 - \alpha_1)}{\cos\gamma_1}.$$

By multiplying together the two left-hand sides and the two right-hand sides of these equations, we get

$$\frac{XY}{DY} = \frac{\tan\gamma_1 - \tan\alpha_1}{\operatorname{cotan}\beta - \tan\alpha_1}.$$

From equations (*a*) and (*b*) of this section and from Fig. 413, we have

$$\tan\alpha_1 = \frac{AC}{HF}, \qquad \tan\gamma_1 = \frac{AS}{HF}, \qquad \cot\beta = \frac{AH}{HF}.$$

Hence
$$\frac{XY}{DY} = \frac{AS - AC}{AH - AC} = \frac{CS}{CH}.$$

Now, this ratio $\frac{CS}{CH}$ represents the fraction of the power supplied to the rotor which is lost in heating the rotor-winding. It must, therefore, represent the ratio of the slip to the synchronous speed. Hence

$$\frac{XY}{DY} = \frac{f}{f_1}.$$

If, now, the perpendicular DZ is so drawn that the length DY is equal to 100 mm, the length XY in mm will give the percentage slip.

130. Practical Example.

We shall now construct the Heyland diagram for an actual motor, viz. a 600 horse-power three-phase motor by the Oerlikon Company of Zürich*. The motor was designed for the abnormally low speed of 75 revolutions per minute. For this reason the number of poles must be relatively large and the leakage consequently above the average (see Section 131). In these circumstances a frequency of 50 would have been too high, and a frequency of 22·5 cycles per second was therefore employed. For the number of pairs of poles we have

$$p = \frac{f_1}{n} = 18.$$

The diameter of the rotor was about 300 cm and the air-gap 0·2 cm.

The terminal pressure V_1 of the delta wound stator was 1900 volts. The resistance R_1 per phase of the stator was 0·4 ohm, while that R_2 of the rotor was 0·016 ohm. The stator had 324 slots each containing 8 wires, so that the total number of wires Z_1 in the primary winding was $324 . 8 = 2592$. The rotor had a winding of bare copper bar, each of the 432 slots containing a single bar, so that $Z_2 = 432$. Since both stator and rotor had ordinary coil-windings, the winding coefficients k_1 and k_2 are equal. The no-load current is given as 62 amperes, and the normal full-load current as 170 amperes. The corresponding currents in the stator-winding can be found by dividing by $\sqrt{3}$. Collecting these data, we have

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* See *E.T.Z.*, 1900, No. 52.

$$R_1 = 0{\cdot}4, \qquad V_1 = 1900, \qquad I_0 = \frac{62}{\sqrt{3}} = 36,$$
$$R_2 = 0{\cdot}016,$$
$$Z_1 = 2592, \qquad I_1 = \frac{170}{\sqrt{3}} = 100 \text{ (normal)}.$$
$$Z_2 = 432,$$

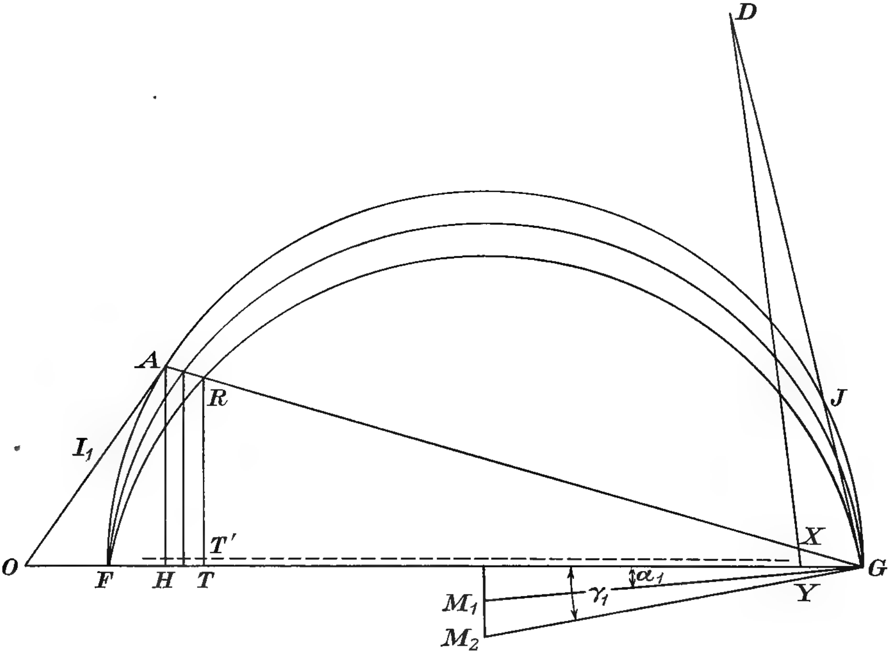

Fig. 414

From experiments which were carried out on the motor, and which will be considered in the next section, the leakage factor under normal conditions was found to be

$$\tau = 0{\cdot}117 \text{ or } \tau_1 = \tau_2 = 0{\cdot}057.$$

For the construction of the diagram, we have

$$OF = I_0 = 36,$$
$$FG = \frac{I_0}{\tau} = \frac{36}{0{\cdot}117} = 306.$$

From equations (193) and (194) on pp. 413 and 414, we have

$$\tan\alpha_1 = \frac{I_0 \,.\, R_1}{V_1} \,.\, \left(2 + \frac{1}{\gamma}\right) = \frac{36 \,.\, 0{\cdot}4}{1900} \,.\, \left(2 + \frac{1}{0{\cdot}117}\right) = 0{\cdot}08,$$

and
$$\tan\gamma_1 - \tan\alpha_1 = \frac{(1+\tau_1)^2 \,.\, I_0 \,.\, Z_1^2 \,.\, R_2}{\tau \,.\, Z_2^2 \,.\, V_1} = 0{\cdot}103.$$

By addition therefore we have

$$\tan\gamma_1 = 0{\cdot}103 + 0{\cdot}08 = 0{\cdot}183.$$

Fig. 414 is drawn from these values of I_0, $\frac{I_0}{\tau}$, α_1 and γ_1. The total no-load losses were found to be $P_0 = 16{,}000$ watts, and the dotted horizontal line is drawn at a corresponding height above the base line, viz.

$$T'T = \frac{P_0}{3\,.\,V_1} = \frac{16{,}000}{3\,.\,1900} = 2{\cdot}8 \text{ amperes.}$$

We now proceed to determine the current, efficiency, power-factor, and speed for different values of the load, from no-load up to the maximum load, at which the motor stops. By load we mean the useful turning-moment M_u as measured on the motor pulley. This is not the turning-moment exerted on the rotor, but that obtained by dividing the useful output P_u by the angular velocity: thus

$$M_u = \frac{P_u}{2\pi\,.\,n\,.\,9{\cdot}81} \text{ met.-kg.}$$

Since the useful output is equal to the product $3\,.\,V_1\,.\,RT'$, where RT' is measured to a scale of amperes, we have

$$M_u = \frac{3\,.\,1900\,.\,RT'}{2\pi\,.\,n\,.\,9{\cdot}81} = 92{\cdot}5\,\frac{RT'}{n} \text{ met.-kg.}$$

The corresponding value of the primary current I_1 is given by the length OA on the scale of amperes.

The watt-component of the primary current is AH, so that the efficiency is given by the ratio

$$\eta = \frac{RT'}{AH}.$$

In order that the speed may be read off directly, the perpendicular from D on to GM_1 is taken so that the length DY, down to the point at which it meets GO, is equal to 75 mm, 75 revolutions per minute being the synchronous speed. It follows from the previous section that the length DX in mm is equal to the speed of the motor in revolutions per minute.

In this way the values given in the following table for seven different values of I_1 can be found very quickly. The third horizontal row represents the normal full-load conditions as shown in Fig. 414. The last vertical column gives the values of the line current I which, for the delta connection, is equal to $\sqrt{3}\,.\,I_1$.

$I_1 = OA$	RT'	AH	$\eta = \frac{RT'}{AH}$	r.p.m. $= DX$ in mm	$\cos\phi = \frac{AH}{OA}$	$M_u = 92{\cdot}5\,.\,\frac{RT'}{n}$	$I = \sqrt{3} I_1$
45	21·2	24·2	0·87	74·4	0·548	1,565	78
60	38·7	42·7	0·91	74	0·718	2,900	104
100	73·5	80·5	0·916	73	0·805	5,600	173
196·3	121·2	142·7	0·85	70·5	0·73	9,520	340
220	123·9	150	0·83	68·3	0·682	10,000	380
243	121·2	152·5	0·793	66·6	0·63	10,100	421
(284	113·5	150·2	0·753	64·8	0·57	9,700	490)

The curves in Fig. 415 have been plotted from this table. The abscissae represent the useful torque. This reaches a maximum value of 10,000 metre-kilogrammes, which is nearly double the normal load of 5600 met.-kg. The motor has, therefore, nearly 100 per cent. overload capacity. The corresponding value of the current is 380 amperes, which is more than double the normal current of 173 amperes. If the load be still further increased the motor comes to rest. It is, therefore, impossible for the motor to run with the point *A* on the right-hand side of the diagram in Fig. 414; when starting without external resistance in the rotor circuit, the point *A* starts from *J* and moves over the semicircle to the left-hand side. By putting suitable starting resistance in the rotor circuit, the point *A* may be made to start on the left-hand side of the diagram; in this way the starting current is greatly reduced.

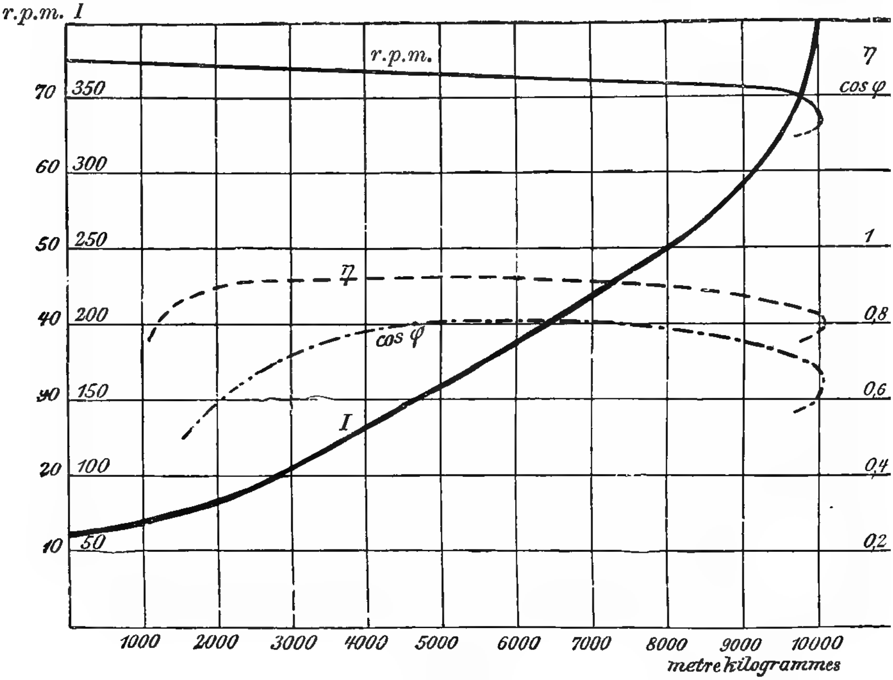

Fig. 415

The speed of the motor under full load is 73, which represents a slip of 2 in 75, or 2·7 per cent. The speed drops continually as the load is increased, and reaches 68·8 revs. per minute at the maximum possible load.

The curves of efficiency and power-factor increase very rapidly at first, reach their maximum values at the normal load, and then fall very gradually as the load is still further increased. Both curves are practically horizontal over a remarkably wide range in the neighbourhood of the normal load. An efficiency of 92 per cent. and a power-factor of 0·81 might appear, at first sight, to be very low values for a motor of 600 horse-power. We shall see in the next section that the low speed results in a relatively great leakage, and affects both the power-factor and the efficiency disadvantageously. An Oerlikon standard motor of the same output for a speed of 370 revolutions per minute, with a frequency of 50 cycles per second, had a power-factor of 0·92, an efficiency of 95 per cent. and 1·5 per cent. slip.

We shall now consider the experimental determination of the leakage factor τ, which we have seen to be of fundamental importance in the principle and working of induction motors. We shall also consider the various factors which affect the leakage, since an accurate knowledge of these factors is essential to the design of an induction motor.

131. The Leakage Factor.

If the motor has a two- or three-phase wound rotor, the leakage factor τ can be determined experimentally in the following manner. A terminal pressure V_1 is applied to the stator, and the pressure across the slip-rings of the open rotor is measured. The motor is simply a transformer on open circuit, and the terminal pressures should be proportional to the numbers of turns Z_1 and Z_2. Due allowance must be made, of course, for any differences between the connections and windings of the stator and rotor.

We should expect the pressure across the rotor terminals to be $V_1 \cdot \frac{Z_2}{Z_1}$. If, however, some of the lines which link the primary do not link the secondary, the pressure V_2 will be smaller than $V_1 \cdot \frac{Z_2}{Z_1}$. The ratio of the expected pressure $V_1 \cdot \frac{Z_2}{Z_1}$ to the actual pressure V_2 is equal to the ratio of the total flux produced by the stator to that actually linking the rotor conductors. Now, at no-load there is no rotor current and consequently no rotor leakage, hence τ_1 will be the ratio of the leakage flux to the flux linking the rotor conductors, or

$$\frac{\Phi}{\Phi_{l_1}} = \frac{1}{\tau_1}.$$

Hence

$$\frac{\Phi_{l_1} + \Phi}{\Phi} = 1 + \tau_1,$$

and

$$1 + \tau_1 = \frac{V_1 \cdot \frac{Z_2}{Z_1}}{V_2}.$$

In the same way a three-phase pressure V_2 can be applied to the rotor, and the resulting pressure V_1 across the disconnected stator terminals measured. It should theoretically be equal to $V_2 \cdot \frac{Z_1}{Z_2}$, but leakage will cause it to be smaller, and we have

$$1 + \tau_2 = \frac{V_2 \cdot \frac{Z_1}{Z_2}}{V_1}.$$

As the difference between the theoretical and the actual value is very small, great care is necessary if reliable results are required. Having found τ_1 and τ_2 we can easily calculate the leakage factor $\tau = \tau_1 + \tau_2 + \tau_1\tau_2$.

This method of finding the leakage factor has the advantage of simplicity and depends directly on the definition of τ_1 and τ_2. Unfortunately, however, it leads to incorrect results in the majority of cases, because the leakage factor is not a constant, as we have assumed, but decreases with large currents owing to the saturation of the tips of the teeth. The current used in the above method is very small, and the normal current is several times as large. The power-factor of the motor is therefore better than one would expect from the above experiments.

For the same reason it is obvious that the Heyland diagram is not strictly correct, except for one special load, at which the assumed values of τ_1 and τ_2 hold true. It may be necessary, in extreme cases, to draw separate diagrams for different values of the load. It is, however, of the greatest importance that the assumed values of τ_1 and τ_2 should be correct at the normal load. To do this, the rotor is short-circuited and fixed, and the primary current I_{sc} determined. If the rotor resistance were negligible, the short-circuit current $OJ = I_{sc}$ in Fig. 405 would coincide with OG. In any case, it will not differ much from it as regards magnitude. Having measured also the no-load current I_0 we have

$$I_{sc} = I_0 + \frac{I_0}{\tau},$$

or

$$\tau = \frac{I_0}{I_{sc} - I_0} \quad \text{..............................(195).}$$

If the short-circuit test were made with the normal primary voltage, the current would be far too large. This current would highly saturate the tips of the teeth across which the leakage mainly occurs. The value of τ obtained by this method would be too small for the normal working condition, and would only be true for the extreme right-hand side of the diagram*. The stator pressure should be adjusted so as to give a short-circuit current approximately equal to the normal working current. The reluctance of the leakage paths will then be the same as under normal working conditions. The theoretical value of I_{sc} at the full pressure can be found by multiplying the measured current by the ratio of the full pressure to the reduced pressure at which the test was made.

To make this clearer, we shall take, as an example, the motor considered in the previous section. The experimental data are taken from the *E.T.Z.*, 1900, No. 52. The short-circuit current was measured with values of the primary P.D. of 600 and 390 volts respectively instead of the normal P.D. of 1900 volts. The values found for the short-circuit current were 200 and 110 amperes in the external leads, which correspond to $200/\sqrt{3} = 116$ and $110/\sqrt{3} = 63{\cdot}7$ amperes respectively.

Considering in the first place the value found with a P.D. of 600 volts, viz. 116 amperes, we see that this would give a short-circuit current I_{sc} at the full P.D. of $116 \cdot \frac{1900}{600} = 368$ amperes. In the second case a current of 63·7 amperes is measured at a P.D. of 390 volts, which corresponds to a current of $63{\cdot}7 \cdot \frac{1900}{390} = 310$ amperes at the normal primary pressure.

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* See *Electrician*, 1904, vol. LIV, p. 87, for an interesting example.

These results are collected in the following table:

V_1	I_0	Observed current	I_{sc} (calc.) for 1900 v.	$\tau = \frac{I_0}{I_{sc} - I_0}$
1900	36	—	—	—
600	—	116	368	0·109
390	—	63·7	310	0·131

It is evident that the leakage factor τ decreases as the current increases, so that the values of τ_1 and τ_2 in the table on page 406 decrease with increasing current. The reason for this, as we have already pointed out, is that the tips of the teeth become saturated, and so prevent an increase of current from causing a corresponding increase of leakage flux. The percentage loss of flux by leakage is therefore continually decreased as the current in stator and rotor is increased.

The value of τ corresponding to the normal full load of the motor can be found by interpolation from the foregoing figures. We have

$$I_{sc} = 116, \qquad \tau = 0{\cdot}109,$$
$$63{\cdot}7, \qquad 0{\cdot}131.$$

Hence, for the normal current of 100 amperes,

$$\tau = 0{\cdot}117.$$

Now, from page 401, we know that $\tau = \tau_1 + \tau_2 + \tau_1\tau_2$, and if $\tau_1 = \tau_2$ this becomes $\tau = 2\tau_1 + \tau_1^2$, which for a value of τ of 0·117, gives

$$\tau_1 = \tau_2 = 0{\cdot}057.$$

Although a large value for a motor of this size, it is by no means excessive when the number of poles is taken into account.

The third method of constructing the diagram is perhaps the most obvious one, viz. by making an actual brake test on the motor at different loads, or by fixing the rotor and varying the regulating resistance in the rotor circuit. We measure the line circuit I, the terminal voltage V, and, by means of one or more wattmeters, the power P_1 supplied to the stator. We have then

$$\cos\phi = \frac{P_1}{\sqrt{3} \cdot V \cdot I}.$$

The primary P.D. is set up vertically, and the angle ϕ set out for each value of the primary current, which is then marked off along its corresponding line. Were τ invariable, the points so obtained should lie on a circle. A circle can generally be drawn through the points in the working part of the figure, and then used for the calculation of τ. The points shown in Fig. 416 were obtained by the author on a $1\frac{1}{4}$ H.P. motor.

Since the power-factor and the overload capacity are both directly dependent upon the leakage factor τ, the designer of an induction motor must strive to obtain as small a value of τ as possible, that is, he must reduce the leakage to the smallest possible limit. It is obviously disadvantageous to place the winding in closed tunnels or holes, on account of the small reluctance offered to the leakage flux. This is especially the case if a considerable thickness of iron is left between the hole and the air-gap, but if this bridge of iron be reduced to a minimum by stamping the holes very near the air-gap, there is little objection to the totally enclosed hole. The best construction, from the point of view of reduced leakage and of ease of manufacture, is the rectangular open slot, into which former-wound coils can be placed. This leads, however, to an unequal distribution of flux in the air-gap, and consequently to an increased magnetising current.

It is necessary in all cases to divide the coil-side between several slots, in order to keep down the leakage flux produced by the current in one slot. A limit is set to the number of slots by the proportionately greater amount of space taken up by the insulation. Generally speaking, the stator has from 3 to 5 slots per coil-side, while the rotor has from 3 to 7. It is obvious why induction motors are often made with relatively large diameters and small axial lengths, since the large pole-pitch enables the coil-side to be distributed among a large number of slots. Unfortunately, however, increasing the diameter at the expense of axial length adds considerably to the cost of the motor. This is obvious from the consideration that the output could be considerably increased by making the motor axially longer, without adding very materially to the cost of manufacture. Another disadvantage of the short axial length is the relative importance of the flank leakage.

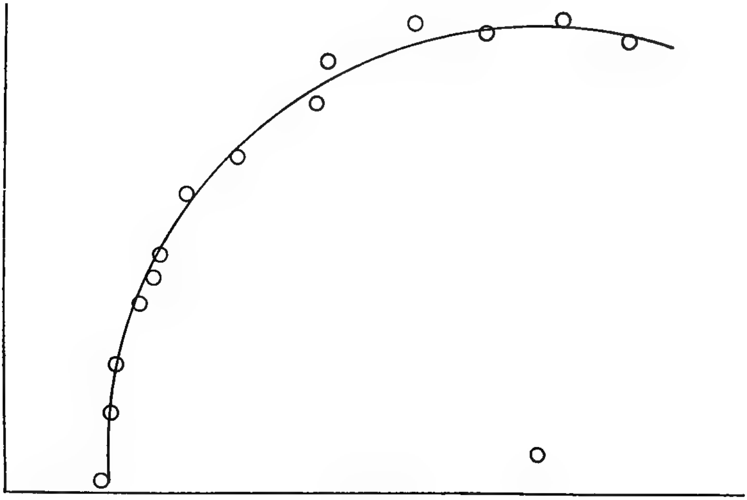

Fig. 416

A second way in which the value of τ can be reduced is by decreasing the air-gap. Since the product $\tau_1\tau_2$ is negligibly small, we may put $\tau = \tau_1 + \tau_2$, and, as we can see from the definition of τ_1 and τ_2 on page 400, the value of τ is directly proportional to the reluctance of the air-gap*.

It is interesting to consider the effect on the operation of the motor of varying the air-gap. This cannot be done by using one motor and successively turning a little off the stator or rotor, since this would not only increase the air-gap, but would also increase the reluctance of the tips of the teeth.

Two motors should be compared which have the same slot dimensions, etc., but different air-gaps. The smaller the air-gap, the smaller is the value of τ, and the larger is the maximum power-factor, in accordance with the equation

$$(\cos\phi)_{\max} = \frac{1}{2\tau + 1}.$$

A decrease in the air-gap will naturally lower the no-load current, since a smaller magnetising current is now required to drive the flux across the reduced air-gap. The maximum torque is dependent on the radius of the Heyland circle, or on $\frac{I_0}{\tau}$. This has remained the same for both motors, showing that the maximum torque is but little affected by the air-gap if the motors are the same in other respects.

The value of τ is very greatly affected by the choice of frequency. In order to make this clear we may consider two motors with stator and rotor cores of the same dimensions and with the same number of slots. The primary P.D., the output and the speed are the same in each case, but one motor is designed for a frequency of 50 cycles per second, the other for 30. We shall assume that the proportion of slot space occupied by copper is the same in each case, as also the current density in the wires, and the number of ampere-turns per cm of periphery.

Since the output and the P.D. are the same, it follows that the current will also be the same in each case, if we neglect any difference in the value of $\cos\phi$. Moreover, since the number of ampere-turns per cm of periphery has been taken as the same in each case, the equality of current leads to the equality of the number of wires on the stator and also of the number of wires per slot.

The motors will differ, however, in the number of pairs of poles, in accordance with the equation

$$p = \frac{f_1}{n}.$$

The motor for a frequency of 50 must have a greater number of poles than that for a frequency of 30, in the ratio of 5 : 3. Since both motors have equal peripheries, their pole-pitches will be in the proportion of 3 : 5. If Fig. 417 represents the first motor with 3 slots per coil-side or 9 per pole-pitch, then the second motor will be represented by Fig. 418, in which there are 5 slots per coil-side or 15 per pole. The number of wires in each slot is the same in each case, so that the wires per coil-side will be as 3 : 5.

The E.M.F. per phase is given by the equation

$$V_1 = k \cdot \Phi_0 \cdot f_1 \cdot z_1' \cdot 10^{-8} \text{ volts.}$$

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* See Behrend, *The Induction Motor*; also Behrend, *E.T.Z.*, 1904, p. 59.

We can put Φ_0 approximately equal to the flux $B_{\text{mean}}.A_g$ which crosses the air-gap, and f_1 equal to $p.n$ neglecting the slip; we thus get

$$V_1 = k.B_{\text{mean}}.A_g.p.n.z_1'.10^{-8} \text{ volts.}$$

In this equation $A_g.p$ is equal to half the cylindrical surface, and is therefore the same in each case. Since, moreover, V_1, k, n, and z_1' are the same in each case, it follows that both motors must work with the same average flux density B_{mean}.

With equal flux density the hysteresis loss is proportional to the frequency, and will therefore be $\frac{5}{3}$ times as great in the 50 cycle motor as in the 30 cycle motor.

To produce the same flux density in each case will require the same number of ampere-turns per coil-side, and as the coil-side of the 50 cycle motor contains only $\frac{3}{5}$ of the wires in the coil-side of the other motor, its no-load or magnetising current will have to be greater in the ratio of 5 : 3, that is, the no-load current will be directly proportional to the frequency*.

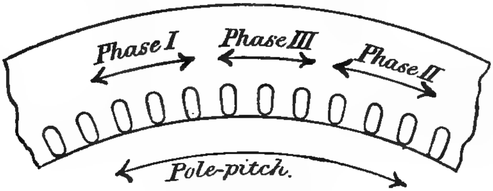

Fig. 417

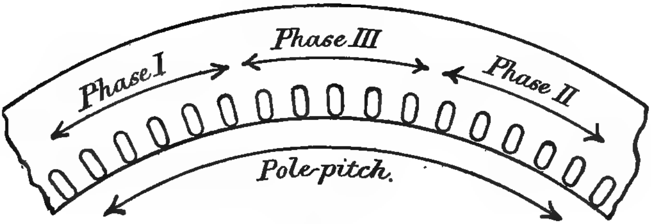

Fig. 418

The question is not nearly so simple when we come to consider the relative values of the leakage factor τ. If we consider the leakage flux due to one coil-side in Fig. 417 we see that it has to cross 3 slots, whereas in Fig. 418 it has to cross 5 slots, or, in other words, the reluctances of the leakage paths are as 3:5. If we assume, temporarily, that the short-circuit current is the same in each case, the ampere-wires per coil-side are in the same ratio, viz. 3 : 5. Since both the ampere-turns and the reluctances have changed in the same proportion, the resulting flux will be the same in each case. In the first case, this flux links 3 slots with a frequency of 50, whereas in the second case it links 5 slots with a frequency of 30; the E.M.F. induced in the coil-side would be the same in each case, but as the first motor has $\frac{5}{3}$ times as many coil-sides as the second, the total E.M.F. of the first motor would be $\frac{5}{3}$ times that of the second. Hence, to produce the same short-circuit current in each case, the primary pressures would have to be in the ratio of 5 : 3, or, on the other hand, with the same P.D. the short-circuit currents of the two motors are as 3 : 5.

---

* When considering any given motor the primary P.D. of which is maintained constant, the no-load current must evidently vary approximately inversely as the frequency.

We have seen above that the no-load currents are in the ratio of 5 : 3. The alterations in both no-load current and short-circuit current are therefore in such a direction as to reduce τ in the second motor, i.e. the one for the lower frequency. If our assumptions were strictly correct, the value of τ would vary as the square of the frequency. An important part of the leakage, which we have not considered, is that around the ends of the coils, where they project beyond the stator and rotor. Since the 30 cycle motor in Fig. 418 would have much longer coil-ends, this part of the leakage would be increased as the frequency was decreased. How far this increased leakage would counteract the decreased slot leakage is not at all certain. If it exactly counterbalanced it, the short-circuit current would be the same in both cases and τ would vary directly as the frequency. The general correctness of these results has been proved experimentally by Behrend.

It is interesting to note that the two motors will have about equal overload capacity. The motor with the greater number of poles has the greater magnetising current, but also the greater leakage factor τ. The radius of the Heyland circle, and therefore the maximum torque, will be approximately the same as in the other motor.

The important points of difference between the two motors lie in the magnetising current and in the power-factor.

CHAPTER XIX

132. Resolution of the primary M.M.F. of the single-phase motor into two constant rotating components.—133. The circle-diagram of the single-phase motor.—134. Single-phase commutator motors.—135. The plain series single-phase motor.—136. The repulsion motor.—137. The Eichberg-Latour motor.

132. Resolution of the Primary M.M.F. of the Single-phase Motor into Two Constant Rotating Components.

If one of the leads supplying current to a three-phase induction motor be broken, it will continue to run as a single-phase motor in the same direction. Coils 1 and 2 of the three-phase motor in Fig. 419 now form a single coil with a breadth equal to two-thirds of the pole-pitch, carrying the same current throughout. If the rotor circuit is open, the primary current produces an alternating flux, the axis of which has a fixed direction. In the figure this direction is vertical. The centres of the poles remain at *A* and *C* and the neutral points at *B* and *D*. It is difficult to see how a torque can be produced by means of this alternating field, and how the direction of this torque can be determined by the initial direction of rotation which is arbitrarily given to the rotor.

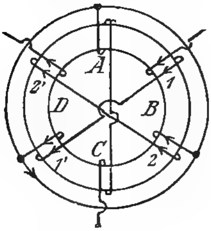

Fig. 419

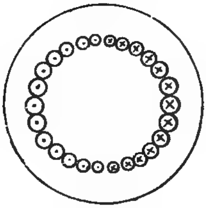

Fig. 420

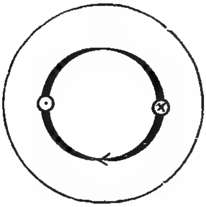

Fig. 421

Ferraris resolved the alternating field into two oppositely rotating fields of constant magnitude. We shall go a step further, and resolve, not only the flux, but also the primary ampere-turns into two oppositely rotating sets of ampere-turns. We must assume, however, that the conductors are distributed around the stator according to the sine law, as is shown in Fig. 420 by circles of various sizes, the larger circles indicating that the wires are more closely packed. Theoretically this would lead to an infinite number of infinitely small wires. We can represent this sinusoidal distribution of the ampere-turns in the manner indicated in Fig. 421.

On this assumption, the effect of the alternating ampere-turns at any moment and at any point is exactly equal to the combined effect of two sinusoidal windings, carrying a constant current and rotating in opposite directions. The constant current is such that the number of ampere-turns in each rotating winding is a half of the total ampere-turns on the actual stator, when the alternating current reaches its maximum value. In Fig. 422 the actual stator current is a maximum. At this moment, those sides of the imaginary rotating coils (shown black) which carry current in the same direction are superposed, and their individual effects can be added. They are therefore equivalent to the stationary alternating ampere-turns, represented by the outer ring of the figure, which have their maximum value at the given moment.

An eighth of a period later the stator current will be

$$\hat{i}.\sin 45^\circ = 0{\cdot}707\,\hat{i}.$$

We see that at this moment the rotating ampere-turns at *A* and *C* neutralise each other (Fig. 423). The addition of the two sine curves in the lower part

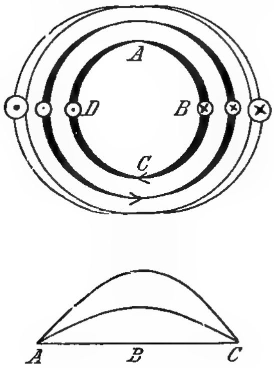

Fig. 422

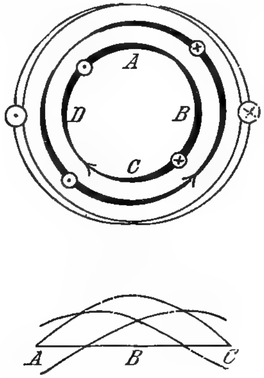

Fig. 423

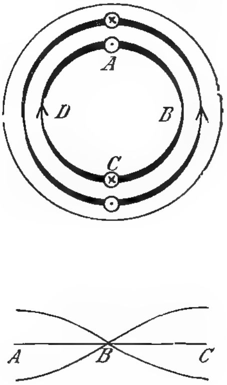

Fig. 424

of the figure gives a sine curve with a maximum at *B*, as before, but with the ordinates reduced in the ratio of 0·707 : 1 as compared with Fig. 422. The magnetic effect of the two rotating windings is still equal to that of the actual stator-winding. The same agreement is found after another $\frac{1}{8}$ period, when the actual stator current is zero (Fig. 424). The rotating ampere-turns now neutralise each other at all points. It must be impressed on the reader that this substitution of two oppositely rotating magnetomotive forces with a constant sine-wave distribution in place of the actual stationary alternating current, is purely fictitious and does not exist except in our imagination. It is, nevertheless, a very useful and fruitful device.

We shall now imagine the motor to be frictionless and running light with a squirrel-cage rotor of negligible resistance. We shall assume the direction of rotation to be clockwise, and we shall designate the imaginary stator-winding which rotates in this direction as the forward ampere-turns, and that in the opposite direction as the backward ampere-turns. The magnetic flux which the forward ampere-turns drive through the rotor rotates synchronously with it and does not cut the rotor bars. This flux is shown in Figs. 425, 426 and 427, for the three moments previously considered. In all that follows, the flux, electromotive force, and rotor currents, etc., for each fictitious rotating magnetomotive force will naturally be distributed around the periphery according to the sine law.

With respect to the backwardly rotating ampere-turns, the rotor has a slip of 200 per cent. In consequence of this, a large short-circuit current flows in the rotor, directly opposing the backwardly rotating ampere-turns, and preventing the flux from entering the rotor. The flux is driven, however, along the secondary leakage path (Figs. 428, 429 and 430). The outer black circles in these figures represent the backwardly rotating ampere-turns, and the inner ones the rotor ampere-turns. On the assumption that the rotor has no resistance, an infinitely small number of lines is sufficient to induce the rotor currents necessary to counterbalance the primary backwardly rotating ampere-turns. When running perfectly light with a rotor of zero resistance, we have therefore the forward rotor field, the backward rotor leakage field, and the backward rotor ampere-turns. Since the rotor itself runs synchronously, the rotor bars must carry an alternating current of double frequency.

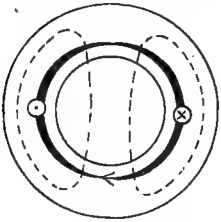

Fig. 425

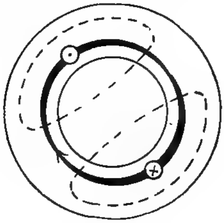

Fig. 426

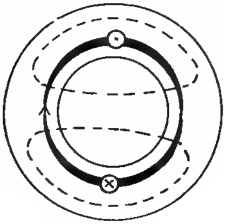

Fig. 427

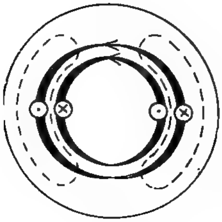

Fig. 428

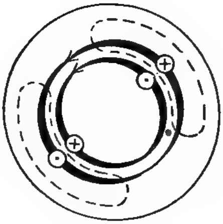

Fig. 429

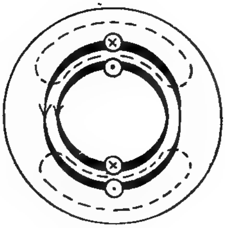

Fig. 430

Turning now to the actual conditions in the motor, we see that the flux in Fig. 425 is produced by the difference between the actual ampere-turns on the stator and those on the rotor. Figs. 425 and 428 represent ampere-turns and fluxes existing simultaneously in the motor; similarly for Figs. 426, 429 and 427, 430. The flux represented in Fig. 427 is 90° out of phase with the actual stator ampere-turns both as regards time and space, for the actual stator current is zero at this moment; hence the flux shown in Fig. 427, which we have ascribed to the fictitious forwardly rotating component, must really be produced by the rotor currents. The rotor currents shown in Fig. 430 at this moment are in the right direction to produce this flux, and will therefore be increased by an amount equivalent to the ampere-turns shown in Fig. 427. The single-phase motor is thus reduced to a two-phase motor, one phase of which, viz. that producing a vertical flux in the figures, is obtained from the difference between the stator and rotor currents, while the other, viz. that producing the horizontal flux, is obtained from the rotor currents alone. These relations are somewhat altered, even on no-load, if the rotor resistance is taken into account. They are considerably altered when the motor is heavily loaded. The single-phase motor does not possess the simplicity and uniformity of the polyphase induction motor. We shall proceed further with the theory of the rotating ampere-turns, and see to what results it leads us.

133. The Circle-diagram of the Single-phase Motor.

In the previous section we imagined the actual stator winding to be replaced by two oppositely rotating windings distributed around the periphery according to a sine law and carrying a direct current. We would arrive at exactly the same result if these two imaginary windings were stationary and uniformly distributed around the periphery, but made up of a great many phases, each turn, in fact, being supplied with alternating current of the correct phase depending on its position on the periphery. Such a distribution of current exists around the periphery of a squirrel-cage rotor.

In what follows we shall imagine that the actual stationary single-phase stator winding is replaced in this way by two stationary polyphase windings, suitably supplied with polyphase currents and producing two magnetic fields rotating in opposite directions. *For the sake of simplicity we shall fix our attention on those turns of these two polyphase windings, which are co-axial with the actual stator winding.* It is obvious from Fig. 422 that the current in these phases must have its maximum value at the moment of maximum current in the actual stator winding. From this it follows that the current in these two phases which we are specially considering is in phase with the actual stator current, and that their vectors will coincide in direction.

With regard to the magnitude of these vectors, we shall draw the diagram to represent, not the effective value of the current in the phases considered, but the ampere-turns of the rotating magnetomotive forces, which are, of course, proportional to the effective value of the current. Each of these rotating magnetomotive forces is equal to a half of the actual maximum ampere-turns on the stator, for which we shall use the symbol I_0 when the rotor circuit is open, and I when the motor is working.

When the rotor circuit is open, the two rotating magnetomotive forces produce two rotating magnetic fluxes, which, together, induce the same E.M.F. in the stator winding as is actually induced by the alternating flux, and which, we know, must be practically equal and opposite to the applied terminal P.D.

When the motor is running, the equality of the two rotating fluxes is destroyed by the rotor currents. *They must, nevertheless, induce in the stator, electromotive forces whose geometrical sum is equal and opposite to the applied terminal* P.D. The fact that the magnetic fluxes are rotating in opposite directions does not affect the vector diagram in any way, for the latter shows only the variation of E.M.F. with time, and it is immaterial whether the flux has been removed from a coil to the right or to the left. If, then, the forwardly rotating flux induces an E.M.F. E_1 in the actual stator, while that induced by the backwardly rotating flux is E_2, the constant terminal pressure V must be the resultant of two vectors drawn equal and opposite to E_1 and E_2 (Fig. 431).

The electromotive forces induced in the two imaginary phase windings which we are specially considering will be in phase with E_1 and E_2 since they are co-axial with the actual stator winding, and, moreover, they will be proportional to their respective magnetising ampere-turns OP_1 and OP_2. Hence, the resultant OF must be at right-angles to the terminal pressure V and must also be constant. This is true for all conditions of load, and even when the rotor is open; in this latter case, however, the two rotating fields are exactly equal, and therefore OP_1 and OP_2 are also equal, and are added algebraically together to give the maximum value of the actual ampere-turns with open rotor. Obviously then OF represents this latter magnitude which we have called I_0.

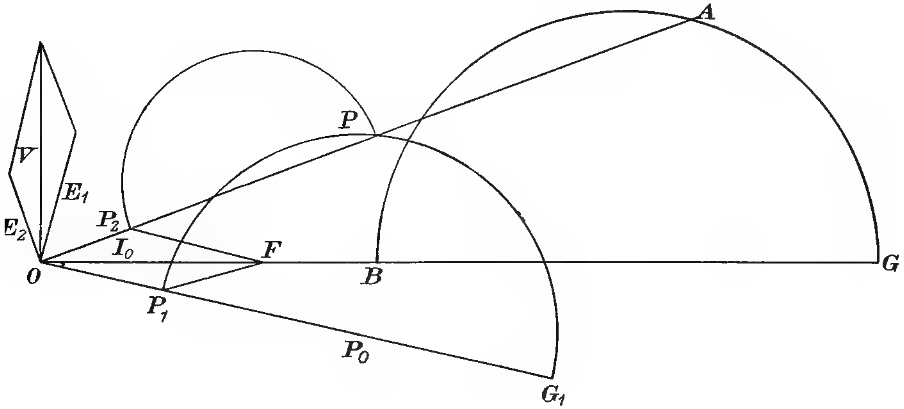

Fig. 431

We now make

$$P_1G_1 = \frac{1}{\tau} . OP_1$$

and

$$P_2P = \frac{1}{\tau} . OP_2$$

and draw the two Heyland circles. Since, if we neglect the rotor resistance, the slip with respect to the backwardly rotating flux is 200 per cent., OP must be the backwardly rotating ampere-turns, and, since the forwardly rotating ampere-turns must be equal both in magnitude and phase, it is obvious that the point P must also lie on the circle on P_1G_1 with centre P_0. The Heyland circles are both drawn with the same value of τ. For the sake of simplicity, let

$$OP_2 = c . OP,$$

and let x and y be the co-ordinates of the point P, and, with the corresponding index, of the points P_1 and P_2. Remembering that $OF = I_0$ we have

$$x_1 = I_0 - x_2 = I_0 - c.x,$$

$$y_1 = -y_2 = -c.y.$$

Again, if $OP_0 = c_1.OP_1$, we have

$$x_0 = c_1.x_1 \text{ and } y_0 = c_1.y_1.$$

Finally, if we let the radius $P_0P_1 = c_2.OP_1$ we get for the equation to the circle about P_0,

$$(x - x_0)^2 + (y - y_0)^2 = P_0P_1^2 = c_2^2.(x_1^2 + y_1^2).$$

If we substitute in this equation the values found for x_0, x_1, y_0 and y_1, we obtain an equation in which x^2 and y^2 have equal coefficients, and in which the xy and y terms vanish, showing that the locus of the point P is a circle with its centre on the base line OG. The same is therefore true of the end point A of the actual stator ampere-turns, since OA is double OP.

The coefficients c, c_1 and c_2 could easily be calculated from the leakage factor τ, but this is not necessary if we introduce the no-load current I_0' taken by the motor when running light. This is equivalent to a current $\frac{I_0'}{2}$ in each of the rotating magnetomotive forces, and seeing that the motor is running nearly synchronously this will be the magnetising current in one case, but the short-circuit current in the other. Since the short-circuit current in the latter case is $\frac{I_0'}{2}$, the corresponding magnetising current will be $\frac{I_0'}{2} \cdot \frac{\tau}{1+\tau}$. Now, we have already proved that the resultant of the two magnetising currents is a constant I_0, and in the present case, where there is no load, these currents are in phase and can be added algebraically. Hence,

$$\frac{I_0'}{2} + \frac{I_0'}{2} \cdot \frac{\tau}{1+\tau} = I_0,$$

or

$$I_0' = 2I_0 \cdot \frac{1+\tau}{1+2\tau} \quad \text{..............................(196).}$$

In the figure OB is made equal to I_0', the no-load current. On short-circuit, that is, when the rotor is closed but stationary, the two imaginary rotating fluxes are again equal. A moves round to G, while P and G_1 coincide on the base line OG; P_1 and P_2 will fall together at the mid-point of OF. Hence, $OG = I_0 + \frac{I_0}{\tau}$, where OG is the current taken by the motor when short-circuited. The diameter of the actual primary current circle is therefore

$$BG = OG - OB = I_0 + \frac{I_0}{\tau} - 2I_0 \frac{1+\tau}{1+2\tau} = \frac{I_0}{\tau} \cdot \frac{1+\tau}{1+2\tau} \quad \text{...(197).}$$

The same result could have been obtained from the equation to the circle already found.

The following characteristic properties of the single-phase motor follow from these equations.

1. As τ is of the order of 0·1, the no-load current *OB* is nearly twice as large as the current with open circuited rotor.
2. The maximum power-factor is smaller than that of polyphase motors.
3. Assuming the rotor to have no resistance, the slip is zero, and the speed of the motor is therefore constant, depending simply on the frequency. The turning-moment is therefore proportional to the power, and is represented by the ordinates of the circle *GB*. The overload capacity is evidently much smaller than that of the polyphase motor.
4. A single-phase motor will not start as such, even though resistance be inserted in the rotor circuits, for, when the rotor is stationary, the two rotating excitations are exactly similar, and there is no reason why the motor should revolve in one direction in preference to the other. In practice single-phase motors are provided with an auxiliary winding, making them practically two-phase motors. Both windings are connected up to the single-phase supply mains, but by various so-called phase-splitting devices, consisting of arrangements of choking-coils and resistances and sometimes also condensers, the currents taken by the two windings are made to differ largely in phase. When the motor has got up speed the auxiliary winding is either disconnected entirely or put in series with and thus made to form a part of the main winding.

Further investigation shows that the end point of the primary current vector lies on a circle, even when the rotor resistance is taken into account. The centre of this new circle lies vertically above that of the circle obtained above by neglecting the rotor resistance. The proof of this point is, however, beyond the scope of this work. It can be shown, moreover, that the power supplied, the mechanical output, the torque, and the rotor losses, for the actual motor, are equal to the sum of the corresponding values obtained from the diagrams for the two rotating excitations. The sum of the two slips must be always 200 per cent., and the torque and mechanical output due to the backward rotating excitation must be reckoned negative.

These results are very remarkable, when we consider the complicated nature of the relations involved. For example, if the rotor resistance be taken into account, the rotor currents are found to have a sine-wave distribution, the position of the maximum rotating in the opposite direction to the motor, while its value fluctuates from moment to moment. It is found also that the actual field which cuts the stator-winding has a sine-wave distribution, and rotates in the same direction as the motor, while its maximum value is subjected to periodic fluctuations (elliptic rotating field).

134. Single-phase Commutator Motors.

On account of the poor characteristics of the single-phase induction motor as regards overload capacity, power-factor, and starting torque, it is very unsuitable for many purposes, notably for electric traction. The polyphase induction motor is superior in all these respects, but has the disadvantages of requiring a number of overhead wires and of offering no simple and efficient means of speed regulation.

The construction of practical single-phase commutator motors must therefore be regarded as an important advance. These can only be operated as series motors since the inductance of the field coils of a shunt motor would cause the flux to lag nearly 90 degrees behind the armature current, thus giving a very small torque.

Single-phase commutator motors may be divided into three types:

1. The ordinary series motor with the armature and field windings connected simply in series.

2. The repulsion motor in which the torque-producing flux is due to one part of the stator winding, whilst the torque-producing current is induced in the short-circuited armature by the other part of the stator winding.

3. The Eichberg-Latour motor in which the field-producing current is supplied directly to the armature, whilst the torque-producing current is induced in the armature by the current in the stator winding.

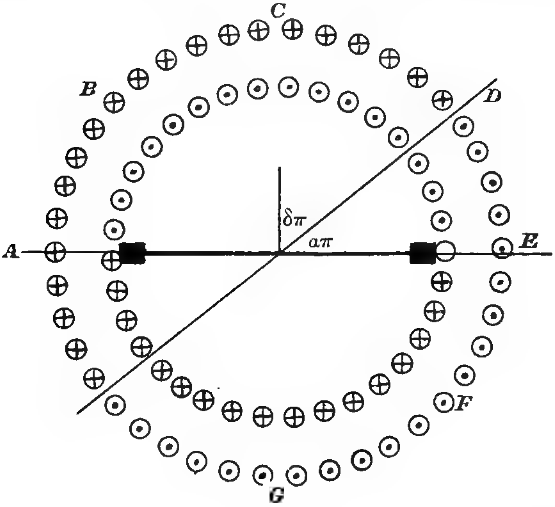

Fig. 432

In order the more readily to understand the action of these motors, we shall first consider the E.M.F. induced in the armature due to transformer action and to rotation. We shall consider the general case in which the axis of the armature winding, i.e. the direction of the field produced thereby, makes an angle a with the stator winding. The stator winding is assumed to be uniformly distributed, and the rotor is assumed to be short-circuited by connecting the brushes (Fig. 432). As in equation (68 *a* on page 88, **the coefficient of mutual induction M is 10^{-8} times the number of stator linkages produced by a rotor current of 1 ampere.** By this we mean a current of 1 ampere in the connection short-circuiting the two brushes. Only those stator coils between *B* and *D*, i.e. subtending an angle $\pi(1-2\alpha) = 2\delta\pi$, need be considered, as the coils between *D* and *F* are not linked with the flux produced by the rotor. Let

T_1 be the number of stator turns in series per pole-pair,

$T_2 = c.T_1$ the number of rotor turns in series per pole-pair.

The maximum value of the flux density produced by a rotor current of 1 ampere will be

$$B_2 = \frac{0{\cdot}4\pi \,.\, T_2}{l_g},$$

where l_g is the double air-gap.

This maximum value occurs at E in Figs. 432, 433, the flux density falling off uniformly on either side and reaching zero at the points C and G. If we assume for the moment that the maximum flux density is unity and that the motor is bipolar, so that each pole subtends an angle π, we see that the flux density at any point in the air gap will be $x \,.\, \frac{2}{\pi}$, where x is the angle between C and the point considered. A coil of width dx, as shown in the lower part of Fig. 433, will be linked by flux partly positive and partly negative, the resultant flux being equal to the shaded area, on the assumption that the motor has an axial length of 1 cm. Calling this flux Φ we have

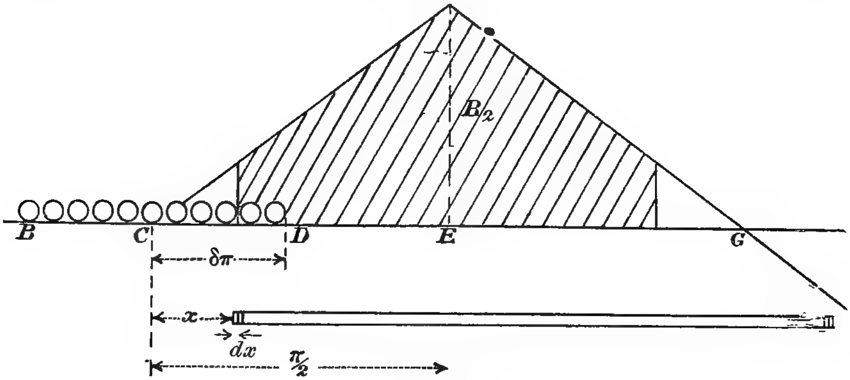

Fig. 433

$$\Phi = \left(1 + \frac{2x}{\pi}\right) . \left(\frac{\pi}{2} - x\right) = \frac{\pi}{2}\left(1 - 4\frac{x^2}{\pi^2}\right).$$

The number of turns in series in this stator coil of width dx will be $\frac{dx}{\pi} . T_1$. The linkages will therefore be

$$\Phi \,.\, \frac{dx}{\pi} \,.\, T_1 = \frac{T_1}{2}\left(1 - 4\frac{x^2}{\pi^2}\right) dx.$$

Now, to find the total stator linkages per pole-pair we have seen that we need consider only those coils between B and D, that is from $x = -\delta\pi$ to $x = +\delta\pi$. Hence we have

$$\frac{T_1}{2}\int_{-\delta\pi}^{+\delta\pi}\left(1 - 4\frac{x^2}{\pi^2}\right) dx = k \,.\, \frac{\pi}{3} \,.\, T_1,$$

where

$$k = 3\delta - 4\delta^3.$$

Now, when we remember that

the axial length is not 1 but L_a,

the maximum density not 1 but $B_2 = \frac{0{\cdot}4\pi . T_2}{l_g}$,

the pairs of poles not 1 but p, and

the pole-pitch not π but $\frac{D}{2p}$ times as great,

we have
$$M = \frac{0{\cdot}2\pi^2}{3} . k . T_1 . T_2 . \frac{D . L_a}{l_g} . 10^{-8}.$$

This is the mutual induction, but from it we can easily determine the coefficient of self-induction of the stator, neglecting leakage, that is, neglecting the flux which links the conductors without crossing the air-gap. We put T_1 instead of T_2 and make $\alpha = 0$; k is then unity, and we have

$$L = \frac{0{\cdot}2\pi^2}{3} . T_1^2 . \frac{D . L_a}{l_g} . 10^{-8}.$$

If the stator carries an alternating current I_1, the E.M.F. induced in the stator due to its own flux, including its leakage flux, will be

$$E_1 = 2\pi f . L . I_1 (1 + \tau_1) = (1 + \tau_1) . X . I_1,$$

where X is put for $2\pi fL$.

The rotor turns $T_2 = cT_1$, hence its self-induction will be c^2 times that of the stator, and the E.M.F. induced in the rotor due to its own flux, including that due to leakage, will be

$$E_2 = c^2 . 2\pi f . L . I_2 (1 + \tau_2) = (1 + \tau_2) . c^2 . X . I_2.$$

The E.M.F. induced in the rotor by the stator will be

$$E_{12} = 2\pi f . M . I_1 = c . k . X . I_1,$$

and similarly that induced in the stator by the rotor,

$$E_{21} = c . k . X . I_2.$$

We have determined the electromotive forces due to transformer action; we must now consider those due to rotation. These cannot be due to rotation in the field produced by the rotor itself, nor by the turns *BCD* (Fig. 432). The only field we need consider is that produced by the turns *DEF*, and this will be distributed as shown in Fig. 434. The maximum value B_1 of the alternating flux will be produced by $2\alpha . T_1 . \hat{i}_1$ ampere-turns, where i_1 is the external current supplied to the stator. Hence

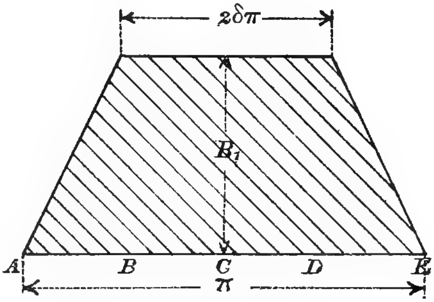

Fig. 434

$$B_1 = \frac{0{\cdot}4\pi}{l_g} . 2\alpha . T_1 . \hat{i}_1.$$

If the axial length be L_a and the pole-pitch $\frac{\pi . D}{2p}$, the maximum flux is evidently (see Fig. 434)

$$\Phi = B_1 \pi \left(\frac{1+2\delta}{2}\right) . \frac{L_a . D}{2p} .$$

The E.M.F. induced by the rotation will have a maximum value (see equation (79) on page 124)

$$\hat{e}_r = \frac{p}{a} . \Phi . n . Z_2 . 10^{-8}.$$

The total number of wires on the armature is Z_2, where

$$Z_2 = 2a . p . 2T_2.$$

If ν is the speed expressed in terms of the synchronous speed $\frac{f}{p}$, we have

$$n = \nu . \frac{f}{p} = \frac{\nu . \omega}{p . 2\pi} .$$

Substituting these values, we have

$$\hat{e}_r = \left(\frac{1+2\delta}{2}\right) . \nu . \omega . T_2 . B_1 . L_a . D.$$

If now we put in the value found above for B_1, and put X for $2\pi f L$ or ωL, we find for the effective or R.M.S. value of the E.M.F.,

$$E_r = \frac{3}{\pi}(1 - 4\delta^2) . c . \nu . X . I_1.$$

Let $$k' = \frac{3}{\pi}(1 - 4\delta^2),$$

then $$E_r = k' . c . \nu . X . I_1.$$

One often puts as an approximation

$$k = \cos \alpha ; \qquad k' = \sin \alpha .$$

The following table shows that the error thereby made is small:

α	$\cos \alpha$	$\sin \alpha$	k	k'
10°	0·985	0·174	0·982	0·2
20°	0·94	0·342	0·933	0·373
30°	0·866	0·5	0·853	0·53

135. The Plain Series Single-phase Motor.

This is essentially the same in principle as the continuous current series motor. If such a motor be supplied with alternating current, the current through armature and field both reverse simultaneously and the torque is consequently always in the same direction. In the actual operation of such motors with alternating currents, the phase displacement due to the large self-inductance of the field and armature windings is a great disadvantage.

The magnetic field produced by the field magnets is essential to the production of torque and must not be neutralised, whereas that due to the armature current is not only non-essential but increases the difficulties of commutation. It can therefore be neutralised by means of a suitable compensating winding. This winding is placed in slots in the pole faces of the stator (Fig. 435) and can be connected in series with the field and armature windings (Fig. 436), or it may be short-circuited on itself, in which case it acts as a short-circuited secondary winding of a transformer, of which the armature winding forms the primary (Fig. 437). The ampere-turns of the two windings are approximately equal and opposite and produce no flux except that due to leakage.

In Fig. 438 the current vector is horizontal and lags behind the applied terminal pressure V by the angle ϕ. If R is the total ohmic resistance of the motor, the ohmic drop is RI; this is represented by the vector AB, in phase with the current. The vector JA represents that part of the applied terminal pressure necessary to counterbalance the E.M.F. induced by the rotation of the armature. This is in phase with the current, since it is a maximum when the flux is a maximum. JB is the total energy component of the terminal pressure. The wattless component BO of the terminal pressure counterbalances the E.M.F. due to the inductance of the field winding and the leakage inductance of the armature. These can be calculated by the methods described in the previous section. Putting $BO = \omega L_r I$, we have $\tan\gamma = \dfrac{\omega L_r}{R}$. Since this is constant, the point A lies on a circle through the points O and J, the centre M of which is on the line through O making an angle $90° - \gamma$ with OJ. Now

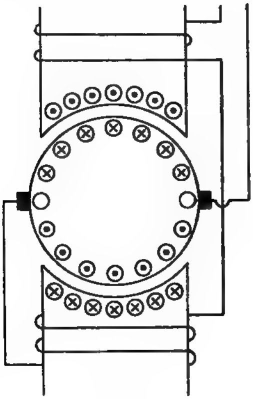

Fig. 435

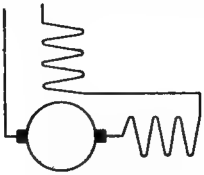

Fig. 436

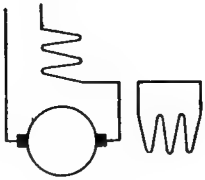

Fig. 437

$$AO = I\sqrt{(R^2 + \omega^2 L_r^2)},$$

and, calling the current which flows when the motor is prevented from rotating I_{sc},

$$I_{sc} = \frac{V}{\sqrt{(R^2 + \omega^2 L_r^2)}}.$$

Hence, if to some suitable scale the line OJ represents the constant current I_{sc} the working current I will to the same scale be represented by the line OA. In Fig. 439 the diagram is re-drawn with the diameter horizontal. Since the angle AOM in Fig. 438 is obviously equal to $90^\circ - \phi$, the vector of the terminal pressure V must be drawn vertically from O. As the load s varied the end A of the current vector moves around the circle.

The torque is proportional to the flux and to the current; for low magnetic densities it is therefore proportional to the square of the current.

The speed is proportional to the quotient of the electromotive force of rotation AJ by the flux; assuming the flux to be proportional to the current, the speed is proportional to AJ/OA, i.e. approximately to cotan ϕ. The power-factor cos ϕ increases therefore with increasing speed. This is obvious from the fact that the energy component AJ becomes increasingly important with increasing speed. The variations of both torque and speed with the load follow curves similar to those obtained for the continuous current series motor.

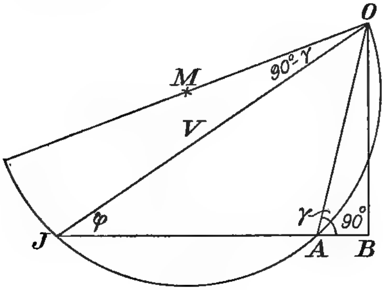

Fig. 438

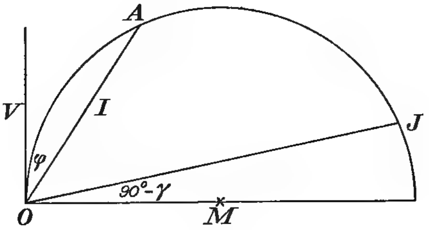

Fig. 439

In motors of this type, commutation is a matter of great importance. As in the continuous current motor an E.M.F. due to self-induction is induced in the short-circuited coil, and is proportional to the current and to the speed. In the present case, however, the current to be commutated is alternating, and the induced E.M.F. will vary in magnitude with the variations of the current. It can be compensated by means of interpoles as in the continuous current motor.

Another E.M.F. is induced in the short-circuited coil due to transformer action between it and the field winding, the latter acting as the primary of a transformer. This induced E.M.F. lags 90° behind the magnetic flux and is proportional to it and to the frequency. It will be greatest at starting since the current and therefore also the flux have then their greatest value.

To minimise the current due to this transformer action, the flux and the frequency should be kept as small as possible. Frequencies of 15 to 25 are more favourable than 50. Further improvement is obtained by the use of narrow hard carbon brushes, and by the insertion of high resistance connections between the armature coils and the commutator segments. In the Siemens-Schuckert motor these resistances are made of copper and are laid along the slots in such a way that the current in them increases the torque.

As compared with the continuous current motor the alternating current motor shows a greater iron loss since both stator and rotor are subjected to reversal of magnetism. On the other hand the latter type can be started and regulated by means of a transformer with tappings without the use of a resistance.

136. The Repulsion Motor.

The distinguishing characteristic of the repulsion motor is the absence of any direct connection between the armature and the source of supply, the current being induced in the short-circuited armature by that in a part of the stator winding, the two being thus mutually compensated. The repulsion motor may be regarded as being obtained from Fig. 437 by making the compensating winding the primary and the rotor the secondary, i.e. by exchanging their functions; in this way we get Fig. 440. The stator winding consists of the exciting or field winding *DF* and the inducing winding *DB*. As a matter of fact the two are combined in a single distributed winding, the axis of which makes an angle with the axis of the rotor winding as shown in Fig. 432. Even in this case, however, the part *DF* and the part *DB* may be regarded as the exciting and the inducing windings respectively. Under the conditions shown in Fig. 432 the torque is clockwise. The speed can be regulated by displacing the brushes. If the angle between the stator and rotor axis is reversed the direction of rotation is reversed.

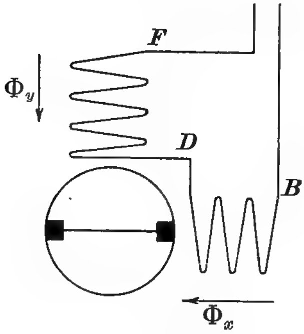

Fig. 440

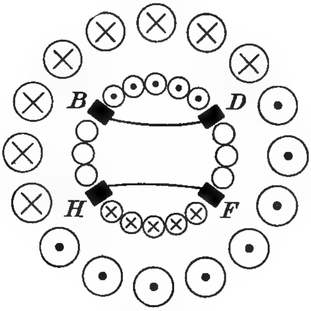

Fig. 441

If the rotor winding be of the short chord type shown in Fig. 111, its effective width is reduced. The effective width can also be reduced by the use of two pairs of brushes per pole pair (Brown-Boveri repulsion motor), as shown in Fig. 441. The portions *DF* and *BH* are not effective. By making the inducing stator winding and the induced rotor winding occupy the same portion of the periphery, a better compensation is obtained. If the brushes *B* and *F* in Fig. 441 are displaced an angle $\Delta\alpha$, the magnetic axis is only displaced $\Delta\alpha/2$; this permits a finer adjustment of the angle α between the stator and rotor axes.

Let Φ_y be the exciting flux produced by the exciting winding *DF* (Fig. 440), and Φ_x the transformer flux produced by the joint action of the in ducing winding *BD* and the rotor. The former induces in the rotor due to its rotation an E.M.F. *OB* (Fig. 442) which is in phase with the stator current since it reaches its maximum when the flux Φ_y is a maximum. As in a continuous current motor this induced E.M.F. is opposed to the armature current and is therefore in the same direction as the inducing current in the stator; for this reason I_1 and *OB* are drawn in the same direction.

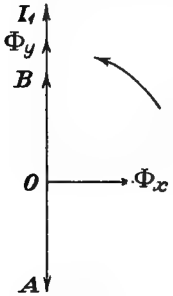

Fig. 442

The transformer E.M.F. induced in the rotor by the flux Φ_x must, if the rotor resistance be neglected, be equal and opposite to the E.M.F. due to rotation. It is represented in Fig. 442 by the line *OA*. Since it must lag 90° behind the flux Φ_x, the vector of the latter is drawn horizontally to the left. The two fluxes which are displaced 90° in space have thus a phase displacement of 90°, giving a resultant rotatory field.

If ν be the ratio of the speed to the synchronous speed, and if it be assumed that, for the same frequency and flux, the rotational E.M.F. and the transformer E.M.F. are equal, it follows from the necessary equality of these two electromotive forces that $\Phi_x = \nu\Phi_y$. It follows from this that at synchronism ($\nu = 1$) the rotating field is constant in magnitude; at rest ($\nu = 0$) the horizontal component vanishes, whereas above synchronous speed it exceeds the vertical component.

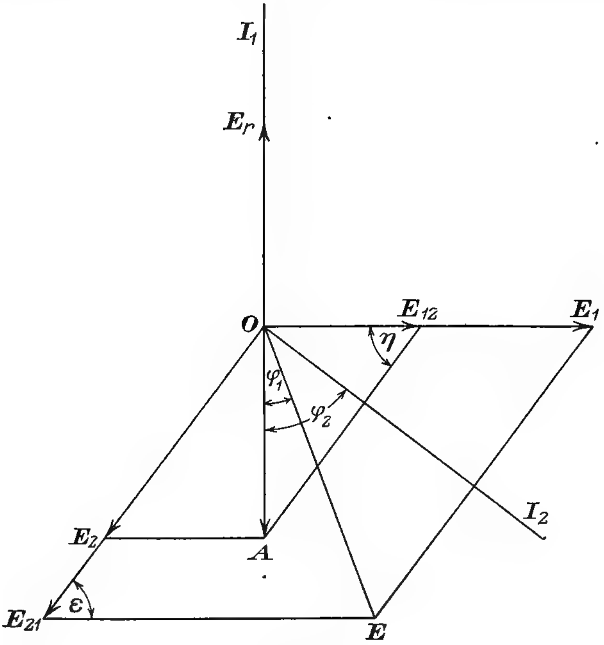

Fig. 443

We will now construct the vector diagram of the repulsion motor (Fig. 443).

The electromotive forces E_1 and E_{12} lag 90° behind the stator current I_1, by the magnetic field of which they are induced. Similarly E_2 and E_{21} lag 90° behind the rotor current I_2. The electromotive force E_r is, however, exactly in phase with I_1, since the E.M.F. due to rotation is evidently a maximum when the stator current is a maximum.

As in any transformer, the secondary current I_2 is more or less in opposition to the primary current I_1 (compare Fig. 432). If the primary current vector be drawn vertically upwards (Fig. 443), the vector representing the rotor current will be drawn in some direction below the horizontal. The vectors E_1 and E_{12} will lag 90° behind the current I_1, while the vectors E_2 and E_{21} will lag 90° behind the current I_2. Since the total E.M.F. in the short-circuited armature must be zero, the resultant of E_2 and E_{12} must be equal and opposite to E_r, that is, it must be vertically downwards in the vector diagram. On the other hand, the resultant of E_1 and E_{21} gives the induced back E.M.F. in the stator, which must be equal and opposite to the terminal pressure. We have then

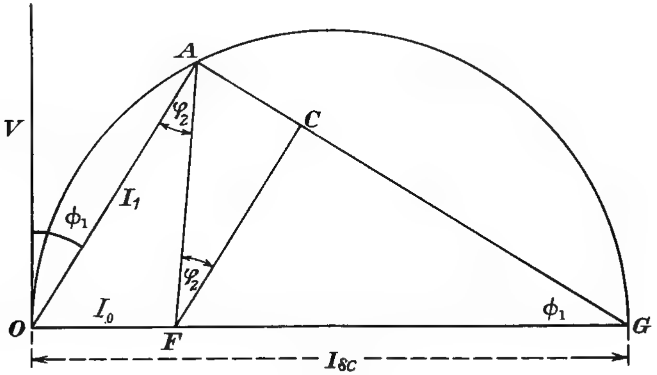

Fig. 444

$$OA = E_r \quad \text{and} \quad OE = V.$$

From the figure and our equations for the electromotive forces, we have

$$V \cos \phi_1 = OA \, . \frac{E_{21}}{E_2} = \frac{kk'\nu}{1 + \tau_2} \, . X \, . I_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Similarly, putting $1 + \tau = (1 + \tau_1)(1 + \tau_2)$, we find that

$$V \sin \phi_1 = E_1 - E_{12} \, . \frac{E_{21}}{E_2} = \frac{1 + \tau - k^2}{1 + \tau_2} \, . X \, . I_1 \quad \ldots\ldots\ldots\ldots (b).$$

We have from these two equations

$$\tan \phi_1 = \frac{1 + \tau - k^2}{kk'\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

When the rotor circuit is open, the stator current I_0 is given by the equation

$$V = I_0 X (1 + \tau_1).$$

Putting this value for V in the above equation for $V \sin \phi_1$, we have

$$\sin \phi_1 = \frac{1 + \tau - k^2}{(1+\tau) \, . \, I_0} \, . \, I_1 .$$

The locus of the end point of the primary current vector is therefore a circle (Fig. 444), the diameter of which is the short-circuit current I_{sc}, where

$$I_{sc} = \frac{(1+\tau) \, . \, I_0}{1 + \tau - k^2} .$$

The turning-moment can be found from the equation

$$M_t = \frac{V \, . \, I_1 \cos \phi_1}{9{\cdot}81 \, . \, 2\pi \, . \, n} = \frac{V \, . \, I_1 \cos \phi_1}{9{\cdot}81 \, . \, 2\pi \, . \, \frac{f}{p} \, . \, \nu} \text{ kg-met.,}$$

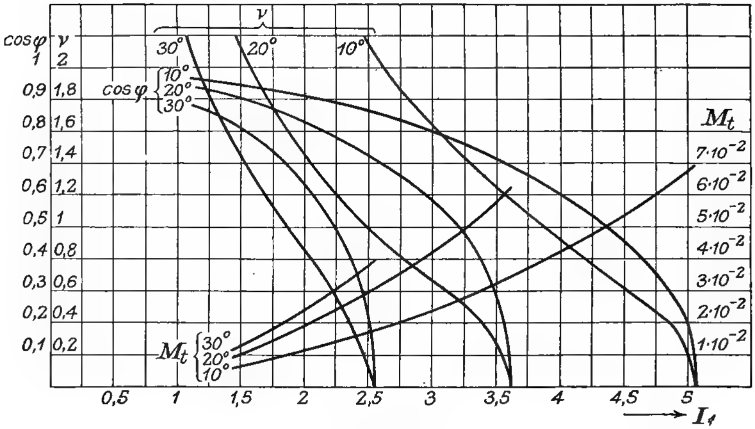

Fig. 445

or, using the above equations for I_0 and $V \cos \phi_1$,

$$M_t = \frac{p \, . \, k \, . \, k'}{9{\cdot}81 \, . \, 2\pi \, . \, f} \cdot \frac{V}{I_0 (1+\tau)} \, . \, I_1^2 \text{ kg-met.}$$

The right-hand side of this equation is constant except for the primary current.

In Fig. 443 the angles ϕ_2 and η are equal, from which it follows that

$$\tan \phi_2 = \frac{OA}{E_{12}} = \frac{E_r}{E_{12}} = \frac{k'}{k} \, . \, \nu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

ϕ_2 is the angle by which I_1 and I_2 depart from exact opposition. If, now, in Fig. 444 AF be drawn making an angle ϕ_2 with AO, then, since the sides of a triangle are proportional to the sines of the opposite angles,

$$\frac{OF}{I_1} = \frac{\sin \phi_2}{\cos (\phi_1 - \phi_2)} .$$

If, now, $\cos (\phi_1 - \phi_2)$ be expanded and divided through by $\sin \phi_2$, we find, with the aid of equations (*a*), (*b*), (*c*) and (*d*), that

$$OF = I_0 .$$

By a further application of the sine law to the triangle OAF, we find that

$$FA = \frac{k \, . \, c}{1 + \tau_1} \, . \, I_2.$$

This is strikingly similar to the corresponding value in the Heyland diagram. In Fig. 445 the power-factor, speed and torque are plotted against the current, for three positions of the brush axis, viz. $\alpha = 10°$, $20°$ and $30°$. To simplify the diagram, I_0 and $p \, . \, V/f$ have been taken as unity whilst τ has been assumed to be 0·2. It will be seen from the curves that the repulsion motor has somewhat similar characteristics to the continuous current series motor.

Turning now to the subject of commutation, it will be seen that, as in the continuous current motor, an E.M.F. is induced in the short-circuited coil due to its self-induction; this E.M.F. is proportional to the speed and to the rotor current. If the motor is working well down on the left-hand side of the semicircle in Fig. 444, the rotor current I_2 is almost constant whilst the speed ν increases continually as the point A approaches O (see equation (c)). The self-induced E.M.F. consequently increases with the speed. To this is added at high speeds a further disadvantage. We have seen that at synchronous speed the field is a rotating one of constant amplitude and therefore without effect on the short circuited coil, but that above synchronism the cross-flux Φ_x is in excess. The rotation at high speed in this flux induces a large E.M.F. in the short-circuited coil, and leads to bad commutation.

Below synchronous speed, on the other hand, the exciting flux Φ_y is the more important. It can be shown that the E.M.F. induced in the short-circuited coils by the variation of this flux neutralises to some extent the E.M.F. due to their self-induction. The commutation of repulsion motors is therefore better at low than at high speeds.

137. The Eichberg-Latour Motor.

The Eichberg-Latour motor, which is shown diagrammatically in Fig. 446, differs from the repulsion motor in that the motor field or excitation is produced by current led into the rotor, whilst the torque producing current is induced in the rotor. In Fig. 446 the vertical flux Φ_y is produced by the rotor current I_y which in this case is the same as the main current I. The stator winding forms the primary of a transformer of which the short-circuited rotor forms the secondary; in this way the current I_x is induced in the

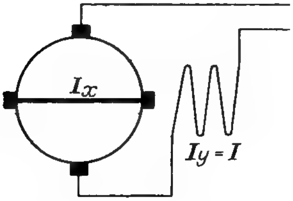

Fig. 446

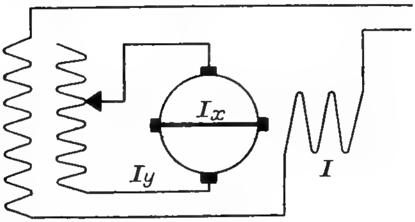

Fig. 447

latter. It is perhaps helpful to picture the rotor as carrying two distinct windings each with its own commutator as shown in Fig. 449.

Although in the Eichberg motor the two currents are superposed in the same winding, Latour has suggested the arrangement shown in Fig. 448 in which the two currents flow in separate parts of the same winding. The portions *AB* and *CD* carry the main current $I = I_y$ and produce the vertical field Φ_y, whilst the current I_x is induced in the short-circuited portions *AC* and *BD*.

In the actual Eichberg motor the rotor is not connected directly in series with the stator winding, but is supplied with the exciting current I_y from one of a number of tappings from the secondary winding of a transformer, the primary of which is connected in series with the stator winding, as shown in Fig. 447. Beyond the possibility of varying the ratio between I and I_y Fig. 447 does not differ fundamentally from Fig. 446.

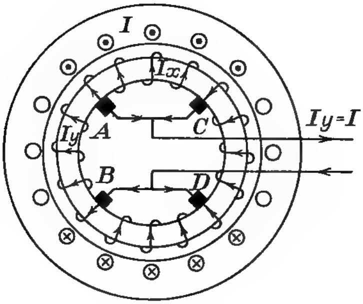

Fig. 448

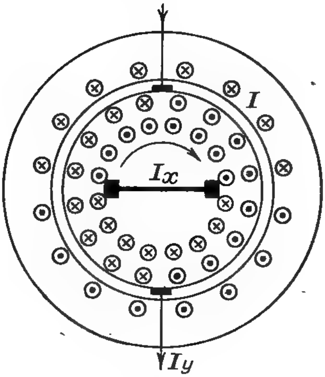

Fig. 449

Let u = the ratio of primary to secondary turns in the transformer,
I = the primary current,
and I_y = the secondary current;
then

$$I_y = u.I.$$

Since the primary and secondary currents are approximately 180° out of phase, the current I_y in the rotor can be assumed in phase with the stator current I (Fig. 450). The electromotive force E_{ry} induced in the short-circuited rotor by its rotation in the flux Φ_y is in phase with Φ_y, and its vector must have the same direction in Fig. 451 as the vectors representing I and Φ_y.

Neglecting the resistance of the rotor, the sum of the electromotive forces induced in the short-circuited winding must be zero, and the E.M.F. E_x due to transformer action must be represented by a vector equal and opposite to the vector E_{ry}. The horizontal flux Φ_x which induces this E.M.F. E_x must be 90° ahead of it and is therefore drawn horizontally to the right. Hence this type of motor possesses a rotary field. As in the repulsion motor so here it can be shown that in the neighbourhood of synchronism the two fields are not only at right angles and 90° out of phase but are equal in amplitude, thus producing a perfect rotating field.

Two electromotive forces are also induced in the armature considered about the vertical axis, one E_y by the vertical flux Φ_y and the other E_{rx} by rotation in the flux Φ_x. The former lags 90° behind the flux Φ_y which induces it and its vector is therefore drawn horizontally to the right; the latter must be a maximum when the flux Φ_x is a maximum. Its direction can be seen from the consideration that at synchronous speed the two fluxes Φ_x and Φ_y produce a perfect rotating field and since the rotor conductors do not cut the field, no E.M.F. is induced. Hence E_y and E_{rx} are in opposition.

It is noteworthy that the difference $E_y - E_{rx}$, which represents the total back E.M.F. in the exciting circuit of the rotor, is wattless. Consequently the primary P.D. of the series transformer which is equal to $u\,(E_y - E_{rx})$ is also wattless.

We now proceed to determine the current, speed, and torque, in relation to the phase displacement.

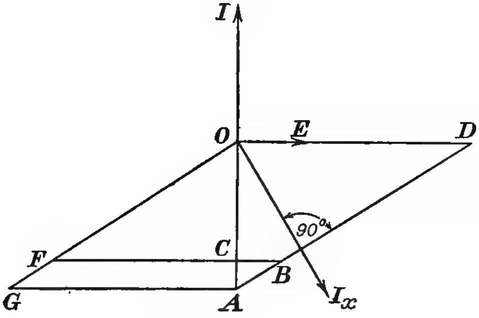

Fig. 450

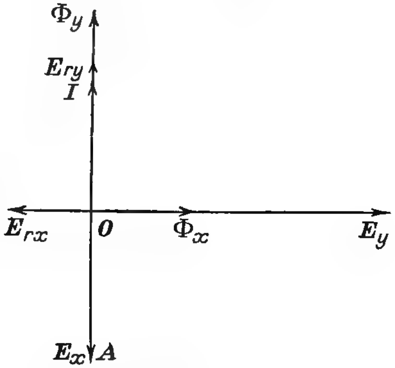

Fig. 451

The stator current I (Fig. 450) induces in the rotor the E.M.F.

$$OD = c.X.I \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

and in the stator itself due to its leakage flux, the E.M.F.

$$OE = \tau_1.X.I \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Both of these lag 90° behind the stator current.

The rotor current I_x induces in the rotor itself an E.M.F. OF due to that part of its flux which links the stator and a further E.M.F. FG due to its leakage flux, where

$$FG = \tau_2 OF.$$

Both of these lag 90° behind I_x.

The resultant OA of OD and OG is the total E.M.F. in the horizontal axis due to self and mutual inductance; it is equal and opposite to the E.M.F. of rotation E_{ry}. Applying the formula established on p. 438 and putting $k' = \frac{3}{\pi}$ and $I_1 = c\,.\,(1 + \tau_2)\,u\,.\,I$, we have

$$OA = E_x = E_{ry} = \frac{3}{\pi}\,.\,c^2\,.\,\nu\,.\,X\,.\,u\,.\,I\,(1 + \tau_2) \quad\ldots\ldots\ldots (c).$$

The resultant OB of the electromotive forces OF and OD is the E.M.F. induced in the rotor by the flux which links both stator and rotor. The E.M.F. induced in the stator by the same flux is OB/c, since c is the ratio of the turns. This E.M.F. has the watt component OC/c and the wattless component CB/c. Since no watt component other than OC/c occurs in the motor, this is the watt component of the supply P.D. and

$$V\cos\phi = \frac{OC}{c} = \frac{OA}{c}\cdot\frac{OF}{OG} = \frac{OA}{c}\cdot\frac{1}{1+\tau_2} \;\ldots\ldots\ldots\ldots\ldots(d).$$

For the wattless component $\frac{CB}{c}$ of the E.M.F. induced in the stator by the rotor flux we have:
$$\frac{CB}{c} = \frac{OD}{c}\cdot\frac{FG}{OG} = \frac{OD}{c}\cdot\frac{\tau_2}{1+\tau_2}.$$

To this must be added the E.M.F. OE induced in the stator by its own leakage flux and the wattless P.D. across the series transformer, viz. $u\,(E_y - E_{rx})$. Hence the total wattless component of the supply P.D. is

$$V\sin\phi = OE + \frac{OD}{c}\cdot\frac{\tau_2}{1+\tau_2} + u\,(E_y - E_{rx}) \;\ldots\ldots\ldots\ldots\ldots(e).$$

From p. 437 putting $k = 1$, we have

$$E_y = c^2.X.u.I.(1+\tau_2) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(f).$$

Since the electromotive forces for the same flux, due respectively to rotation and to inductance, are in the ratio $k'\nu/k$, which in the present case is equal to $3\nu/\pi$, we have

$$E_{rx} = \frac{3}{\pi}.\nu.E_x = \frac{3}{\pi}.\nu.OA.$$

Substituting in equation (*d*) from equations (*a*), (*b*) and (*c*) and putting

$$\tau = \tau_1 + \tau_2 + \tau_1\tau_2,$$
$$k_0 = \frac{3}{\pi}.u.\frac{cX}{V},$$
$$k_1 = \frac{\tau X}{(1+\tau_2)V} + \frac{u^2.c^2.X(1+\tau_2)}{V},$$
$$k_2 = \frac{X}{V(1+\tau_2)},$$

we get
$$\nu = \frac{\cos\phi}{k_0 I},$$

which substituted in equation (*e*) gives

$$\frac{1}{I}.\sin\phi = k_1 - \frac{1}{k_2}.\frac{1}{I^2}.\cos^2\phi.$$

In Fig. 452 the vector of the terminal P.D. is drawn vertically upwards. The current vector lags behind it by the angle ϕ. The reciprocal $1/I$ is drawn in phase with I. Then $x = \frac{1}{I}\sin\phi$ and $y = \frac{1}{I}\cos\phi$; hence from the last equation,

$$x = k_1 - \frac{1}{k_2}y^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(g).$$

The locus of the extremity of $1/I$ is a parabola with its focus at a distance $k_2/4$ to the left of the apex S.

At starting $\phi = 90°$, $y = 0$, and the reciprocal of the starting current I_{sc} is equal to x. Hence

$$1/I_{sc} = OS = k_1.$$

This fixes the distance OS. It changes with any change in the ratio of the series transformer. The parabola itself remains unchanged however.

The ratio ν of the speed to the synchronous speed can be obtained from Fig. 452. We saw above that

$$\nu = \frac{\cos\phi}{k_0} \cdot \frac{1}{I} = \frac{y}{k_0}.$$

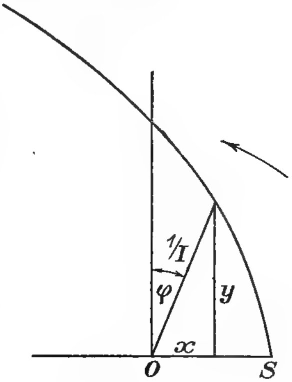

Fig. 452

The torque or turning-moment is given by the equation

$$M_t = \frac{V.I.\cos\phi}{2\pi . 9{\cdot}81 . n} = \frac{V.I.\cos\phi}{2\pi . 9{\cdot}81 . \dfrac{f}{p} . \nu} \text{ kg-met.}$$

Substituting the value just found for ν we obtain

$$M_t = \frac{p . k_0 . V}{2\pi . 9{\cdot}81 . f} . I^2.$$

Thus for each value of ϕ one can determine successively $\cos\phi$, $\frac{1}{I}$, I, ν, and M_t. The results thus obtained are represented in Fig. 453; it has been assumed that

$$\tau_2 = 0{\cdot}1, \quad \tau = 0{\cdot}2, \quad \frac{3}{\pi} . \frac{X}{V} = 1, \quad \text{and} \quad \frac{p . V}{2\pi . 9{\cdot}81 . f} = 1,$$

and uc has been taken as 0·2 for one set of curves and as 0·4 for the other.

The curves for M_t and ν are similar to those obtained for continuous current series motors. They agree in general with the curves obtained from tests on Eichberg motors. Naturally some differences are to be expected since we have neglected the saturation of the iron.

The curves show that with a constant load the speed increases as the transformation ratio u is decreased. This is to be expected since the exciting current I_y is equal to $u.I$, and decreasing u weakens the motor field.

To find the speed ν_0 at which the power-factor is unity, we put $x = 0$ in equation (*g*) (see Fig. 452). This gives $y^2 = k_1 . k_2$.

On the other hand we have already seen that $\nu = y/k_0$. Combining these two equations, we get

$$\nu_0^2 = \frac{k_1 . k_2}{k_0^2} = \frac{\pi^2}{9}\left(\frac{\tau}{(1+\tau_2)^2 . u^2 . c^2} + 1\right).$$

By changing the transformer ratio u the power-factor can be made unity for any speed. If the torque is also given, however, the supply voltage V would have to be varied.

A point of special interest is the distribution of the supply voltage between the stator and rotor (Fig. 446) or between the transformer and the stator (Fig. 447). When the motor is at rest it is equivalent to two choking coils in series. On account of the short-circuited rotor the stator winding has a very small apparent inductance, the flux being confined to the leakage paths, whereas the current I_y passing through the rotor is free to produce a large flux without any such restriction. For this reason the supply P.D. comes almost entirely across the rotor brushes.

When the motor rotates the voltage across the brushes decreases to the value $E_y - E_{rx}$ and, at a certain speed, becomes zero. The whole supply P.D. is then being applied to the stator. At higher speeds $E_y - E_{rx}$ becomes negative and the motor is thereby enabled not only to compensate for the wattless component of the stator voltage ($\phi = 0$), but to take a leading current from the mains (ϕ negative).

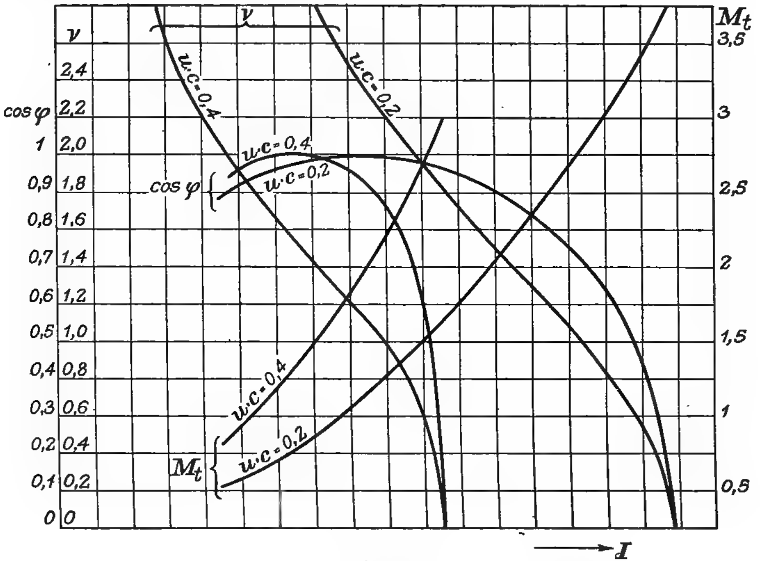

Fig. 453

The commutation at the short-circuiting brushes is similar to that in the repulsion motor. An interesting point arises, however, in connection with the commutation at the exciting brushes. Two electromotive forces will be induced in the coils short-circuited by these brushes, one due to the horizontal field and the other due to rotation in the vertical field (Fig. 446). These electromotive forces are in the ratio $\Phi_x/\nu\Phi_y$. Now the electromotive forces in the horizontal and vertical axes due to flux pulsation are proportional to the respective flux; hence, making use of equations (*c*) and (*f*) we have

$$\frac{\Phi_x}{\Phi_y} = \frac{E_x}{E_y} = \frac{3}{\pi} \cdot \nu,$$

so that, neglecting the factor $3/\pi$, $\Phi_x = \nu\Phi_y$. Hence the two electromotive forces in the short-circuited coils are equal to each other at all speeds. That they are in opposite directions follows from the consideration of the pure rotary field which exists at synchronous speed.

CHAPTER XX

138. Relation between direct and alternating currents in rotary converters.—139. The voltage regulation of rotary converters.—140. Armature copper loss in rotary converters.—141. Comparison between rotary converters and direct-current generators with regard to armature copper loss.—142. Six-phase rotary converters. 143. La Cour's motor converter.

138. Relation between Direct and Alternating Currents in Rotary Converters.

The principle of the rotary converter has been already considered in Section 95. If suitable points on the armature-winding of a direct-current motor are connected to slip-rings, an alternating current may be taken from these rings when the motor is running in the usual manner. If this arrangement is inverted, that is, if alternating current is supplied to the motor by means of the slip-rings, it runs as a synchronous motor, and direct current can be collected from the commutator. The peculiarity of this type of machine is the simultaneous existence of both direct and alternating currents in the same armature conductors, so that the alternating current may sometimes be added to the direct, while at other times it is flowing in the opposite direction and must therefore be subtracted. The heating of the armature-winding of a rotary converter is consequently entirely different from that of the same machine when mechanically driven as a direct-current dynamo on the same load.

In order to compare the heat developed in both cases, we must know the relation between the maximum value $\hat{i}$ of the alternating current I_a and the value I_c of the direct current. These are the currents in the armature conductors, and not those in the external circuit. The relation between them may be expressed by the equation

$$\eta = \frac{\hat{i}}{I_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (198)$$

Now, neglecting losses, the alternating current power supplied must be equal to the direct-current output. Let

ν = the number of armature sections per pair of poles; thus, $\nu = 2$ for single-phase, 3 for three-phase, and so on,

z' = the number of wires per armature section,

$I_a = \frac{\hat{i}}{\sqrt{2}}$ = the alternating current in the armature conductors,

E = the R.M.S. value of the alternating E.M.F. in each armature section,

ϕ = the phase difference between E and I_a.

If P be the total alternating current power, we have

$$P = p.v.E.I_a.\cos\phi.$$

By putting $E = k.\Phi f z' 10^{-8}$ and $I_a = \frac{\hat{i}}{\sqrt{2}} = \frac{\eta I_c}{\sqrt{2}}$, and substituting Z, the total number of wires on the armature, for pvz', we get

$$P = k.\Phi f Z \frac{\eta I_c}{\sqrt{2}}.\cos\phi.10^{-8}.$$

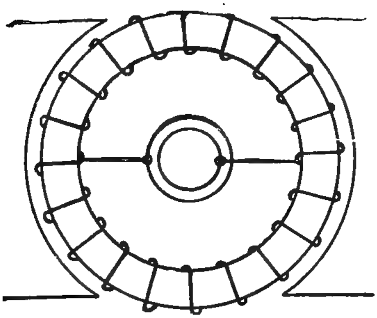

Fig. 454

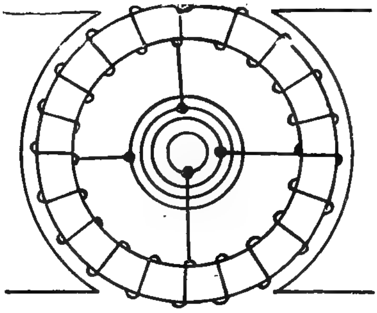

Fig. 455

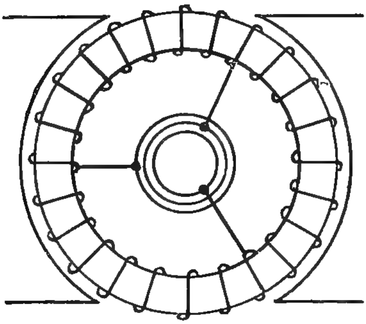

Fig. 456

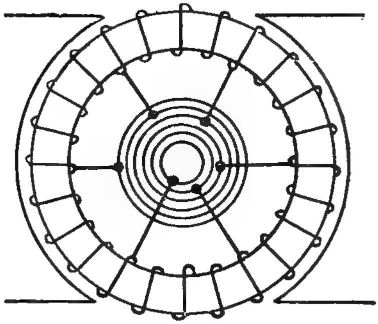

Fig. 457

On the other hand, the direct-current power is given by the equation

$$P = \frac{p}{a}.\Phi.n.Z.10^{-8}.I,$$

where I is the external direct current. If there are $2a$ paths through the armature between the negative and positive terminals on the direct current side, then $I = 2aI_c$. By putting $p.n = f$ and equating the two values found above for P, we obtain the relation

$$\eta = \frac{2\sqrt{2}}{k\cos\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots(199).$$

In this equation the value of k is largely dependent upon the breadth of the coil-sides or armature sections. In the single-phase converter (Fig. 454) the ratio of the breadth 2γ of the section to the pole-pitch is unity. In the two- or four-phase converter (Fig. 455) the breadth ratio 2γ is equal to $\frac{1}{2}$. In three-phase converters (Fig. 456) $2\gamma = \frac{2}{3}$, while in the six-phase converter it is only $\frac{1}{3}$ (Fig. 457). The connections of a six-phase converter are dealt with in Section 142.

The coefficient k depends also upon the distribution of the magnetic field. As before, we shall consider three cases, firstly, a sinusoidal field, secondly, a pole breadth equal to two-thirds of the pitch, and lastly, a pole breadth of half the pitch. For a sinusoidal field we have equation (144) on page 297

$$k = 2{\cdot}22 \,.\, \frac{\sin \gamma\pi}{\gamma\pi} = \frac{\pi}{\sqrt{2}} \,.\, \frac{\sin \gamma\pi}{\gamma\pi} \,.$$

For pole breadths of $\frac{2}{3}$ or $\frac{1}{2}$ the pitch, equations (152) and (153) on page 302 can be employed for single- or three-phase converters, since the breadth of the coil 2γ is greater than the polar arc β. In four- and six-phase converters the breadth of the coil is less than the polar arc, and equations (147) and (154) on pages 298 and 303 must be employed. The values thus obtained are given in the following table:

	2γ	Values of the coefficient k*		
		Sinusoidal field	$\beta = \frac{2}{3}$	$\beta = \frac{1}{2}$
single-phase	1	$\sqrt{2} = 1{\cdot}414$	$\frac{2}{3}\sqrt{5} = 1{\cdot}49$	$\frac{2}{3}\sqrt{6} = 1{\cdot}63$
three-phase	$\frac{2}{3}$	$\frac{3}{4}\sqrt{6} = 1{\cdot}84$	$\frac{1}{2}\sqrt{15} = 1{\cdot}94$	$\frac{2}{3}\sqrt{10} = 2{\cdot}11$
four-phase	$\frac{1}{2}$	2	$\frac{2}{3}\sqrt{10} = 2{\cdot}11$	$\frac{4}{3}\sqrt{3} = 2{\cdot}31$
six-phase	$\frac{1}{3}$	$\frac{3}{2}\sqrt{2} = 2{\cdot}121$	$\sqrt{5} = 2{\cdot}24$	$\frac{2}{3}\sqrt{14} = 2{\cdot}49$

From these values of k, the value of $\eta = \frac{2\sqrt{2}}{k \cos \phi}$ can easily be found. We shall make use of these values in Section 140 in order to determine the copper loss in the armatures of rotary converters.

This coefficient k enables us, moreover, to determine the relation between the P.D. between the brushes on the D.C. side and that between two adjacent slip-rings on the A.C. side. If we neglect losses in the windings, we have

$$E_{\text{D.C.}} = \frac{p}{a} \,.\, \Phi \,.\, n \,.\, Z \,.\, 10^{-8},$$

and

$$E_{\text{A.C.}} = k \,.\, \Phi f Z' \,.\, 10^{-8}.$$

If we put

$$\frac{p}{a} = 1, \quad pn = f, \quad \text{and} \quad Z = z'vp,$$

we get

$$\frac{E_{\text{A.C.}}}{E_{\text{D.C.}}} = \frac{k}{v} \,.$$

---

* The last two columns have been calculated on the assumption that the flux-wave is rectangular. Fringing and rounded pole-tips will considerably reduce the value of k.

and by putting $\nu = 2, 3, 4$ or 6 as the case may be, and substituting the values of k from the table above, we obtain the following values for this ratio:

	Values of $\dfrac{\text{voltage between adjacent slip-rings}}{\text{voltage between D.C. brushes}}$		
	Sinusoidal field	$\beta=\frac{2}{3}$	$\beta=\frac{1}{2}$
single-phase	0·707	0·745	0·815
three-phase	0·61	0·65	0·70
four-phase	0·5	0·53	0·58
six-phase	0·353	0·37	0·415

139. The Voltage Regulation of Rotary Converters.

It would appear from the foregoing that for a given alternating supply voltage, the continuous current P.D. was incapable of regulation or adjustment. The voltage on the continuous current side of a rotary converter can, however, be regulated if a choking coil be inserted in series with the converter on the A.C. side. The alternating voltage applied to the converter is no longer identical with the supply voltage. If the converter is supplied through a transformer, as is usually the case on account of the high voltages used for A.C. transmission, a special choking coil may not be necessary if the transformer has a large magnetic leakage.

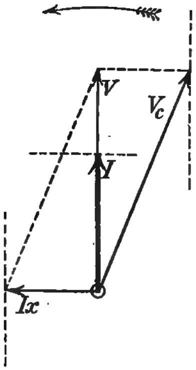

Fig. 458

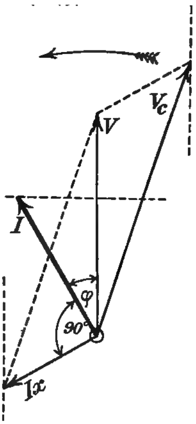

Fig. 459

Let V be the alternating supply voltage and x the reactance of the choking coil. If the converter be over-excited—by cutting resistance out of the field rheostat—we have seen that it will take a leading current (see p. 355). If, on the other hand, it be under-excited, the current taken will lag behind the terminal P.D. Assuming that the power remains unaltered $V.I.\cos\phi$ will be constant, so that the locus of the end of the current vector in Fig. 458 will be a horizontal line. The current I will be a minimum when it is in phase with V (Fig. 458). The vector V may be split up into two components, $I.x$, 90° ahead of I, representing the P.D. across the choking coil, and V_c, the P.D. between the slip-rings of the rotary converter. Now Ix being always proportional to, and 90° ahead of, the current, its locus must be a vertical line as shown in Fig. 459 from which it is obvious that V_c increases rapidly as the current is caused to lead and decreases as the current is caused to lag. Now V_c is the P.D. between the slip-rings and every change in V_c causes a proportional change in the P.D. on the continuous current side. Hence to increase the latter it is only necessary to cut resistance out of the field rheostat and thus cause the converter to take a heavier leading current. By compounding the machine, that is, by passing the main continuous current through a few series turns, wound so as to strengthen the field due to the shunt turns, its voltage regulation can be made similar to that of a level or over-compounded dynamo, the reasons for the regulation being, however, very different in the two cases. In the compound-wound rotary converter, an increase in the load, that is, in the continuous current taken from the machine, causes the alternating current taken by the converter to lead ahead of the alternating supply P.D. The E.M.F. induced in the choking coil by this leading current is of such a phase that the P.D. between the slip-rings of the rotary converter is increased. Since the P.D. on the continuous current side bears a fixed relation to that on the A.C. side, the former is also increased, or, if not increased, maintained constant in spite of the drop in the armature.

The ratio of the continuous to the alternating P.D. can be adjusted in a type of machine known as the split-pole converter. This is little used in practice but is interesting enough to merit a brief description. The poles are split into two or three parts and each part carries a winding in addition to the main winding embracing the whole pole, although the latter winding may also be subdivided between the various parts of the pole. The principle will best be understood from Fig. 460 where, for the sake of clearness, there are two distinct sets of poles, whereas, in practice, a set of three auxiliary poles would constitute the main pole. The diagram represents a two-pole 3-phase converter. The auxiliary poles are alternately N and S and being six in number they have no effect on the 3-phase winding, the three tappings being symmetrical at every moment with regard to the six poles. The alternating E.M.F. induced in the 3-phase winding is therefore independent of the excitation of the auxiliary poles and depends only on the main poles. With regard to the brushes rubbing on the commutator, each half of the armature has two auxiliary poles assisting the main pole and only one opposing it, so that the P.D. between the brushes will depend on the degree of excitation of the auxiliary poles.

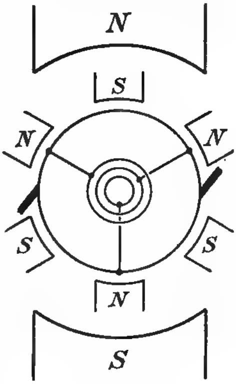

Fig. 460

Hence by varying this excitation, the ratio of the continuous to the alternating voltages can be adjusted within certain limits. It is easily seen that the main poles are unnecessary, for if the three upper auxiliary poles are equally excited in the same direction, viz. as N poles, and the lower three as equal S poles, they will replace the main poles. Any difference now in the excitation of the poles, such as weakening those marked S in Fig. 460, will have the same effect as the assumed superposition of a six pole field on the bipolar field.

In the ordinary form of rotary converter, the P.D. between the brushes on the commutator can be varied by shifting the brushes. If the brushes are shifted 90° (assuming a bipolar machine) the P.D. between them will be zero, but owing to the brushes short-circuiting coils moving in a strong field, the sparking will be excessive. This can be obviated by making a gap in the middle of each pole, that is, by splitting the poles and placing the brushes so that, when short-circuited, the coil is moving under the gap between the two halves of the pole. The P.D. between the brushes is now zero, *if the two halves of each split pole are equally excited*, but by exciting the two halves to a different degree, the P.D. can be made to differ from zero over a considerable range in either direction. This is also applicable to a continuous current converter, that is, a machine which is supplied with D.C. at one voltage and converts it into a D.C. supply at another voltage. In the above example, the original brushes would be left in their normal position and supplied with continuous current; the machine would run as a continuous current motor, unaffected by the distribution of the total flux between the two halves of the split poles. The P.D. of the supply which could now be taken from the new brushes, half way between the main brushes, would depend entirely on the difference in excitation between the two halves of the split poles. There is another way of using such a converter, for, by suitably exciting the parts of the split poles, the auxiliary brushes can be made to have the same potential, and may therefore be connected together, while the P.D. between the auxiliary brushes and either main brush may be varied considerably by exciting the two halves of the split poles to different degrees*.

140. Armature Copper Loss in Rotary Converters.

As a general rule alternating current is supplied to the rotary converter, while direct current is taken from its commutator. It runs, therefore, as a combination of A.C. synchronous motor and D.C. generator. The E.M.F. induced in the armature opposes the alternating current, whereas the direct current, being produced by the E.M.F., is in the same direction. It follows from this that the direct and alternating currents within the armature must neutralise each other to a large degree, thus modifying the amount of heat developed in the armature-winding.

It does not follow that the heat developed will be necessarily less than in the ordinary dynamo. This question will depend very largely on the phase difference between the E.M.F. and the alternating current. Moreover, each coil-side or section is at times under two unlike poles, and it is evident

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* For details of such split-pole continuous current converters see *The Electrician*, vol. LXIII. 1909, pp. 498, 506.

that, at such moments, the alternating and direct currents in some wires must be added together, while in other wires they must be subtracted. The ideal arrangement would have an infinite number of commutator segments and of phases, so that each coil-side or section would be infinitely narrow.

To make the problem as clear and as simple as possible, we shall assume that the number of commutator segments is very large, and we shall trace the variations of current in a single armature conductor during a complete period. We shall choose, in the first place, a conductor in the middle of a section, i.e. midway between two adjacent slip-ring connections, and we shall assume that the phase difference between E.M.F. and current is zero, which is a condition easily obtained by suitable excitation.

At the moment when the conductor under consideration passes the neutral zone (Fig. 461) both the direct and alternating currents in it are reversed. If we assume that the direct current taken from the machine is constant, the curve I_c in Fig. 462 will represent the direct current in the conductor, plotted to a time base. Similarly, curve i will represent the alternating current. The resultant current curve is given in Fig. 463. To obtain the power lost in the given conductor at any moment, the corresponding ordinate in Fig. 463 must be squared and multiplied by the resistance R of the conductor.

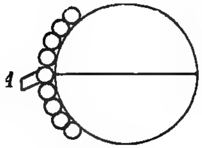

Fig. 461

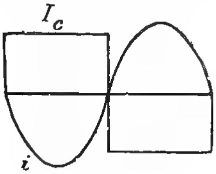

Fig. 462

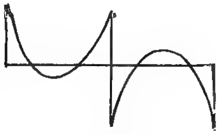

Fig. 463

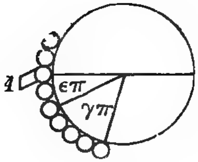

Fig. 464

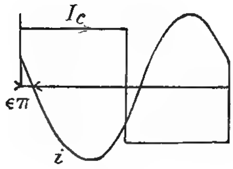

Fig. 465

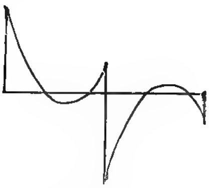

Fig. 466

The conditions are not so favourable for a conductor separated by an angle $\epsilon\pi$ from the centre of the armature section. As this conductor passes the neutral zone at A (Fig. 464) the direct current in it is reversed. The alternating current, however, being in phase with the E.M.F., does not reverse until the middle point of the section passes through the neutral zone, i.e. an angle $\epsilon\pi$ later. The alternating current curve in Fig. 465 is displaced, therefore, to the right, and by combining the ordinates, Fig. 466 is obtained as the curve of resultant current in the given conductor.

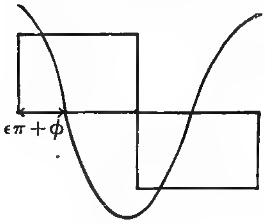

Fig. 467

If, now, the current lags behind the E.M.F. by the angle ϕ, the current curve will be displaced still further to the right (Fig. 467). For the sake of simplicity the angle α has been plotted as abscissae instead of the time t. The instantaneous value of the current in the given conductor is

$$I_c - \hat{i}\sin(\alpha-\epsilon-\phi) = I_c\{1-\eta\sin(\alpha-\epsilon-\phi)\}.$$

We square the instantaneous value of the current and take the mean of all such values between 0 and π. If R be the resistance of the given conductor, the mean power lost in it will be

$$\frac{I_c^2R}{\pi}\int_0^\pi\{1-\eta\,.\sin(\alpha-\epsilon-\phi)\}^2.\,d\alpha = I_c^2R\left\{1+\frac{\eta^2}{2}-\frac{4\eta}{\pi}.\cos(\epsilon+\phi)\right\}.$$

In carrying out the above integration we must square out the bracket, and integrate separately the three terms thus obtained, remembering that $(\epsilon+\phi)$ is constant. On examining the right-hand side of this equation, we see that the mean loss of power in any conductor contains two constant terms, independent of the position of the conductor, and a third term which varies with the position of the conductor in the coil-side. The average value of this third term, taken over the whole coil-side, is

$$\frac{1}{2\gamma}\int_{-\gamma\pi}^{+\gamma\pi}\frac{4\eta}{\pi}.\cos(\epsilon+\phi).\,d\epsilon = \frac{4\eta}{\pi}.\frac{\sin\gamma\pi}{\gamma\pi}.\cos\phi.$$

The mean loss of power per conductor, taken over the whole armature, is therefore

$$P_l = I_c^2R\left(1+\frac{\eta^2}{2}-\frac{4\eta}{\pi}.\frac{\sin\gamma\pi}{\gamma\pi}.\cos\phi\right).$$

If we substitute for η its value $\frac{2\sqrt{2}}{k\cos\phi}$, we have

$$P_l = I_c^2R\left(1+\frac{4}{k^2\cos^2\phi}-\frac{3{\cdot}6}{k}.\frac{\sin\gamma\pi}{\gamma\pi}\right) \quad \ldots\ldots\ldots\ldots\ldots(200).$$

We shall use this equation in the next section, in order to compare the converter with the direct-current generator, in respect of the copper loss in the armature.

141. Comparison between Rotary Converters and Direct-current Generators with regard to Armature Copper Loss.

The product I_c^2R in equation (200) is the loss of power per armature conductor when the machine is working as a direct-current generator with a current I_c in the armature-winding. The expression in the brackets,

$$\Gamma = 1+\frac{4}{k^2\cos^2\phi}-\frac{3{\cdot}6}{k}.\frac{\sin\gamma\pi}{\gamma\pi},$$

gives us the relation between the armature copper loss in the converter, and that in the same machine when driven as a D.C. dynamo with the same output. The greater the phase difference ϕ, that is, the smaller the value of $\cos\phi$, the larger will be the second term in the above expression for Γ, and the greater the loss. The influence of the width of coil-side or section is also plainly seen in the above equation. If, for example, the field be sinusoidal, the coefficient k is proportional to $\frac{\sin\gamma\pi}{\gamma\pi}$, and the last term in the equation becomes a constant. The width of coil-side can then affect the second term only. The coefficient k increases as the width of coil-side is decreased. With narrow sections the second term is therefore small, and consequently the loss is small. The superiority of the six-phase converter, with its narrow sections, is plainly evident. Any practical deviation of the field from the sine form will not materially alter these relations.

The following table contains the values of Γ, i.e. the ratio of the losses in the converter to those in the equally loaded dynamo, under various conditions.

The table shows how considerably the losses can be reduced by so adjusting the field current that the power-factor is unity. It shows, further, the enormous superiority of the polyphase converter, as compared with the single-phase type. The influence of pole-form is seen to be very small, and is, in reality, still smaller, owing to the rounding off of the pole-shoes and the fringing of the magnetic field.

	$\cos\phi$	$\Gamma = 1 + \frac{4}{k^2 . \cos^2\phi} - \frac{3{\cdot}6}{k} . \frac{\sin\gamma\pi}{\gamma\pi}$		
		Sinusoidal field	$\beta = \frac{2}{3}$	$\beta = \frac{1}{2}$
single-phase	1	1·38	1·26	1·10
	0·8	2·50	2·28	1·94
three-phase	1	0·56	0·53	0·49
	0·8	1·23	1·13	0·99
four-phase	1	0·38	0·36	0·35
	0·8	0·94	0·87	0·77
six-phase	1	0·27	0·26	0·26
	0·8	0·77	0·71	0·63

A more important question from the practical point of view is: How great a load can be put on the converter without overheating its armature?

Assuming equal losses in both cases, let

I_c = the direct current in each conductor of the converter armature, and

I_d = the current in each armature conductor when driven as a dynamo.

The loss per armature conductor in the dynamo is I_d^2R, whereas in the converter it is ΓI_c^2R (equation 200). The loss being the same in each case, we have

$$I_d^2R = \Gamma I_c^2R,$$

$$\frac{I_c}{I_d} = \frac{1}{\sqrt{\Gamma}}.$$

On the assumption of equal losses in the armature, the relation between the D.C. load on the converter and that on the dynamo is therefore equal to $1/\sqrt{\Gamma}$. The following table gives the value of $1/\sqrt{\Gamma}$ for the various cases considered:

	$\cos\phi$	$\frac{1}{\sqrt{\Gamma}}$		
		Sinusoidal field	$\beta=\frac{2}{3}$	$\beta=\frac{1}{2}$
single-phase	1	0·85	0·89	0·95
	0·8	0·63	0·66	0·72
three-phase	1	1·33	1·37	1·43
	0·8	0·90	0·94	1·00
four-phase	1	1·62	1·66	1·70
	0·8	1·03	1·07	1·14
six-phase	1	1·93	1·95	1·94
	0·8	1·14	1·18	1·26

Hence, with unity power-factor the output of a single-phase converter is 85 per cent. of the output of the same machine, used as a dynamo, and allowing the same armature heating. Under the same condition, however, the six-phase converter has almost double the dynamo output.

142. Six-phase Rotary Converters.

In the preceding sections we have dealt with rotary converters, not only for single-phase, two-phase and three-phase, but also for six-phase working. This is the first time we have mentioned more than three phases, and it will be advisable to consider exactly what we understand by a six-phase converter.

Every phase winding has of course two ends, or, if we prefer it, a beginning and an end. Hence, a single-phase machine has two terminals or slip-rings, and a two-phase machine four. In the same way a three-phase machine should have six terminals, and would have, were it not that the windings are usually interconnected in either the star or mesh arrangement, thus reducing the number of terminals to three. In a two-phase converter each phase of the supply is connected across diametrically opposite points of the armature, the axes of the two phases being at right angles as in Fig. 455. The two phases are thus superposed on the same armature winding. If each of the three phases in a three-phase system be kept quite distinct and connected across diametrically opposite points on the armature winding, the three axes being at 120°, we obtain Fig. 457, which, after all, is only six-phase in the same sense that Fig. 455 is four-phase.

At first it would appear that a so-called six-phase converter would require the three-phases of the supply mains to be kept quite distinct, thus necessitating the use of six conductors, but this is not so, for by means of transformers we can split up the ordinary three-phase supply with three conductors into three separate phases with six conductors. Such an arrangement is shown in Figs. 468 and 469. In the former the primaries are star connected, while in the latter they are mesh connected. In Fig. 468 three separate single-phase transformers are shown, whereas in Fig. 469 a three-phase transformer is employed, in which each iron limb acts, in turn, as the magnetic return path for the flux in the other two limbs.

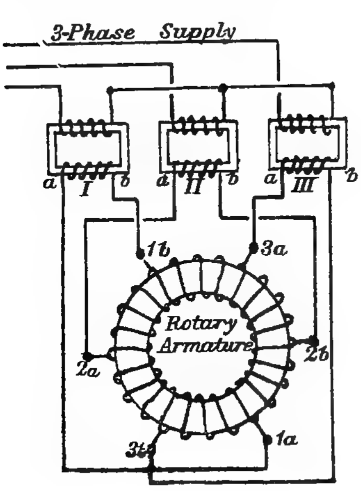

Fig. 468

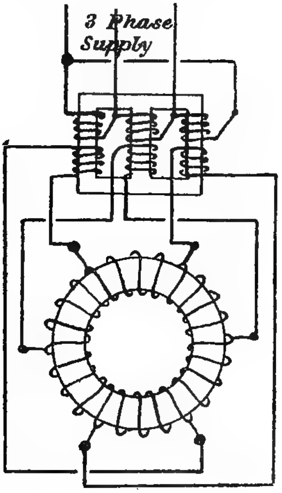

Fig. 469

There is another way, however, of looking at these six-phase converters. In the method just considered we had three separate single-phase supply circuits, 120° out of phase with each other. Now, Fig. 456 represents the ordinary three-phase converter with three supply conductors; if we can obtain another similar three-phase supply 180° out of phase with the one shown in Fig. 456 then by superposing the two supply systems we obtain Fig. 457. Fig. 470 shows how this is accomplished in practice. Each of the three single-phase transformers, or each limb of the three-phase transformer, has two distinct but similar secondary windings. The primary windings may be mesh or star connected, and the same is true of the secondary windings. In the figure we have shown mesh connection on the primary and star connection on the secondary side. Fig. 471 shows the same arrangement as Fig. 470 but is more diagrammatic, and by separating the two sets of secondary windings the principle of the arrangement is more easily understood.

The figures have been similarly numbered and lettered to assist the student in tracing out the connections.

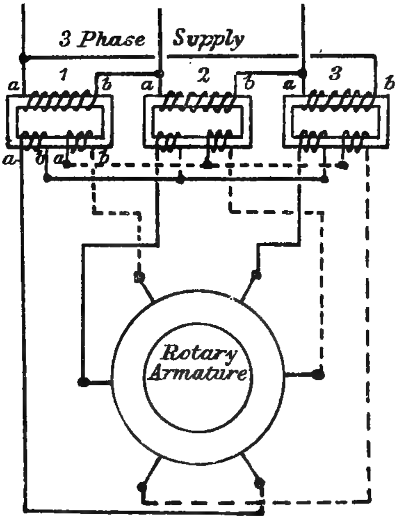

Fig. 470

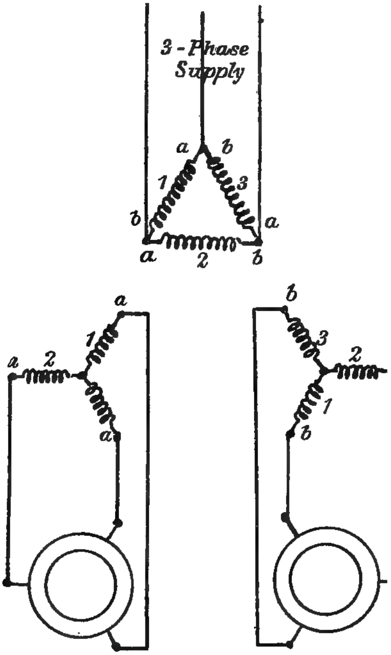

Fig. 471

143. La Cour's Motor Converter.

The disadvantages of the rotary converter with regard both to starting and regulation of the terminal pressure on the D.C. side are obvious. Further disadvantages are the necessity of static transformers for reducing the alternating supply pressure, and the difficulty of commutation for any but low frequencies. These difficulties are overcome to a certain extent by using motor-generators, in which the two machines are quite distinct but coupled mechanically together. We then have a free hand in the design of each machine, and the number of poles in the dynamo is not affected by the frequency of the A.C. supply. The motor can be designed to take the full supply pressure, thus dispensing with transformers, and the D.C. terminal pressure can be regulated in the usual way. On the other hand, we have two machines each of the same output, which is also the full output of the set. This increases the capital cost, requires more floor space, and decreases the efficiency. The motor can be synchronous or asynchronous. With the induction motor the question of starting is quite simple, but we then have the disadvantage of a low and non-adjustable power-factor, whereas the synchronous motor has all the advantages and disadvantages of the converter with regard to starting and power-factor.

A type of converter which has been used to some extent during the last few years is the so-called motor converter of La Cour. It combines more or less all the advantages of the rotary converter, the synchronous motor-generator, and the induction motor-generator. It consists essentially of an induction motor and D.C. dynamo side by side on the same shaft. In addition to being keyed on the same shaft, the rotor and armature are connected electrically, and the power is transmitted partly mechanically along the shaft, as in a motor-generator, and partly electrically from rotor to armature, so that they are together equivalent to the armature of a rotary converter. The three-phase current is supplied to the stator at any pressure up to about 10,000 volts, and the rotor revolves at a speed much below synchronism. Currents are induced in the rotor with a frequency equal to the slip, which, if both machines have the same number of poles, will be 50 per cent. These rotor currents pass across into the armature of the dynamo, and are of just the correct frequency to drive it as a synchronous motor at the speed corresponding to 50 per cent. slip in the induction motor. Hence, half the total power passes mechanically along the shaft, while the other half passes electrically across from the rotor to the armature. The power-factor can be adjusted by varying the field current of the dynamo.

The machine is started from the A.C. side as an induction motor. To enable this to be done the rotor phases are disconnected at the neutral point, and a three-phase starting resistance introduced into three of the phases. When the correct speed is attained the rotor phases are all joined at the neutral point. The rotor and armature can be nine or twelve-phase, since the connections are short and direct without the intervention of slip-rings. This large number of phases improves the synchronising power and the efficiency. This latter is slightly lower than that of the ordinary converter, but much better than that of a motor-generator. Since only a part of the power is converted into mechanical energy each machine is smaller than the corresponding machine of a motor-generator of the same output. A motor-converter with nine-phase connection between rotor and armature is represented diagrammatically in Fig. 472.

If the motor has p_m pairs of poles and the dynamo p_d, the field of the motor will make $\frac{f_1}{p_m}$ revs. per second. The rotor will, in this case, run at $\frac{f_1}{p_m + p_d}$ revs. per second. The currents induced in the rotor, and supplied by it to the armature of the dynamo, will have a frequency $f_1 \frac{p_d}{p_m + p_d}$. Since the dynamo has p_d pairs of poles, and rotates at $\frac{f_1}{p_m + p_d}$ revs. per second, its frequency is also $f_1 \frac{p_d}{p_m + p_d}$, and the current supplied to it from the rotor has the correct frequency.

Equation (179) on p. 390, which is the equation for the slip of an induction motor, may be put into the following form:

$$\frac{f_1 - \text{slip}}{f_1} = \frac{\text{energy converted into mechanical form}}{\text{total energy supplied to rotor}}.$$

It is immaterial whether the difference between the total supply of energy and that part of it which is converted into mechanical energy is dissipated as heat, as in the induction motor, or used to drive a synchronous motor or rotary converter, as in the present case. If we substitute for the frequency of slip its value $f_1 \frac{p_d}{p_m + p_d}$, we get

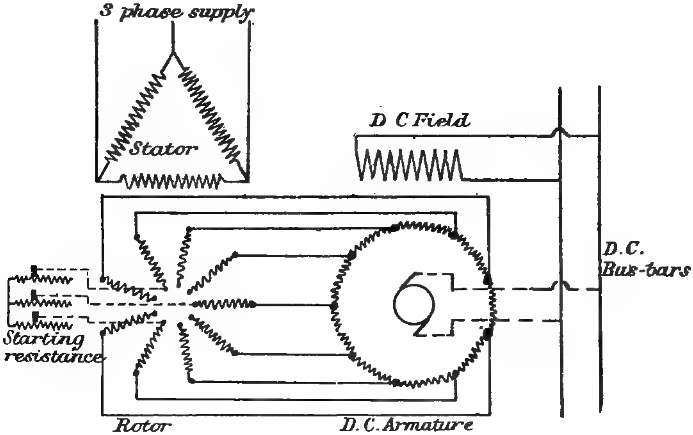

Fig. 472

$$\frac{f_1 - f_1 \cdot \frac{p_d}{p_m + p_d}}{f_1} = \frac{p_m}{p_m + p_d} = \frac{\text{energy supplied mechanically}}{\text{total energy}}.$$

Hence, of the total output, a fraction $\frac{p_m}{p_m + p_d}$ is transmitted mechanically along the shaft, while the remainder $\frac{p_d}{p_m + p_d}$ is transmitted electrically*.

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* For further information on motor-converters, see H. S. Hallo, *Trans. of the Inst. of Elec. Eng.*, 1909.

APPENDIX

THE SYMBOLIC METHOD
OF SOLVING ALTERNATING-CURRENT PROBLEMS

1. The Polar Diagram.

In considering alternating currents we have used rotating vectors, the lengths of which were equal to the maximum values of the current or potential difference. The direction of the vector corresponded to the plane of an imaginary coil which rotated in a uniform vertical field, and the projection of the vector on a vertical line represented the momentary value of the E.M.F. induced in the rotating coil.

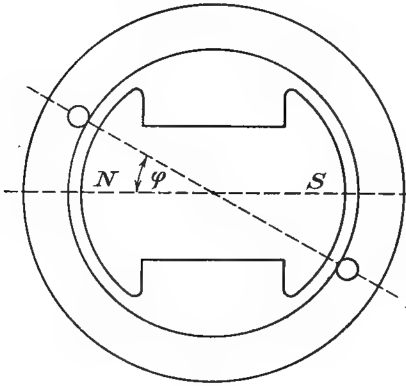

Fig. 473

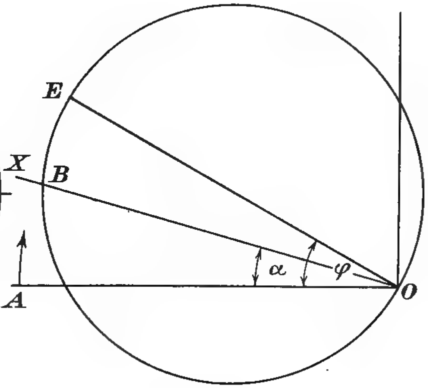

Fig. 474

We shall now look at the matter from another point of view. We shall imagine the armature coil to be at rest and the poles to rotate, as shown in Fig. 473. As the initial position of the polar axis we arbitrarily choose the horizontal. In the polar diagram (Fig. 474) we shall also take the horizontal OA as the initial line or line of reference. The vector $OE = \hat{e}$ is set out at the angle ϕ which the fixed armature coil makes with the horizontal. A circle is drawn on OE as diameter. Contrary to our previous method, we keep everything stationary except the line OX which rotates in the clockwise direction. At the moment when this rotating line makes an angle α

with the initial line OA, it will cut the circle at the point B. The value of the E.M.F. at this moment will be

$$\hat{e} \,.\, \cos(\phi - \alpha) = OB.$$

OB is the projection of $\hat{e}$ on the rotating line OX. $\hat{e}$ is called the amplitude of the sine wave, and ϕ is called its phase. In polar co-ordinates the instantaneous value is expressed as a function of the variable angle α. It reaches its maximum value at the moment when OX coincides with OE.

In Fig. 475, for example, the electromotive forces E_1 and E_2 are of different phase, because their maximum values OE_1 and OE_2 make different angles ϕ_1 and ϕ_2 with the horizontal. The electromotive force E_2 is behind E_1 because the rotating line OX passes through OE_1 before passing through OE_2. It is evident that rotating OX in a clockwise direction round the stationary vector diagram gives exactly the same result as our former procedure of giving the whole vector diagram an anti-clockwise rotation, and projecting on to a fixed vertical line.

2. Geometrical Addition.

We have seen in Fig. 474 that the momentary value is represented by the length OB, that is, by the projection of the diameter OE on the rotating line OX. As in Section 74 (page 240), it can be shown that the two electromotive forces OE_1 and OE_2 in Fig. 475 can be replaced by a resultant with an amplitude OR and with a phase angle ϕ.

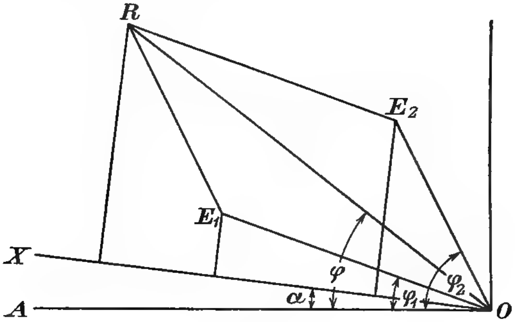

Fig. 475

In the following considerations we shall not trouble about momentary values, but shall consider simply the maximum value, or amplitude, and the phase of the wave. We can, naturally, use effective values instead of maximum values, since this simply alters the scale. The wave is completely determined by its amplitude and phase. Instead of giving the length OE and the angle ϕ, we can give the rectangular co-ordinates of the point E. In Fig. 476, for example, the wave represented by the diameter OE is completely determined by the abscissa $a = 3$ and the ordinate $b = 4$. We have further

$$OE = \sqrt{a^2 + b^2}$$

and

$$\tan\phi = \frac{b}{a}.$$

We shall follow the usual plan of calling those abscissae positive which lie to the right of O, while those to the left we shall call negative. Ordinates which are above O we shall call positive, and those below, negative. We shall multiply all ordinates (not abscissae) by the imaginary quantity $\sqrt{-1}$, which, for the sake of shortness we shall represent by j. On this assumption $+3$ represents a right-hand abscissa, while $+j.4$ represents an upward ordinate, and the wave represented by Fig. 476, which has an amplitude of E, is represented completely by the equation

$$E = 3 + j \cdot 4.$$

We shall follow the plan adopted by Steinmetz and use capital letters to denote amplitudes, and small letters to denote the projections on the co-ordinate axes.

We saw on page 241 that the projection of the resultant was equal to the sum of the projections of the components. If we are given two waves of different phase, we are now in a position to carry out the geometrical addition to find their resultant by simple calculation instead of the graphical method previously employed. The two electromotive forces E_1 and E_2, the phases of which are ϕ_1 and ϕ_2 (Fig. 477), have a resultant E of phase ϕ. The trigonometrical calculation of this resultant would be very laborious, and the graphical solution is not very easy when we have to set out the angles ϕ_1 and ϕ_2. The symbolic method enables us to solve the problem in the following manner:

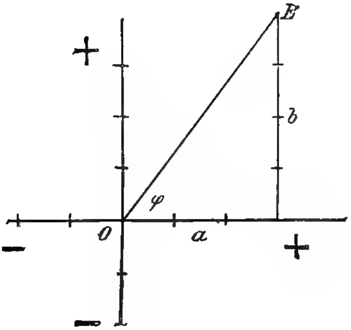

Fig. 476

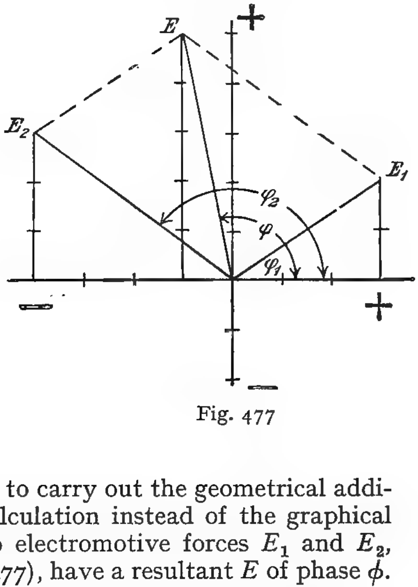

Fig. 477

$$\begin{aligned} E_1 &= +3 + j.2 \\ E_2 &= -4 + j.3 \\ \hline E = E_1 + E_2 &= -1 + j.5 \end{aligned}$$

We see that geometrical addition is no longer a mere name but an actual operation of addition. The + sign between E_1 and E_2 signifies their geometrical addition, and a glance at Fig. 477 shows us that the geometrical resultant has the co-ordinates -1 and $+j.5$. Passing now from imaginary or complex quantities to the calculation of real quantities, we have

$$E = \sqrt{(-1)^2 + 5^2} = \sqrt{26},$$

and $$\tan\phi = \frac{+5}{-1} = -5.$$

3. Rotation of a Vector.

Fig. 478 represents two electromotive forces of equal amplitude, one of which, viz. E_1, lags 90° behind the other. These are evidently represented by the equations

$$E_1 = +4 + j.3,$$

and $$E_2 = -3 + j.4.$$

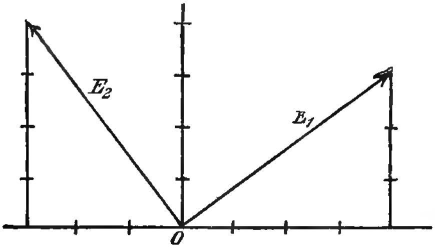

Fig. 478

Since $(j)^2$ is equal to -1, it follows that the wave E_2 can be obtained from E_1 by multiplying by j, thus

$$E_2 = j.E_1 = j.4 - 3.$$

Hence, multiplication by j simply rotates the diameter of the circle in Fig. 474 in the opposite direction to that in which the line OX is rotating, and gives a wave leading 90° ahead of the original wave.

In a similar manner we can obtain the wave E_1 by multiplying the wave E_2 by $-j$:

$$E_1 = -j.E_2 = -j(-3 + j.4),$$

or, since $(-j)(+j) = +1$, $$E_1 = 4 + j.3.$$

Hence, multiplication by – j denotes a rotation of the diameter in the direction in which the line OX is rotating, and gives a wave lagging 90° behind the original wave.

If two waves have a phase difference of 180°, it is evident that the co-ordinates of the one are the same as those of the other, except that they have the opposite sign. **Multiplication by – 1 will therefore rotate a wave through 180°.**

If we are given the amplitude and phase of the current, we are now in a position to express the ohmic pressure and the E.M.F. of self-induction in the symbolic method. Let the current be expressed by the equation

$$I = 3 - j.2,$$

let the resistance r be 2 ohms and the reactance $2\pi fL$, which for the sake of simplicity we shall denote by x, 1·5 ohms. The pressure required to overcome the simple ohmic resistance is found by multiplying I by r, so that

$$I.r = 2.(3 - j.2) = 6 - j.4.$$

Since the E.M.F. induced owing to the self-induction lags 90° behind the current, its equation is found by multiplying the current equation, first by x to get the magnitude, and then by $-j$ to put it back 90°. Hence

$$E_s = -j.x.I,$$

or, in the present example,

$$E_s = -j.1{\cdot}5.(3 - j.2) = -3 - j.4{\cdot}5.$$

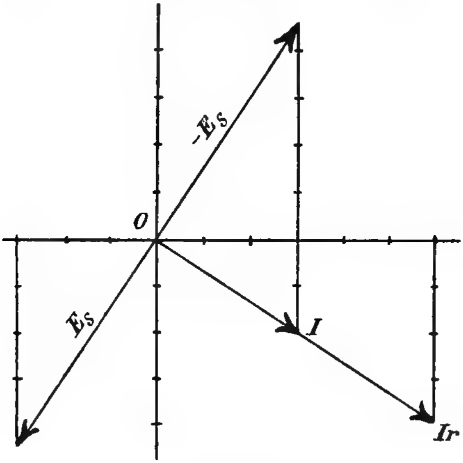

Fig. 479

This is confirmed by Fig. 479. If, instead of the self-induced E.M.F., we require the component of the terminal pressure which has to counterbalance the induced E.M.F., we must multiply the equation for E_s by -1. This gives us

$$-E_s = 3 + j.4{\cdot}5.$$

4. Inductance and Resistance in Series.

The component of the applied terminal pressure which is equal and opposite to the self-induced E.M.F., which it counterbalances, leads 90° ahead of the current. It is obtained by multiplying the current I by the reactance x, and also by j. This gives $j.x.I$. The applied terminal pressure is the resultant of this and the component which overcomes the resistance. We obtain by geometrical addition

$$V = I.r + j.x.I = I.(r + j.x).$$

The apparent total resistance $r + j.x$ is called the impedance and is denoted by Z, so that

$$Z = r + j.x.$$

The impedance is therefore made up of the two sides of a right-angled triangle, the hypotenuse of which gives its real value, thus

$$z = \sqrt{r^2 + x^2}.$$

Ohm's law for the alternating current circuit may be expressed as follows:

In the symbolic method

$$I = \frac{V}{Z} = \frac{E}{r + j \, . \, x}.$$

In the usual method

$$I = \frac{V}{z} = \frac{V}{\sqrt{r^2 + x^2}}.$$

We shall now determine the terminal pressure of a generator, the resistance of which is r_0 and the reactance x_0. The external circuit has a resistance r and a reactance x. The internal impedance Z_0 is equal to $r_0 + j . x_0$, and the external Z to $r + j . x$ (compare Fig. 249 on page 254). If we call the constant E.M.F. induced in the generator E_0 and the terminal pressure V, both expressed in the symbolic method, we have

$$I = \frac{E_0}{Z_0 + Z} = \frac{E_0}{r_0 + r + j \, . \, (x_0 + x)},$$

and

$$V = I \, . \, Z = \frac{E_0 \, . \, Z}{r_0 + r + j \, . \, (x_0 + x)}.$$

If we proceed now to find the real values, we have

$$V = \frac{E_0 \, . \, z}{\sqrt{(r_0 + r)^2 + (x_0 + x)^2}} = \frac{E_0 \, . \, z}{\sqrt{r_0^2 + 2r_0 r + r^2 + x_0^2 + 2x_0 x + x^2}}.$$

Now,

$$r_0^2 + x_0^2 = z_0^2,$$

and

$$r^2 + x^2 = z^2.$$

Hence,

$$V = \frac{E_0 \, . \, z}{\sqrt{z_0^2 + z^2 + 2\,(r_0 r + x_0 x)}}.$$

The same result could have been derived directly from Fig. 249.

5. Capacity and Resistance in Series.

We saw in Section 80 that the back E.M.F. of a condenser led 90° ahead of the current. From equation (126) on page 263 it is evident that this back pressure E_c is to be obtained from the current I by dividing by $2\pi fC$ and multiplying by j. Hence

$$E_c = j \, . \, \frac{I}{2\pi fC}.$$

The component of the terminal pressure which exactly counterbalances this back pressure is therefore $- j \, . \, \frac{I}{2\pi fC}$. If capacity be connected in series with resistance, the terminal pressure will be given by the equation

$$E = I \, . \, r - j \, . \, \frac{I}{2\pi fC} = I\left(r - j \, . \, \frac{1}{2\pi fC}\right).$$

For the impedance of the circuit we have therefore

$$Z = r - j \, . \, \frac{1}{2\pi fC}.$$

For the actual values we have

$$z = \sqrt{r^2 + \left(\frac{1}{2\pi fC}\right)^2}.$$

If we compare the two equations

$$Z = r + j \, . \, 2\pi fL$$

and

$$Z = r - j \, . \, \frac{1}{2\pi fC},$$

we see that $\frac{1}{2\pi fC}$ may be looked upon as a negative reactance. If we let the reactance x represent the component of the impedance which is at right angles to the current, whether due to self-induction or to capacity, the formula

$$Z = r + j \, . \, x$$

will be quite general. For inductance we have $x = 2\pi fL$, and for capacity $x = -\frac{1}{2\pi fC}$; Steinmetz calls $\frac{1}{2\pi fC}$ the condensance.

6. Resistance, Inductance and Capacity in Series.

The equation

$$I = \frac{V}{r + j \, . \, x}$$

can now be used when the circuit contains both inductance and capacity, if x be taken to represent $2\pi fL - \frac{1}{2\pi fC}$. The equation then becomes

$$I = \frac{V}{r + j \, . \left(2\pi fL - \frac{1}{2\pi fC}\right)}.$$

Passing from complex to real quantities, we have

$$I = \frac{V}{\sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}}.$$

The equation which we established for the terminal pressure of a generator was

$$V = \frac{E_0 \, . \, z}{\sqrt{z_0^2 + z^2 + 2\,(r_0 r + x_0 x)}}.$$

It can be seen from this equation that there is a certain value of the external impedance z, for which the terminal pressure is equal to the E.M.F. The exact value of this impedance depends on the internal impedance of the generator. For V to equal E_0 in the above equation, it is necessary that the denominator becomes z, that is, that

$$z_0^2 + 2\,(r_0 r + x_0 x) = 0.$$

If we neglect the small internal resistance r_0 we can put $x_0 = z_0$, and we get

$$x_0^2 + 2x_0 x = 0,$$

or

$$x = -\frac{x_0}{2}.$$

The negative sign shows that the condensance of the external circuit must exceed its reactance by an amount equal to half the internal reactance. When this condition is fulfilled the terminal pressure of the generator will be the same as its E.M.F. If the capacity of the external circuit is still further increased, the terminal pressure will exceed the E.M.F. induced in the armature. (Compare Fig. 314.)

7. Inductance and Capacity in Parallel.

Fig. 480 represents an inductive resistance shunted by means of a condenser. For the upper branch we have

$$I_1 = \frac{V}{r + j\,.\,x}.$$

The current I_2 in the condenser branch leads 90° ahead of the terminal pressure V, and is therefore given by the equation

$$I_2 = j\,.\,2\pi f C\,.\,V.$$

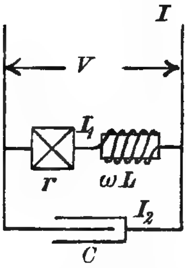

Fig. 480

For the total current I we have, therefore,

$$I = I_1 + I_2 = \frac{V}{r + j\,.\,x} + j\,.\,2\pi f C\,.\,V.$$

By putting V outside, and multiplying top and bottom by $r - j.x$, in order to remove the complex quantities from the denominator, we get

$$I = V\,.\left\{\frac{r - j\,.\,x}{r^2 + x^2} + j\,.\,\omega C\right\}.$$

We have put ω for $2\pi f$, the angular velocity of the vector. The equation may be altered so as to separate horizontal from vertical components, thus

$$I = V\,.\left\{\frac{r}{r^2 + x^2} - j\,.\left(\frac{x}{r^2 + x^2} - \omega C\right)\right\} \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

To obtain real values we must add the squares of the two components and extract the root. We thus obtain

$$I = V\,.\sqrt{\left(\frac{r}{r^2 + x^2}\right)^2 + \left(\frac{x}{r^2 + x^2} - \omega C\right)^2}.$$

By suitably choosing the capacity it is possible to bring the external or total current I into phase with the terminal pressure V, so that the whole arrangement appears non-inductive. The condition for equality of phase between I and V is evidently the disappearance of the imaginary component in equation (*a*). This can only occur if

$$2\pi f C = \frac{x}{r^2 + x^2}.$$

We notice that the necessary capacity depends on the frequency, on the inductance, and also on the resistance. Wattless current in the external circuit can therefore be eliminated by the use of suitable capacity, but a change in either the frequency or the resistance upsets the compensation and causes a phase difference between current and pressure.

8. Admittance, Conductance and Susceptance.

As we pointed out on page 10, it is often more convenient to consider conductance than its reciprocal, resistance. In calculating the combined resistance of several parallel paths, we obtained the equation

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\cdots\cdots$$

This equation expresses the result directly and simply in terms of the conductances of the various paths. Steinmetz introduced this and other reciprocals into the symbolic system. The reciprocal of impedance he termed *admittance*, and represented it by the letter Y. We have then

$$Y=\frac{1}{Z}, \qquad y=\frac{1}{z},$$

and

$$Y=\frac{1}{Z}=\frac{1}{r+j.x}=\frac{r-j.x}{r^2+x^2}=\frac{r}{r^2+x^2}-\frac{j.x}{r^2+x^2}.$$

We see then that the admittance consists of two components, the one real, the other imaginary. The real component is called the conductance and is denoted by g, while the imaginary component is called the susceptance and is denoted by b. Hence

$$g=\frac{r}{r^2+x^2}=\frac{r}{z^2} \quad \cdots\cdots\cdots\cdots\cdots\cdots(a),$$

$$b=\frac{x}{r^2+x^2}=\frac{-x}{z_2} \quad \cdots\cdots\cdots\cdots\cdots\cdots(b),$$

$$Y=g+j.b.$$

Ohm's law may now be written thus:

$$I=\frac{E}{Z}=E.Y,$$

or

$$I=E.(g+j.b).$$

The actual value of the admittance is given by the equation

$$y=\sqrt{g^2+b^2}=\sqrt{\frac{r^2}{z^4}+\frac{x^2}{z^4}},$$

or, since $r^2+x^2=z^2$,

$$y=\sqrt{\frac{1}{z^2}}=\frac{1}{\sqrt{r^2+x^2}}.$$

To illustrate the use of these equations, we shall again consider the problem of making the external current in Fig. 480 in phase with the terminal pressure by suitably choosing the capacity of the condenser. To do this we must make the imaginary component in the equation

$$I=E.(g+j.b) \quad \cdots\cdots\cdots\cdots\cdots\cdots(c)$$

vanish. That is, the susceptance b must vanish.

If the upper branch in Fig. 480 has a reactance x_1, its susceptance is given by equation (b) as

$$b_1=\frac{-x_1}{z^2}=\frac{-x_1}{{x_1}^2+{r_1}^2}.$$

In the lower branch, $x = -\frac{1}{\omega C}$ and $r = 0$, so that equation (*b*) gives

$$b_2 = +\omega C.$$

For the combined susceptance we have

$$b = b_1 + b_2 = \frac{-x_1}{x_1^2 + r_1^2} + \omega C.$$

If there is to be no phase displacement between E and I, the second term in equation (*c*) must vanish, since it is this component which causes the displacement. Hence

$$b = \frac{x_1}{x_1^2 + r_1^2} - \omega C = 0,$$

or

$$\omega C = \frac{x_1}{x_1^2 + r_1^2}.$$

It is noteworthy—

(1) that the conductance and susceptance of the individual branches are not the reciprocals of their resistance and reactance respectively, but are given by the equations

$$g = \frac{r}{z^2} \text{ and } b = \frac{-x}{z^2};$$

(2) that neither the resistance nor the reactance of the combined circuit is equal to the sum of the resistances or reactances of the separate parallel branches.

In order to find b in the above example, it was necessary to find b_1 and b_2 for each branch separately, and then to add them together.

LIST OF SYMBOLS USED

CAPITAL LETTERS

A	area of cross-section in sq. cms.
B	flux density per sq. cm.
C	capacity in farads; a constant, see p. 403
D	diameter of armature in cms.
E	electromotive force in volts (occasionally in C.G.S. units)
E_{12}	E.M.F. induced in winding 2 by current in winding 1
F	force in kilogrammes
H	strength of magnetic field
I	current in amperes
I'	current in absolute units
L	axial length of armature; self-induction in henries
M	mass in grammes; coefficient of mutual induction
M_i	moment of inertia
M_t	turning moment or torque
M_u	useful or nett turning moment
P	power in watts; power output from rotor
P_1	total power supplied to stator
P_2	power transmitted to rotor
Q	quantity of electricity in coulombs
Q_h	quantity of heat in calories
R	resistance in ohms; reluctance
S	speed of car in kms. per hour; number of slots in armature
T	temperature Centigrade; duration of short circuit; number of turns in coil
U	number of coil-sides on armature
V	volume in cubic cms.; potential difference in volts
W	weight in kilogrammes; work or energy
X	ampere-turns
Y	admittance
Z	impedance; total number of wires on periphery

SMALL LETTERS

a	acceleration; atomic weight; pairs of parallel paths
b	breadth of slot; susceptance
c	current density; various coefficients
d	depth of slot
e	instantaneous value of electromotive force
f	force in dynes; tractive force in kgs. per 1000 kgs.; frequency in cycles per second; frequency of slip
f_1	frequency of supply in induction motors

g	acceleration due to gravity; conductance; slots per coil-side
h	horizontal component of earth's magnetism
i	instantaneous value of current in amperes
j	$\sqrt{-1}$ in symbolic treatment
k	valency; various coefficients
l	length
m	strength of pole
n	revolutions per second
p	pairs of poles
q	instantaneous value of quantity of electricity
q'	wires per coil-side, i.e. per pole per phase
r	resistance in ohms; radius in metres; distance in cms.
s	slope in parts per thousand; distance in cms.; specific heat
t	time in seconds; natural period of oscillating alternator
t_1	forced period of oscillating alternator
u	ratio of primary to secondary turns of transformer
v	velocity; instantaneous value of terminal pressure in volts
w	weight of deposit
x	reactance
y	admittance; winding step
z	impedance
z'	wires in series or wires per phase

GREEK LETTERS

α	twice the angular brush displacement; variable angle; temp. coeff.
β	ratio of pole-arc to pole-pitch
γ	ratio of half width of coil-side to pole-pitch; phase angle between pole and terminal pressure of alternator
Γ	output of converter ÷ output of dynamo
δ	$2\gamma + \beta - 1$
ϵ	phase displacement due to engine and alternator; 2·71828; emissivity
ϵ_1	phase displacement due to engine only (see p. 341)
η	efficiency; hysteresis coefficient
μ	permeability
ν	number of tappings per pole pair in rotary converter; leakage coefficient; ratio of speed to synchronous speed
ρ	specific resistance (see p. 8)
τ_1	air-gap reluctance ÷ primary leakage reluctance
τ_2	air-gap reluctance ÷ secondary leakage reluctance
τ	$\tau_1 + \tau_2 + \tau_1\tau_2$
Φ	magnetic flux; flux per pole
ϕ	angular phase displacement
ϕ_1	angle between current and E.M.F. of alternator
ω	angular velocity; $2\pi f$

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